LIE GROUPS, ASSIGNMENT 2, DUE OCTOBER 5

In this assignment (and hopefully later in class), I will write $Sp(n) = \{g \in GL_n(\mathbb{H}) : gg^* = I\}$.

(1) Prove that
$$O(n) = \{g \in M_n(\mathbb{R}) : |gv| = |v| \text{ for all } v \in \mathbb{R}^n\}$$

(2) If we identify $\mathbb{C}^n = \mathbb{R}^{2n}$ and $\mathbb{H}^n = \mathbb{R}^{4n}$, prove that $U(n) \subset O(2n)$ and $Sp(n) \subset O(4n)$ and hence both of these groups are compact.

(3) After identifying $\mathbb{H}^n = \mathbb{C}^{2n}$, prove that $Sp(n) = U(2n) \cap Sp_{2n}(\mathbb{C})$.

(4) Prove that $\mathfrak{sp}(n)_\mathbb{C} = \mathfrak{sp}_{2n}(\mathbb{C})$ (here the left hand side is the complexification of the Lie algebra of $Sp(n)$ and the right hand side is the complex symplectic Lie algebra.) Optional: prove the same thing about the corresponding groups.

(5) Consider the map $\exp : \mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$. Prove that
$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$
is not in the image of exp. Describe precisely the image of exp.

(6) Consider the action of $SL_2(\mathbb{R})$ on $\mathbb{RP}^1$ by
$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right][x, y] = [ax + by, cx + dy]$$
Write explicitly the vector fields corresponding to the Lie algebra elements
$$E = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right],\ H = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right],\ F = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$$
on the open subset $\mathbb{R} = \mathbb{RP}^1 \setminus \{[0, 1]\}$.

(7) Let $n \in \mathbb{N}$. Consider the Heisenberg Lie algebra $\mathfrak{h}_n$. It is an $2n+1$ dimensional real vector space with basis $p_1, \dots, p_n, q_1, \dots, q_n, c$ and with the only non-trivial Lie bracket of the generators given by $[p_i, q_i] = 1$ for $i = 1, \dots, n$.

Find a faithful representation of $\mathfrak{h}_n$, i.e. realize $\mathfrak{h}_n$ as a Lie subalgebra of $\mathfrak{gl}_m(\mathbb{R})$ for some m. Find a Lie group H_n integrating $\mathfrak{h}_n$. (An aside: why is it called the Heisenberg Lie algebra?)

(8) Find all 2-dimensional Lie algebras over $\mathbb{R}$. Find all connected 2-dimensional Lie groups.

(9) Construct an isogeny $SL_4(\mathbb{C}) \to SO_6(\mathbb{C})$. Hint: think about $\Lambda^2\mathbb{C}^4$.