LIE GROUPS, ASSIGNMENT 4, DUE NOVEMBER 9

(1) For $n \in \mathbb{N}$, let $V(n)$ be the n-dimensional representation of $SU(2)$. Let $n, m \in \mathbb{N}$. Find the decomposition of $V(n) \otimes V(m)$ into irreducible representations.

(2) A ballot sequence is a sequence $(a_1, \dots, a_n)$ where each $a_i \in \{+1, -1\}$ and $a_1 + \cdots + a_k \geq 0$ for $k = 1, \dots, n$. Prove that the multiplicity of $V(m)$ in $V(1)^{\otimes n}$ equals the number of ballot sequences $(a_1, \dots, a_n)$ such that $a_1 + \cdots + a_n = m$.

(3) Prove that $\dim(V(1)^{\otimes 2n})^{SU(2)}$ equals $\frac{1}{n+1}\binom{2n}{n}$ (the nth Catalan number).

(4) Let $\lambda \in \mathbb{C}$. Consider the infinite-dimensional representation $M(\lambda)$ of $\mathfrak{sl}_2$ freely generated by a vector v_0 satisfying $Ev_0 = 0, Hv_0 = \lambda v_0$. More precisely, we can define $M(\lambda) := U\mathfrak{sl}_2/J$ where J is the left ideal generated by E and $H - \lambda$.

 (a) Prove that $M(\lambda)$ is irreducible if λ is not a positive integer.

 (b) When λ is a positive integer, find a non-trivial subrepresentation.

 (c) Fix an isomorphism $M(\lambda) \cong \mathbb{C}[x]$ where E acts on $\mathbb{C}[x]$ by ∂_x and find expressions for H and F as vector fields.

(5) Prove that $V(n)$ always carries a invariant bilinear form. For which n is this form symmetric / skew-symmetric?

(6) Let k be a field of characterstic 2. Prove that $\mathfrak{sl}_2(k)$ is nilpotent.

(7) Let $\mathfrak{h}$ be a semisimple Lie algebra and let V be an irreducible representation. Define a Lie algebra $\mathfrak{g}$ which equals $\mathfrak{h} \oplus V$ as a vector space, and where the Lie bracket is given by

$$[(X_1, v_1), (X_2, v_2)] := ([X_1, X_2], X_1 v_2 - X_2 v_1)$$

Prove that $\mathfrak{g}$ is perfect and that $rad(\mathfrak{g}) = V$.

(8) Prove that a Lie algebra $\mathfrak{g}$ is semisimple if and only if it has no non-zero abelian ideals.

(9) Prove that every irreducible representation of a solvable Lie algebra is 1-dimensional and is given by an element of $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.