Parameterization of invariant manifolds for periodic orbits (II): a-posteriori analysis and computer assisted error bounds

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Abstract

In this paper we develop mathematically rigorous computer assisted techniques for studying high order Fourier-Taylor parameterizations of local stable/unstable manifolds for hyperbolic periodic orbits of analytic vector fields. The parameterizations studied here are not required to be graphs over the stable/unstable eigenspace and can follow folds in the embedding. In addition to providing the embedding of the manifold, the parameterization also gives the dynamics on the manifold in terms of an explicit conjugacy relation. We exploit the numerical methods developed in [1] in order to obtain a high order Fourier-Taylor series expansion of the parameterization. There is no a-priori theory which guarantees the accuracy of this approximation far from the periodic orbit, and the main result of the present work is an a-posteriori Theorem which provides mathematically rigorous error bounds. The hypotheses of the theorem are checked with computer assistance. The argument relies on a sequence of preliminary computer assisted proofs where we validate the numerical approximation of the periodic orbit, its stable/unstable normal bundles, and the jets of the manifold to some desired order M. The validation of the orbit and bundles is based on existing computer assisted methods, but the validation of the jets is new and explained in detail here. We illustrate our method by implementing validated computations of some two dimensional manifold in $\mathbb{R}^3$ and a three dimensional manifold in $\mathbb{R}^4$.

1 Introduction

The present work is the second paper in a series started in [1]. The purpose of this series is to study the partial differential equation (1) below. Our interest in Equation (1) is due to the fact that its solutions parametrize local stable/unstable manifolds associated with hyperbolic periodic orbits of ordinary differential equations. We focus on ordinary differential equations given by analytic vector fields, so the manifolds and hence the solutions of Equation (1) we consider are analytic.

Paper (I) [1] is devoted to efficient numerical solution of the partial differential equation. Since solutions of Equation (1) parametrize embedded cylinders in phase space, it is natural

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to look for a formal solution expressed as a Fourier-Taylor series. The numerical scheme developed in Paper (I) exploits Floquet theory to find elementary recursion relations for the Fourier-Taylor coefficients of the formal series. These recursion relations can be rapidly solved to any desired order.

The present paper is concerned with convergence of the formal series just mentioned. Our main result is Theorem 2.4, an a-posteriori theorem which allows us to establish the existence of a true solution of Equation (1) supposing we have a "good enough" approximate solution. The theorem also provides mathematically rigorous C^0 bounds between the approximate and true solution. The hypotheses of the theorem are checked via finite numerical computations. In order to illustrate the use and performance of our method we implement computer assisted convergence proofs for some example applications, namely we compute some two dimensional manifolds for the Lorenz system as well as some three dimensional manifolds for a simplified suspension bridge equation.

Both papers in this series are based on a functional analytic framework for studying invariant manifolds known as the Parameterization Method. The theoretical core of this method is developed in the work of [2, 3, 4, 5], and we refer the reader back to paper (I) for more complete discussion of the method and its literature. The interested reader may also want to consult the recent book of [6]. Presently we recall only the main philosophy of the Parameterization Method, which is that many of the smooth invariant manifolds of dynamical systems theory are characterized by an (infinitesimal) conjugacy, or *invariance equation*. The invariance equation, which often takes the form of a nonlinear operator on a Banach space, may be approximately solved on the digital computer using existing tools of numerical analysis. The operator equation may also be susceptible to a-posteriori analysis, that is once we obtain a good numerical approximation it may be possible to obtain mathematically rigorous error bounds via a computer assisted Newton-Kantorovich argument in the tradition of [7].

The reader interested in the Parameterization Method as a framework for computer assisted proof can consult the works of [8, 9, 10, 11, 12, 13] and the references discussed therein. We remark that the works just cited deal with stable/unstable manifolds of fixed points for maps/equilibria for differential equations, and also with invariant circles for area preserving maps and their bundles. The present work is a contribution in this vein, where we extend existing validated numerical methods based on the Parameterization Method to hyperbolic periodic orbits of differential equations.

In Section 1.1 we refine the discussion above, and review as much of the Parameterization Method as is needed for the present work. In the section just mentioned we also outline the main steps of the validation argument developed in the remainder of the paper. Before moving on however, several remarks are in order.

Remark 1.1 (First order constraints)**.** Solutions of Equation (1) below are dynamically meaningful only after the imposition of certain first order constraints. These constraints require that the periodic orbit, its stable or unstable Floquet exponents, and its stable or unstable normal bundles are known "exactly". Here *exactly* is interpreted in the sense of validated numerics, that is we need numerical approximations of this data along with explicit mathematically rigorous error bounds.

Since our goal is to validate parameterizations of local stable/unstable manifolds using Fourier spectral methods, we require the first order data is given as Fourier series. Indeed, the methods of the present work exploit complex analytic properties of the first order data and we actually require some knowledge about domains of analyticity of the periodic orbit and the stable/unstable bundles (bounds on the size of a strip about the real axis in the complex plane into which the periodic functions can be extended analytically). In practice this information is not readily available and some preliminary work is required.

Recent advances in computer assisted Fourier analysis of periodic solutions of analytic differential equations [14, 15] and computer assisted Floquet analysis [16, 17] allow us to obtain the desired representation of the first order data. The methods of the works just cited also provide the lower bounds on the domain of analyticity mentioned in the previous paragraph. See Remark 2.9 for more detailed discussion of this point.

Remark 1.2 (Features of the Parameterization Method)**.** Many functional analytic methods for proving stable manifold theorems are based on "graph transform" type arguments, Lyapunov-Perron operators, or sequence space arguments ala Irwin (see a standard text on dynamical systems such as [18, 19, 20]). While it is possible to adapt such arguments for a-posteriori computer assisted analysis this approach has some draw backs, such as the need to compute the composition of the unknown parameterization with itself, and the requirement that the representation of the manifold is expressed as the graph of a function.

The Parameterization Method on the other hand, requires only composition of the unknown function with the known vector field. Moreover there is no requirement that the parameterization be the graph of a function, hence it is possible to follow folds in the embedding. Another advantage is that the parameterization of the manifold satisfies a conjugacy relation which recovers the dynamics on the manifold in addition to the embedding.

The price we pay is the appearance of some non-resonance conditions between the Floquet exponents of the periodic orbit. These non-resonance conditions have no analogue in the classical approaches mentioned in the first paragraph of this remark. However we must point out that (a)-the non-resonance conditions are satisfied "generically" that is for open sets in the parameter space of the vector field, and (b)-that it is possible to modify the underlying conjugacy relation in the Parameterization Method (that is conjugate to a polynomial rather than a linear vector field) in order to treat the resonant cases as well. We do not consider such degeneracies further in the present work. The interested reader can consult the works of [2, 3, 4] for more complete discussion of the resonant case, and can also see the work of [10] for computer aided proofs for resonant stable/unstable manifolds attached to equilibrium solutions of differential equations.

Remark 1.3 (Geometric methods: covering relations and cone conditions)**.** Another approach to stable manifold theory is based on topological degree theory and cone conditions applied directly in the phase space. See for example the work of [21] on stable manifolds of fixed points of maps and the work of [22] on normally hyperbolic invariant manifolds. These geometric methods are well suited for adaptation to computer assisted proof, as is illustrated by the work of [23, 24, 25] (see also the references discussed therein).

Geometric methods have been used to give computer assisted proofs of a number of conjectures in celestial mechanics involving the existence and intersection of stable/unstable manifolds for periodic orbits [26, 27]. One of the advantage of the geometric methods is that they require only C^1 or C^2 assumptions on the vector field. Geometric methods also make only weak hyperbolicity assumptions, so for example there no non-resonance assumptions to check. In fact the geometric methods have been used to give elementary computer assisted proofs of the existence of center manifolds, see for example the work of [28], again in the context of celestial mechanics.

Of course the geometric methods result in only C^1 (or sometimes only Lipschitz) information about the manifold under consideration. Moreover these methods show the existence of the manifold (often that the manifold is contained somewhere inside the union of a collection of polygons) but do not explicitly recover the dynamics, that is one obtains conclusions about the asymptotics of orbits but no conjugacy is obtained. Analytic properties of the embedding such as decay rates of jets or domain of analyticity seem to be unavailable using these methods.

The geometric and analytic methods (such as the analytic methods developed in the present work) complement one another, and the choice of method in a particular problem depends on the desired results. This state of affairs (not surprisingly) mirrors the state of affairs in the classical qualitative "pen and paper" theory of dynamical systems, where many of the most important results have both geometric and functional analytic proofs.

Remark 1.4 (Automatic differentiation for Fourier series)**.** In the present work we implement our argument for vector fields with polynomial nonlinearities. This simplifies the technical details somewhat but is not a fundamental limitation. By exploiting ideas from "automatic differentiation" it is possible to apply the Fourier-Taylor methods of the present work to study analytic vector fields with nonlinearities given by elementary functions. For more complete discussion of automatic differentiation as a numerical tool in dynamical systems theory we refer to the works of [29, 30, 31, 32] and the references therein, though the list is by no means complete. The reader interested in automatic differentiation as a tool for computer assisted proof in Fourier analysis might consult the work of [15].

Remark 1.5 (Computer assisted proof of connecting orbits)**.** While the invariant manifolds studied in the present work are of interest in their own right, we also remark that intersections of stable/unstable manifolds play a central role in the global study of nonlinear systems. By studying the intersections of stable/unstable manifolds it is possible to learn about orbits which connect invariant sets to one another. In addition to being a critical component of Melnikov theory (see for example the classical works of [33, 34, 35, 36]) the intersections of stable/unstable manifolds explain global phenomena such as Arnold diffusion [37] and transport in celestial mechanics and fluid systems [38, 39, 40, 41, 42]. Of course this list of references does not even scratch the surface of the literature, and is only meant to point in the direction of further reading.

Given the importance of connecting orbits in dynamical systems theory it is not surprising that many authors have developed numerical computational methods. We refer for example to the works of [43, 44, 45, 46, 47, 48] and the references discussed therein for much more complete discussion, though again this is by no means a complete list of references. Existence for connecting orbits occupies a central place in the computer assisted proof literature and we refer the interested reader to the works of [49, 50, 51, 52, 53, 54, 55, 9, 56, 8, 57] and the references discussed therein.

Moreover we remark that the last two references just cited incorporate the Parameterization Method in a fundamental way, obtaining computer aided proofs of connecting orbits between equilibria of differential equations and fixed points of maps. A feature of the approach developed in these two papers is that the method of proof obtains the transversality of the connecting orbit "for free" (that is if the method succeeds then the intersection of the stable/unstable manifolds are transverse). Combining the methods of the present work with a mathematically rigorous method for computer assisted analysis of boundary value problems as in [57] would be a natural extension, and will make the topic of a future study. The proposed method would give automatic transversality for connecting orbits between periodic orbits.

1.1 Review of the parameterization method for stable/unstable manifolds of periodic orbits

In this section, and throughout the remainder of the paper, we assume the reader is familiar with classical stability analysis/Floquet theory for periodic orbits of differential equations (an excellent reference for this material is the book of [18]). We also remark that the material which we briefly review in this section is meant to provide context for the work done in the remainder of the present work. The reader interested in more thorough discussion may also want to consult the works of [2, 3, 58, 59, 6] for more general coverage of the Parameterization Method, and [60, 61, 62, 4, 1] for more discussion of the Parameterization Method in the context of periodic orbits.

Now a little notation. With $w \in \mathbb{C}$, let $|w| = \sqrt{\text{real}(w)^2 + \text{imag}(w)^2}$ denote the complex absolute value. We endow $\mathbb{C}^d$ with the max norm, so that if $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ then

$$\|z\|_d = \max_{1 \leq j \leq d} |z_j|.$$

Let

$$\mathbb{A}_r \stackrel{\text{def}}{=} \{w \in \mathbb{C} : |\text{imag}(w)| < r\},$$

denote the complex strip of width r about the real axis. With $T \in \mathbb{R}$, $T > 0$ we say that the complex function $\gamma \colon \mathbb{A}_r \to \mathbb{C}^d$ is T-periodic on $\mathbb{A}_r$ if

$$\gamma(w + T) = \gamma(w),$$

for all $w \in \mathbb{A}_r$. A function $\gamma \colon \mathbb{A}_r \to \mathbb{C}^d$ is said to be analytic on $\mathbb{A}_r$ if each component of γ is complex differentiable at each $w \in \mathbb{A}_r$.

For $k \in \mathbb{N}$, $k \geq 1$ let

$$\mathbb{D}_\nu^k \stackrel{\text{def}}{=} \left\{ z = (z_1, \ldots, z_k) \in \mathbb{C}^k : \|z\|_k < \nu \right\},$$

denote the poly-disk of radius ν about the origin in $\mathbb{C}^k$. Throughout the sequel we are interested in functions $P \colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ with $d \geq k + 1$. We say that P is T-periodic in the first variable (or simply that P is T-periodic) if

$$P(w + T, z_1, \ldots, z_k) = P(w, z_1, \ldots, z_k),$$

for all $(w, z_1, \ldots, z_k) \in \mathbb{A}_r \times \mathbb{D}_\nu^k$. We say that P is analytic on $\mathbb{A}_r \times \mathbb{D}_\nu^k$ if each component of P is complex differentiable in each variable separately at each point $(w, z_1, \ldots, z_k) \in \mathbb{A}_r \times \mathbb{D}_\nu^k$.

Let $T > 0$, $f \colon \mathbb{C}^d \to \mathbb{C}^d$ be an analytic vector field, and fix the complex numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$. We are interested in $2T$-periodic, analytic functions $P \colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ solving the partial differential equation

$$\frac{\partial}{\partial w} P(w, z_1, \ldots, z_k) + \sum_{j=1}^{k} \lambda_j z_j \frac{\partial}{\partial z_j} P(w, z_1, \ldots, z_k) = f[P(w, z_1, \ldots, z_k)]. \tag{1}$$

When properly constrained, Equation (1) has special dynamical significance for the vector field f. In order to make this precise we state the following assumptions.

Assumption 1. Suppose that $\gamma \colon \mathbb{A}_r \to \mathbb{C}^d$ is an analytic, T-periodic solution for the vector field f, that is

$$\frac{d}{dw} \gamma(w) = f[\gamma(w)],$$

and $\gamma(w + T) = \gamma(w)$ for all $w \in \mathbb{A}_r$.

Assumption 2. Suppose that $\lambda_1, \ldots, \lambda_k$ are the stable (or respectively unstable) Floquet exponents for γ. Suppose in addition that these are *all* of the stable (respectively unstable) exponents, so that any remaining exponents are unstable or neutral (respectively stable or neutral). Suppose now that $\xi_j \colon \mathbb{A}_r \to \mathbb{C}^d$ are analytic, $2T$-periodic functions parameterizing the stable (respectively unstable) normal bundle of γ. More precisely suppose that

$$\text{real}(\lambda_j) < 0, \quad \text{for } 1 \leq j \leq k$$

(respectively that $\text{real}(\lambda_j) > 0$, for $1 \leq j \leq k$) and that $\xi_j(w + 2T) = \xi_j(w)$ for all $w \in \mathbb{A}_r$ and solve the eigenvalue problem

$$\frac{d}{dw}\xi_j(w) + \lambda_j \xi_j(w) - Df[\gamma(w)]\xi_j(w) = 0, \quad \text{for } 1 \leq j \leq k.$$

Remark 1.6 (**Orientation of the stable/unstable normal bundles**). The functions $\xi_j(w)$ are $2T$-periodic as the stable/unstable bundles need not be orientable. Moreover, we can find a T-periodic basis function $\xi_j(w)$ if and only if the associated bundle is orientable.

We now recall several properties enjoyed by solutions of Equation (1).

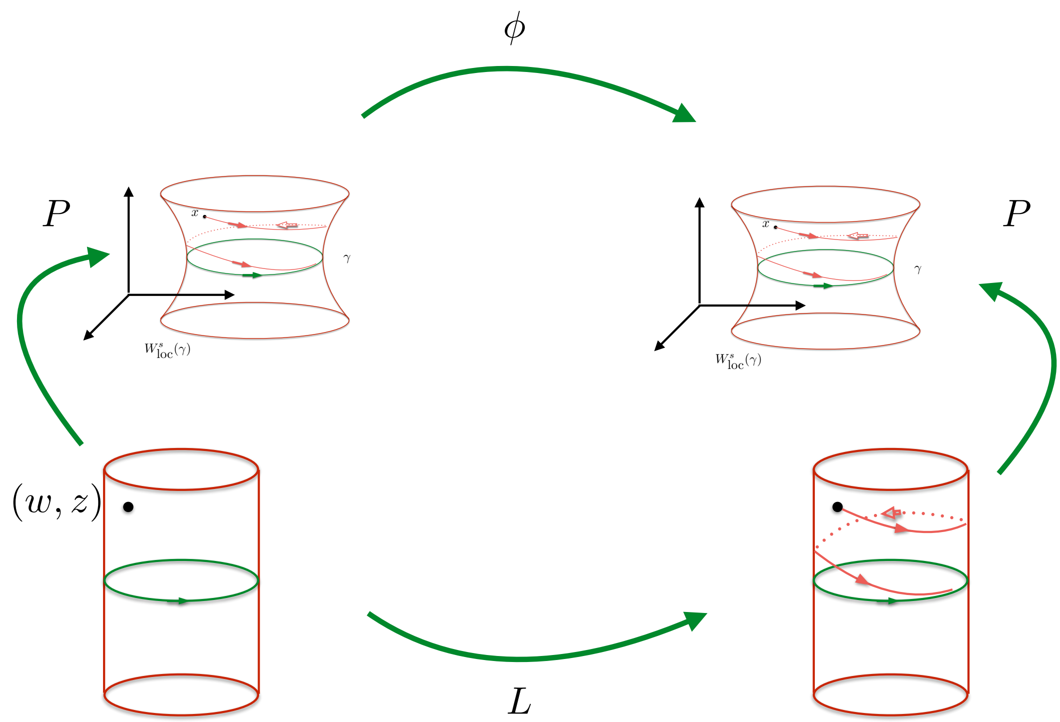

Figure 1: Cartoon of the flow conjugacy Equation (4) satisfied by solutions of Equation (1).

Claim 1. (**Flow conjugacy**) Let $\phi \colon \mathbb{R} \times \mathbb{C}^d \to \mathbb{C}^d$ denote the flow generated by f. (In fact we only need that ϕ is defined in a neighborhood of γ). Suppose that $P \colon \mathbb{A}_r \times \mathbb{D}^k_\nu \to \mathbb{C}^d$ is an analytic $2T$-periodic solution of Equation (1) which satisfies the first order constraints

$$P(w, 0, \ldots, 0) = \gamma(w), \tag{2}$$

and

$$\frac{\partial}{\partial z_j} P(w, 0, \ldots, 0) = \xi_j(w), \quad \text{for } 1 \leq j \leq k. \tag{3}$$

Then the image of P is a local stable (respectively unstable) manifold for γ. In fact, P satisfies the flow conjugacy relation

$$\phi\left[P(w, z_1, \ldots, z_k), t\right] = P(w + t, e^{\lambda_1 t} z_1, \ldots, e^{\lambda_k t} z_k), \tag{4}$$

for all $t \geq 0$ (respectively $t \leq 0$), that is the dynamics on the local manifolds parameterized by P are conjugate to the linear flow $L\colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{A}_r \times \mathbb{D}_\nu^k$ given by

$$L(w, z_1, \ldots, z_n, t) \overset{\text{def}}{=} \left(w + t, e^{\lambda_1 t} z_1, \ldots, e^{\lambda_k t} z_k\right).$$

The geometric meaning of this conjugacy is illustrated in Figure 1. For the elementary proof see Theorem 2.6 in [1].

Remark 1.7 (**Real vector fields, orbits, bundles, and parameterizations).** The case of a real analytic vector field is of special interest, that is when $x \in \mathbb{R}^d \subset \mathbb{C}^d$ implies that $f(x) \in \mathbb{R}^d$. In this case we are especially interested in real analytic periodic orbits, that is $\gamma\colon \mathbb{A}_r \to \mathbb{C}^d$ having that $\gamma(t) \in \mathbb{R}^d$ when $t \in \mathbb{R}$. If γ is real analytic and $\lambda_j \in \mathbb{R}$ for $1 \leq j \leq k$ then the basis functions ξ_j can be chosen real analytic. In this case the solution P of Equation (1) can be taken real analytic. Another case of interest is that λ_j and λ_{j+1} are a complex conjugate pair. In this case the associated basis functions ξ_j and ξ_{j+1} can be taken as complex conjugates and arrange that P maps associated complex conjugate variables into $\mathbb{R}^d$. In other words, P is no longer real analytic, but there is a canonical method for obtaining the real image of P and hence the real stable (unstable) manifold associated with $\gamma \subset \mathbb{R}^d$. See [1] for more complete discussion.

Claim 2. (**Non-uniqueness**) Solutions of Equation (1) are not unique. Indeed suppose that P is a solution of Equation (1) constrained by Equations (2) and (3), and consider any collection $\Gamma = \{\tau_1, \ldots, \tau_k\}$ of non-zero positive real scalars. Define the disk

$$\mathbb{D}_{\Gamma,\nu}^k \overset{\text{def}}{=} \left\{(z_1, \ldots, z_k) \in \mathbb{C}^k : |z_j| \leq \frac{\nu}{\tau^j} \text{ and } 1 \leq j \leq k\right\},$$

and the function $Q\colon \mathbb{A}_r \times \mathbb{D}_{\Gamma,\nu}^k \to \mathbb{C}^d$ by

$$Q(w, z_1, \ldots, z_s) \overset{\text{def}}{=} P(w, \tau_1 z_1, \ldots, \tau_k z_k).$$

By differentiating and evaluating at zero we see that

$$Q(w, 0, \ldots, 0) = P(w, 0, \ldots, 0) = \gamma(w),$$

and that

$$\frac{\partial}{\partial z_j} Q(w, 0, \ldots, 0) = \tau_j \frac{\partial}{\partial z_j} P(w, 0, \ldots, 0) = \tau_j \xi_j(w). \tag{5}$$

Moreover

$$\begin{aligned} f[Q(w, z_1, \ldots, z_k)] &= f[P(w, \tau_1 z_1, \ldots, \tau_k z_k)] \\ &= \frac{\partial}{\partial w} P(w, \tau_1 z_1, \ldots, \tau_k z_k) + \sum_{j=1}^{k} \lambda_j \tau_j z_j \frac{\partial}{\partial z_j} P(w, \tau_1 z_1, \ldots, \tau_k z_k) \\ &= \frac{\partial}{\partial w} Q(w, z_1, \ldots, z_k) + \sum_{j=1}^{k} \lambda_j z_j \frac{\partial}{\partial z_j} Q(w, z_1, \ldots, z_k), \end{aligned}$$

as P is a solution of Equation (1). Then Q is an $2T$ periodic solution of Equation (1) on $\mathbb{A}_r \times \mathbb{D}^k_{\Gamma,\nu}$, satisfying constraint Equations (2) and (3) (with a rescaled choice of basis functions for the normal bundle). Since P is analytic so is Q. If $\tau_j < 1$ for $1 \leq j \leq k$ then Q is also a solution of Equation (1) on $\mathbb{A}_r \times \mathbb{D}^k_{\nu}$, hence the non-uniqueness. If the $\tau_j > 1$ then the question of whether or not Q is a solution on $\mathbb{A}_r \times \mathbb{D}^k_{\nu}$ is a question we return to momentarily, indeed it is one of the main concerns of the present work.

Remark 1.8 (Rescaling the basis of the stable/unstable normal bundle). Equation (5) shows that rescaling the domain by Γ leads to a corresponding rescaling of the basis functions for the stable (respectively unstable) normal bundle of the periodic orbits γ. In fact this is the *only* source of non-uniqueness in the problem, that is once the scalings of the ξ_j are fixed then the solution of Equation (1), if it exists, is unique. See Claim 3 below and also the discussion in Section 5 of [4]. We remark that the freedom in the choice of the scaling of the basis functions can be exploited in numerical computations. Numerical implications of non-uniqueness were discussed in [1], and will play a role in the sequel.

Suppose now that we look for a solution P of Equation (1) as a Taylor series

$$P(w,z) = \sum_{|\alpha|=0}^{\infty} a_\alpha(w) z^\alpha.$$

Here $\alpha = (\alpha, \dots, \alpha_k) \in \mathbb{N}^k$ is the k-dimensional multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_k$, $z = (z_1, \dots, z_k) \in \mathbb{D}^k_\nu$, and $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_k^{\alpha_k}$, and for each $\alpha \in \mathbb{N}^k$ the functions $a_\alpha \colon \mathbb{A}_r \to \mathbb{C}^d$ are analytic, $2T$-periodic functions. Imposing the first order constraints (2) and (3) leads to

$$a_{\mathbf{0}}(w) = \gamma(w) \quad \text{and} \quad a_{e_j}(w) = \xi_j(w),$$

where $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^k$ and $e_j = (0, \dots, 1, \dots, 0)$ is the j^{th} vector of the canonical basis of $\mathbb{R}^k$. Plugging the Taylor expansion into Equation (1) and matching like powers of z shows that the periodic functions $a_\alpha \colon \mathbb{A}_r \to \mathbb{C}^d$ must solve the *homological equations*

$$\frac{d}{dw} a_\alpha(w) + (\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k) a_\alpha(w) - Df(\gamma(w)) a_\alpha(w) = R_\alpha(w), \tag{6}$$

for $|\alpha| \geq 2$. Here R_α is a function only of the a_β with $|\beta| < |\alpha|$, that is the homological equations are inhomogeneous linear differential equations which can be solved recursively to any order. Computation of the a_α is illustrated for a number of specific example problems in [1], and one sees that for a given vector field f the functions $R_\alpha(w)$ can be worked out explicitly.

Definition 1.9. We say that the Floquet exponents $\lambda_1, \dots, \lambda_k$ are *non-resonant* if

$$\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k \neq \lambda_j, \tag{7}$$

for each $|\alpha| \geq 2$ and $1 \leq j \leq k$. Note that since the real part of the λ_j are all negative (respectively positive) this reduces to only a finite number of conditions, that is for $|\alpha|$ large enough the non-resonance condition is automatically met.

Claim 3. If $\lambda_1, \dots, \lambda_k$ are non-resonant then a_α exists and is unique for all $|\alpha| \geq 2$. This is due to the fact that the homological equations given by (6) are linear with periodic coefficients. Hence the classical Floquet theorem gives that these equations have unique analytic $2T$ periodic solution assuming that Equation (7) holds. See also Section 2.3 of [1]. Only the convergence of the formal solution is in question.

For fixed choice of $\xi_j(w)$, $1 \leq j \leq k$ assume that the Floquet exponents $\lambda_1, \ldots, \lambda_k$ are non-resonant and let

$$P(w, z) = \sum_{|\alpha|=0}^{\infty} a_\alpha(w) z^\alpha,$$

be the associated formal solution of Equation (1). By Claim 1 we obtain another formal solution by

$$\begin{aligned} Q(w, z) &\stackrel{\text{def}}{=} P(w, \tau_1 z_1, \ldots, \tau_k z_k) \\ &= \sum_{|\alpha|=0}^{\infty} a_\alpha(w)(\tau_1 z_1, \ldots, \tau_k z_k)^{(\alpha_1, \ldots, \alpha_k)} \\ &= \sum_{|\alpha|=0}^{\infty} a_\alpha(w) \tau_1^{\alpha_1} \cdot \ldots \cdot \tau_k^{\alpha_k} z^\alpha. \end{aligned}$$

Moreover, as seen in Claim 1, this rescaling of the domain is equivalent to rescaling the basis of the stable (respectively unstable) normal bundle. Since by Claim 3 the coefficients with $|\alpha| \geq 2$ are uniquely determined we see that: given the Taylor coefficients $\{a_\alpha(w)\}_{|\alpha|=0}^{\infty}$ of a particular formal solution of Equation (1) all other formal solutions are of the form

$$q_\alpha(w) = \tau_1^{\alpha_1} \cdot \ldots \cdot \tau_k^{\alpha_k} a_\alpha(w), \tag{8}$$

for some choice of the scalars $\tau_1, \ldots, \tau_k$.

Claim 4. (**A-priori existence for small scalings**) Fix first order data $\gamma, \xi_j \colon \mathbb{A}_r \to \mathbb{C}^d$ for $1 \leq j \leq k$, and let

$$\tau_j = \sup_{w \in \mathbb{A}_r} \|\xi_j(w)\|_d \qquad \text{and} \qquad s = \max_{1 \leq j \leq k} \{\tau_j\}.$$

Suppose we fix also a $\nu > 0$. The results of [2, 4] give that: if the Floquet multipliers are non-resonant then there exists an $\epsilon > 0$ so that for all $s \leq \epsilon$ the formal solution associated with this choice of first order constraints converges, that is for small enough s the solution $P \colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ of Equation (1) subject to these constraints exists. The solution is unique, again up to the choice of the scalings $\xi_1(w), \ldots, \xi_k(w)$. Then for different choices of $\tau_j \leq \epsilon$ we parametrize larger or smaller portions of the local stable (respectively unstable) manifold of the periodic orbits.

The dependence of the Taylor coefficients on the scalings illustrated in Equation (8) make it clear that by choosing larger scalings for the basis functions $\xi_j(w)$ we obtain slower convergence of the series P. Hence for a fixed domain disk $\mathbb{D}_\nu^k$ larger scalings correspond to a larger image of the parameterization in phase space. On the other hand, smaller choice of scalings make it more likely that the formal solution converges to a true solution. This is the fundamental balancing act inherent in the Parameterization Method: namely we want the image of P as large as possible in phase space so that the series still converges.

Assumption 3. Suppose that the stable (respectively unstable) Floquet exponents $\lambda_1, \ldots, \lambda_k$ are non-resonant. Choose $\tau_1, \ldots, \tau_k > 0$ and suppose that the basis functions $\xi_1(w), \ldots, \xi_k(w)$ are scaled so that

$$\tau_j = \sup_{w \in \mathbb{A}_r} \|\xi_j(w)\|_d.$$

Fix $N \in \mathbb{N}$ with $N \geq 2$. For $2 \leq |\alpha| \leq N$ assume that $a_\alpha \colon \mathbb{A}_r \to \mathbb{C}^d$ are analytic, $2T$-periodic solutions of the Homological Equation (6). Define the approximate solution $P_N \colon \mathbb{A}_r \times \mathbb{C}^k \to \mathbb{C}^d$ of Equation (1) by

$$P_N(w, z) = \sum_{|\alpha|=0}^{N} a_\alpha(w) z^\alpha. \tag{9}$$

Main question – Existence and approximation: With the scalings $\tau_1, \ldots, \tau_k$, $N \geq 2$, and $P_N(w, z)$ as in Assumption 3, let

$$H(w, z) = \sum_{|\alpha|=N+1}^{\infty} a_\alpha(w) z^\alpha,$$

be the tail function defined by the formal series solutions of the homological equations. For a particular choice of $\nu > 0$, does H converge on $\mathbb{A}_r \times \mathbb{D}_\nu^k$? If so, can we bound H on $\mathbb{A}_r \times \mathbb{D}_\nu^k$?

The remainder of the paper is organized as follows. In Section 2 we develop the main result of the present work. This is Theorem 2.4 which, when its a-posteriori hypotheses are satisfied, answers the existence and approximation questions at the same time. Section 3 deals with computer assisted validation of the solutions $a_\alpha(w)$ of the homological equations for $2 \leq |\alpha| \leq N$, that is with obtaining the data postulated in Assumption 3. Sections 4 and 5 are devoted to applications: validation of manifolds with one and two stable/unstable Floquet exponents respectively. Appendix A contains some technical details associated with the application problem of Section 5.

2 A-posteriori analysis of Equation (1)

We begin somewhat informally in order to introduce the main idea of the argument. Suppose that $f \colon \mathbb{C}^d \to \mathbb{C}^d$ is an analytic vector field and that T, $\gamma(w)$, $\lambda_1, \ldots, \lambda_k$, $\xi_1(w), \ldots, \xi_k(w)$, and $P_N(w, z)$ are as in Assumptions 1, 2 and 3 of Section 1.1. We seek a $2T$-periodic function $H \colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ so that

$$P(w, z) = P_N(w, z) + H(w, z), \tag{10}$$

is an exact solution of Equation (1) for all $(w, z) \in \mathbb{A}_r \times \mathbb{D}_\nu^k$.

Plugging P from Equation (10) into Equation (1) gives

$$f[P_N(w, z) + H(w, z)] = \frac{\partial}{\partial w}[P_N(w, z) + H(w, z)] + D_z[P_N(w, z) + H(w, z)]\Lambda z, \tag{11}$$

where

$$\Lambda \stackrel{\text{def}}{=} \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_k \end{pmatrix},$$

is the $k \times k$ diagonal matrix with the Floquet exponents on the diagonal and zeros elsewhere.

We expand the vector field to second order about the image of the approximate solution P_N so that the left hand side of (11) becomes

$$f[P_N(w, z) + H(w, z)] = f[P_N(w, z)] + Df[P_N(w, z)]H(w, z) + R[P_N(w, z), H(w, z)]. \tag{12}$$

Here R is the second order Taylor remainder associated with the vector field f at the point $P_N(w,z)$. Define the a-posteriori error function $E_N\colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ associated with P_N to be the function given by

$$E_N(w,z) \stackrel{\text{def}}{=} f[P_N(w,z)] - \frac{\partial}{\partial w} P_N(w,z) - D_z P_N(w,z)\Lambda z. \tag{13}$$

Plugging (12) and (13) into Equation (11) and rearranging gives

$$\frac{\partial}{\partial w} H(w,z) + D_z H(w,z)\Lambda z - Df[P_N(w,z)]H(w,z) = E_N(w,z) + R[P_N(w,z), H(w,z)]. \tag{14}$$

We introduce the linear operator

$$\mathcal{L}[H](w,z) \stackrel{\text{def}}{=} \frac{\partial}{\partial w} H(w,z) + D_z H(w,z)\Lambda z - Df[P_N(w,z)]H(w,z), \tag{15}$$

and rewrite (14) as

$$\mathcal{L}[H](w,z) = E_N(w,z) + R[P_N(w,z), H(w,z)].$$

Assuming that $\mathcal{L}$ is invertible gives

$$H(w,z) = \mathcal{L}^{-1}\left(E_N(w,z) + R[P_N(w,z), H(w,z)]\right),$$

and we introduce the nonlinear operator

$$\Phi[H](w,z) \stackrel{\text{def}}{=} \mathcal{L}^{-1}\left[E_N(w,z) + R[P_N(w,z), H(w,z)\right]. \tag{16}$$

We see that H is the truncation error associated with P_N if and only if H is a fixed point of Equation (16). The remainder of the section is devoted to the study of Φ.

We want to show that Φ has a unique fixed point in a small neighborhood of P_N, and we solve the problem in three steps.

Step 1: Show that the linear operator $\mathcal{L}$ is invertible. We will see that invertibility of $\mathcal{L}$ follows if N is "large enough".

Step 2: Establish some quadratic estimates for the nonlinear function $R[P_N(w,z), H(w,z)]$ given by Equation (12).

Step 3: Obtain that the operator Φ is a contraction mapping in a certain neighborhood U_δ.

In order to formalize these steps we need to define appropriate Banach space norms.

2.1 Background: analytic functions and $2T$-periodic families of analytic N-tails

Let $\mathrm{Mat}_{m\times n}(\mathbb{C})$ denote the collection of all $m \times n$ matrices with complex entries. For $A \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ we employ the norm

$$\|A\|_M = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|,$$

where $|a_{ij}|$ is the usual complex absolute value.

Let $U \subset \mathbb{C}$ be a simply connected open domain in the complex plane, and $f\colon U \to \mathbb{C}$ an analytic function. We say that f is *bounded* on U if

$$\sup_{w \in U} |f(w)| < \infty.$$

The set of all bounded analytic functions on U is a Banach space under the norm

$$\|f\|_U^\infty \stackrel{\text{def}}{=} \sup_{w \in U} |f(w)|.$$

The set of all functions $f = (f_1, \ldots, f_d)$ such that each $f_j\colon \mathbb{A}_r \to \mathbb{C}$, $1 \leq j \leq d$ is analytic, bounded, and T-periodic is a Banach space under the product space (maximum) norm

$$\|f\|_r^\infty \stackrel{\text{def}}{=} \max_{1 \leq j \leq d} \|f_j\|_{\mathbb{A}_r}^\infty.$$

Similarly, we say that $P\colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ is *bounded* if

$$\max_{1 \leq j \leq d} \|P_j\|_{\mathbb{A}_r \times \mathbb{D}_\nu^k}^\infty < \infty.$$

For $P\colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ given by the Taylor series

$$P(w, z_1, \ldots, z_k) = \sum_{|\alpha|=0}^\infty a_\alpha(w) z_1^{\alpha_1} \ldots z_k^{\alpha_k},$$

with $a_\alpha\colon \mathbb{A}_r \to \mathbb{C}^d$ bounded analytic functions, we define the two norms

$$\|P\|_{r,\nu} \stackrel{\text{def}}{=} \sum_{|\alpha|=0}^\infty \|a_\alpha\|_r^\infty \nu^{|\alpha|},$$
$$\|P\|_{r,\nu}^\infty \stackrel{\text{def}}{=} \max_{1 \leq j \leq d} \|P_j\|_{\mathbb{A}_r \times \mathbb{D}_\nu^k}^\infty.$$

Note that it is always the case that $\|P\|_{r,\nu}^\infty \leq \|P\|_{r,\nu}$ even when the latter quantity is infinite.

Now for $1 \leq i, j \leq N$ let $a_{ij}\colon \mathbb{A}_r \to \mathbb{C}$ be bounded and $2T$-periodic analytic functions, and consider the matrix

$$A(w) = \begin{pmatrix} a_{11}(w) & \ldots & a_{1N}(w) \\ \vdots & \ddots & \vdots \\ a_{N1}(w) & \ldots & a_{NN}(w) \end{pmatrix}.$$

Define the norm

$$\|A\|_r^\infty \stackrel{\text{def}}{=} \max_{1 \leq i \leq N} \sum_{j=1}^N \|a_{ij}\|_{\mathbb{A}_r}^\infty. \tag{17}$$

Then if $g\colon \mathbb{A}_r \to \mathbb{C}^d$ is a bounded, analytic, T-periodic function then we obtain a new function $Ag\colon \mathbb{A}_r \to \mathbb{C}^d$ by the matrix multiplication $A(w)g(w)$ and have that

$$\|Ag\|_r^\infty \leq \|A\|_r^\infty \|g\|_r^\infty.$$

We say that an analytic function $h\colon \mathbb{D}_\nu^k \to \mathbb{C}$ is an *analytic N-tail* if

$$h(0) = \frac{\partial^{|\alpha|}}{\partial z^\alpha} h(0) = 0, \qquad \text{for all} \quad \alpha \in \mathbb{N}^k, \qquad 0 \leq |\alpha| \leq N.$$

Note that an analytic N-tail has Taylor series

$$h(z) = \sum_{|\alpha|=N+1}^{\infty} h_\alpha z^\alpha,$$

converging absolutely and uniformly for $|z| < \nu$. Analytic N-tails enjoy the following estimate.

Lemma 2.1. *Suppose that $h\colon \mathbb{D}_\nu^k \to \mathbb{C}$ is an analytic N-tail with*

$$\|h\|_{\mathbb{D}_\nu^k}^\infty = \sup_{z\in\mathbb{D}_\nu^k} |h(z)| < \infty.$$

Fix $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and suppose that $|\lambda_j| \le \mu < 1$ for $1 \le j \le k$ for some $\mu > 0$. Then

$$\sup_{z\in\mathbb{D}_\nu^k} |h(\lambda_1 z_1, \ldots, \lambda_k z_k)| \le \mu^{N+1} \sup_{z\in\mathbb{D}_\nu^k} |h(z)|. \tag{18}$$

An elementary proof of (18) is given in [9]. The estimate above extends trivially to functions $h\colon \mathbb{D}_\nu^k \to \mathbb{C}^d$ whose component functions are analytic N-tails.

In the sequel we are interested in analytic N-tails with coefficients which are $2T$-periodic on $\mathbb{A}_r$ and analytic on $\mathbb{A}_r \times \mathbb{D}_\nu^k$. Such an $H\colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ is given by

$$H(w, z) = \sum_{|\alpha|=N+1}^{\infty} h_\alpha(w) z^\alpha,$$

where the sum converges absolutely for $|\mathrm{imag}(w)| < r, |z| < \nu$. Then for each fixed $w_0 \in \mathbb{A}_r$ the analytic function $H(w_0, z)$ is an analytic N-tail in z. We call such a function H a *$2T$-periodic family of analytic N-tails*. The space of $2T$-periodic families of analytic N-tails is a Banach space under both the $\|\cdot\|_{r,\nu}$ and $\|\cdot\|_{r,\nu}^\infty$ norms. Suppose that $\lambda_1, \ldots, \lambda_k$ satisfy the hypothesis of Lemma 2.1 and let Λ be the diagonal matrix with $\lambda_1, \ldots, \lambda_k$ as diagonal entries and zeros elsewhere. Then $\|H\|_{r,\nu}^\infty < \infty$ implies

$$\|H(w, \Lambda z)\|_{r,\nu}^\infty \le \mu^{N+1} \|H\|_{r,\nu}^\infty, \tag{19}$$

as Lemma 2.1 applies uniformly for each fixed $w \in \mathbb{A}_r$.

For bounded analytic functions $f\colon \mathbb{D}_\nu^k \to \mathbb{C}^d$ we have the following bounds on derivatives. The result is standard (and we refer to [8] for the proof).

Lemma 2.2 (Cauchy Bounds). *Suppose that $f : \mathbb{D}_\nu^k \subset \mathbb{C}^k \to \mathbb{C}^d$ is bounded and analytic. Then for any $0 < \sigma \le 1$ we have that*

$$\|\partial_i f\|_{\nu e^{-\sigma}}^\infty \le \frac{2\pi}{\nu\sigma} \|f\|_\nu^\infty.$$

2.2 Validation values and the main theorem

Take $f\colon \mathbb{C}^d \to \mathbb{C}^d$, T, N, $\gamma\colon \mathbb{A}_r \to \mathbb{C}^d$, $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$, $\xi_j\colon \mathbb{A}_r \to \mathbb{C}^d$ for $1 \le j \le k$, $a_\alpha\colon \mathbb{A}_r \to \mathbb{C}^d$ for $2 \le |\alpha| \le N$, and $P_N\colon \mathbb{A}_r \times \mathbb{C}^k \to \mathbb{C}^d$ as in Assumptions $1, 2, 3$ of Section 1.1.

We define the "tube" of radius ρ about γ to be

$$U_\rho(\gamma) = \bigcup_{w\in\mathbb{A}_r} \mathbb{D}_\rho^d[\gamma(w)].$$

Suppose that f is bounded and complex analytic on the tube $U_\rho(\gamma) \subset \mathbb{C}^n$. Consider

$$Df[P_N(w,z)] = Df[\gamma(w) + P_N^1(w,z)],$$

where $P_N^1(w,z) \stackrel{\text{def}}{=} \sum_{|\alpha|=1}^N a_\alpha(w) z^\alpha$. Let

$$\tilde{A}(w,z) \stackrel{\text{def}}{=} Df[P_N(w,z)] - Df[\gamma(w)].$$

For the applications considered in the present work f is always an M-th order polynomial with $M = 2, 3$. Then, since P_N is an N-th order polynomial in z we have that

$$\tilde{A}(w,z) = \sum_{|\alpha|=1}^{(M-1)N} A_\alpha(w) z^\alpha \tag{20}$$

is an $(M-1)N$-th order polynomial with $2T$-periodic matrix coefficients $A_\alpha(w)$. The explicit form of the $A_\alpha(w)$ is a problem dependent computation illustrated in examples.

Now fix $\nu > 0$. The following definition tabulates the constants which need to be computed in order to obtain validated bounds on the tail of P_N.

Definition 2.3 (**Validation values for an approximate solution of Equation** (1)**).** The positive constants κ, $\tilde{C}$, μ_*, ρ, ρ', M_1, M_2, and ϵ are called *validation values* for the approximate parameterization P_N if

(i)
$$\|Df(\gamma)\|_r^\infty \leq \kappa,$$

(ii)
$$\exp\left(\sum_{|\alpha|=1}^{(M-1)N} \frac{\|A_\alpha\|_r^\infty}{|\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k|} \nu^{|\alpha|} \right) \leq \tilde{C},$$

where the $A_\alpha = A_\alpha(w)$ are the coefficients of $\tilde{A}(w,z)$ as given in (20),

(iii)
$$\mu_* \leq \min_{1\leq i\leq k} (|\text{real}(\lambda_1)|, \ldots, |\text{real}(\lambda_k)|), \tag{21}$$

(iv)
$$\max_{1\leq j\leq n} \# \left\{ (k,\ell) \,|\, 1 \leq k, \ell \leq n \text{ such that } \partial^k \partial^\ell f_j \not\equiv 0 \right\} \leq M_1, \tag{22}$$

(v)
$$\sup_{z\in U_\rho(\gamma)} \max_{1\leq i\leq n} \max_{|\beta|=2} \left| \frac{\partial^2}{\partial z^\beta} f_i(z) \right| \leq M_2, \tag{23}$$

(vi) $0 < \rho' < \rho$ and
$$\sum_{|\alpha|=1}^N \|a_\alpha\|_r^\infty \nu^{|\alpha|} \leq \rho',$$
(insuring that $\text{image}(P_N) \subset U_{\rho'}(\gamma)$).

(vii) Finally assume that
$$\|E_N\|_{r,\nu}^\infty \leq \epsilon,$$
where $E_N(w,z)$ is as defined in (13).

Theorem 2.4 (A-Posteriori Validation of a High Order Approximation of the Stable Manifold of a Periodic Orbit). *Suppose that κ, $\tilde{C}$, μ_*, ρ', M_1, M_2, and ϵ are validation values for P_N. Assume that $N \in \mathbb{N}$ and $\delta > 0$ have that*

- $$N + 1 > \frac{\kappa}{\mu_*}, \tag{24}$$
- $$\delta < e^{-1} \min \left\{ \frac{(N+1)\mu_* - \kappa}{2n\pi M_1 M_2 \tilde{C}}, \rho - \rho' \right\}, \tag{25}$$
- $$\frac{2\tilde{C}}{(N+1)\mu_* - \kappa} \epsilon < \delta. \tag{26}$$

Then there is a unique periodic family of analytic N-tails $H \colon \mathbb{A}_r \times \mathbb{D}^k_\nu \to \mathbb{C}^n$ with

$$\|H\|^\infty_{r,\nu} \leq \delta,$$

having that

$$P(w,z) = P_N(w,z) + H(w,z),$$

is the exact solution of Equation (1) on $\mathbb{A}_r \times \mathbb{D}^k_\nu$.

2.3 Spaces and Lemmas

First, we will exploit the following bound.

Lemma 2.5. *Suppose that κ, $\tilde{C}$ are real positive constants with $\|Df(\gamma)\|^\infty_r \leq \kappa$ and*

$$\exp\left(\sum_{|\alpha|=1}^{(M-1)N} \frac{\|A_\alpha\|^\infty_r}{|\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k|} \nu^{|\alpha|} \right) \leq \tilde{C}. \tag{27}$$

Then for any $w \in \mathbb{A}_r$, $z \in \mathbb{D}^k_\nu$, and $t \geq 0$ we have the bound

$$\left| \exp\left(\int_0^t Df\left(P_N\left(w+s, e^{\Lambda s} z\right)\right) \, ds \right) \right| \leq \tilde{C} e^{\kappa t}.$$

Proof. Letting $\tilde{N} = (M-1)N$ note that

$$\begin{aligned}
\left| \int_0^t \tilde{A}(w+s, e^{\Lambda s} z) \, ds \right| &\leq \int_0^t \left| \sum_{|\alpha|=1}^{\tilde{N}} A_\alpha(w+s) \left(e^{\Lambda s} z\right)^\alpha \right| ds \\
&= \int_0^t \sum_{|\alpha|=1}^{\tilde{N}} |A_\alpha(w+s)| \left| e^{(\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k)s} z^\alpha \right| ds \\
&\leq \sum_{|\alpha|=1}^{\tilde{N}} \|A_\alpha\|^\infty_r \left(\int_0^t e^{(\alpha_1\mu_1 + \ldots + \alpha_k\mu_k)s} \, ds \right) \nu^{|\alpha|} \\
&= \sum_{|\alpha|=1}^{\tilde{N}} \|A_\alpha\|^\infty_r \frac{1}{\alpha_1\mu_1 + \ldots + \alpha_k\mu_k} \left(e^{(\alpha_1\mu_1 + \ldots + \alpha_k\mu_k)t} - 1 \right) \nu^{|\alpha|} \\
&\leq \sum_{|\alpha|=1}^{\tilde{N}} \frac{\|A_\alpha\|^\infty_r}{|\alpha_1\mu_1 + \ldots + \alpha_k\mu_k|} \, \nu^{|\alpha|}.
\end{aligned}$$

Then

$$\begin{aligned}\left|\exp\left(\int_0^t Df\left(P_N\left(w+s,e^{\Lambda s}z\right)\right)\,ds\right)\right| &= \exp\left|\int_0^t Df(\gamma(w+s))\,ds+\int_0^t \tilde{A}\left(w+s,e^{\Lambda s}z\right)\,ds\right|\\ &\le \exp\left(\int_0^t |Df(\gamma(w+s))|\,ds\right)\exp\left|\int_0^t \tilde{A}(w+s,e^{\Lambda s}z)\,ds\right|\\ &\le e^{\kappa t}\exp\left(\sum_{|\alpha|=1}^{\tilde{N}}\frac{\|A_\alpha\|_r^\infty}{|\alpha_1\mu_1+\ldots+\alpha_k\mu_k|}\;\nu^{|\alpha|}\right)\\ &\le \tilde{C}e^{\kappa t},\end{aligned}$$

as desired. □

Remark 2.6 (**General vector fields).** If f is not a polynomial then of course the bound for $\tilde{C}$ will be more complicated. We can always obtain an expression similar to Equation (27), plus a small remainder, by considering a Taylor expansion for f (about any convenient point). On the other hand, if the nonlinearity of f is given by elementary functions, then it will be possible to bound $\tilde{C}$ using techniques of automatic differentiation.

We now define our function space. Let

$$\mathcal{X} \stackrel{\text{def}}{=} \left\{H\colon \mathbb{A}_r\times\mathbb{D}_\nu^k\to\mathbb{C}^d \;:\; H \text{ is a } 2T\text{-periodic family of analytic } N\text{-tails and } \|H\|_{\nu,r}^\infty<\infty\right\},$$

and recall that $\mathcal{X}$ is a Banach Space under the $\|\cdot\|_{r,\nu}^\infty$ norm. We denote the closed delta neighborhood of the origin in $\mathcal{X}$ by

$$U_\delta=\left\{H\in\mathcal{X}\;:\;\|H\|_{r,\nu}^\infty\le\delta\right\}.$$

Lemma 2.7. *Let $\mathcal{L}$ as defined by Equation (15). If*

$$N+1>\frac{\kappa}{\mu_*},\tag{28}$$

then $\mathcal{L}$ is boundedly invertible on $\mathcal{X}$ and

$$\|\mathcal{L}^{-1}\|_{\mathcal{X}}\le\frac{\tilde{C}}{(N+1)\mu_*-\kappa}.\tag{29}$$

Proof. Let $S\in\mathcal{X}$. We will show the existence of $H\in\mathcal{X}$ such that $\mathcal{L}[H]=S$. Assuming it exists (it does and it is given by Equation (33) below), consider the equation

$$\mathcal{L}[H](w,z)=\frac{\partial}{\partial w}H(w,z)+D_zH(w,z)\Lambda z-Df[P_N(w,z)]H(w,z)=S(w,z).\tag{30}$$

For $t\in\mathbb{R}^+$ we fix $w\in\mathbb{A}_r$, $z\in\mathbb{D}_\nu^k$ and define the curve $\Gamma:\mathbb{R}^+\to\mathbb{C}^{k+1}$ given by

$$\Gamma(t)=\begin{pmatrix} w+t\\ e^{\Lambda t}z\end{pmatrix}.$$

We now make the change of variables

$$\begin{aligned}\tilde{H}(t)&\stackrel{\text{def}}{=} H\circ\Gamma(t)=H\left(w+t,e^{\Lambda t}z\right),\\ \Psi(t)&\stackrel{\text{def}}{=} Df[P_N\circ\Gamma](t)=Df[P_N(w+t,e^{\Lambda t}z)],\\ \tilde{S}(t)&\stackrel{\text{def}}{=} S\circ\Gamma(t)=S\left(w+t,e^{\Lambda t}z\right),\end{aligned}$$

and observe that

$$\begin{aligned}\frac{d}{dt}\tilde{H}(t) &= DH(\Gamma(t)) \cdot \Gamma'(t) \\ &= [\partial_w H(\Gamma(t)) | D_z H(\Gamma(t))] \begin{pmatrix} 1 \\ e^{\Lambda t}\Lambda z \end{pmatrix} \\ &= \frac{\partial}{\partial w} H\left(w+t, e^{\Lambda t} z\right) + D_z H\left(w+t, e^{\Lambda t} z\right) e^{\Lambda t}\Lambda z.\end{aligned}$$

Then we consider the ordinary differential equation

$$\frac{d}{dt}\tilde{H}(t) - \Psi(t)\tilde{H}(t) = \tilde{S}(t),$$

and note that $\tilde{H}(0)$ is the solution of Equation (30). We introduce the integrating factor

$$\Omega(t) \stackrel{\text{def}}{=} e^{-\int_0^t \Psi(\tau)\, d\tau},$$

and note that $\Omega(0) = I$ and

$$\Omega^{-1}(t) = e^{\int_0^t \Psi(\tau)\, d\tau}.$$

Then for any $t, a \in \mathbb{R}$ we have that

$$\Omega(t)\tilde{H}(t) = \Omega(a)\tilde{H}(a) + \int_a^t \Omega(\beta)\tilde{S}(\beta)\, d\beta. \tag{31}$$

The next step is to evaluate the limit as a goes to infinity. We assume that H is bounded (an assumption which will be justified momentarily) and see (recalling the definition of μ_* in (21)) that

$$\begin{aligned}|\Omega(a)\tilde{H}(a)| &= \left| e^{-\int_0^a \Psi(\tau)\, d\tau} H(w+a, e^{\Lambda a} z) \right| \\ &\stackrel{(a)}{\leq} e^{\int_0^a |\Psi(\tau)|\, d\tau} \left(e^{-\mu_* a}\right)^{N+1} |H(w+a, z)| \\ &\stackrel{(b)}{\leq} \tilde{C} e^{\kappa a} e^{-(N+1)\mu_* a} \|H\|_{r,\nu}^{\infty} \\ &= \tilde{C} \|H\|_{r,\nu}^{\infty} e^{[-(N+1)\mu_* + \kappa]a},\end{aligned} \tag{32}$$

where (a) follows from Lemma 2.1 and (b) follows from Lemma 2.5. Now the hypothesis of Equation (28) implies that $(N+1)\mu_* > \kappa$ and we have

$$\lim_{a \to \infty} |\Omega(a)\tilde{H}(a)| = \tilde{C}\|H\|_{r,\nu}^{\infty} \lim_{a \to \infty} e^{[-(N+1)\mu_* + \kappa]a} = 0.$$

Then (at least formally) we have that Equation (31) becomes

$$\tilde{H}(t) = -\Omega^{-1}(t) \int_t^{\infty} \Omega(\beta)\tilde{S}(\beta)\, d\beta.$$

Taking $t = 0$ gives

$$H(w, z) = \tilde{H}(0) = -\int_0^{\infty} \left[e^{-\int_0^{\beta} Df\left[P_N\left(w+\tau, e^{\Lambda\tau} z\right)\right] d\tau} \right] S\left(w+\beta, e^{\Lambda\beta} z\right)\, d\beta. \tag{33}$$

Now we take Equation (33) as *the definition* of H, and by running the argument backwards we see that H solves Equation (30), assuming that we can show that H is bounded.

It is also clear upon inspection that H so defined is T-periodic on $\mathbb{A}_r$ in the variable w, owing to the fact that P_N and $\tilde{S}$ are. What remains is to show that H is bounded and analytic in both w and z and that H is a T-periodic family of analytic N-tails.

By arguing as in Equation (32) we have

$$|\Omega(\beta)\tilde{S}(\beta)| \leq \tilde{C}e^{[-(N+1)\mu_*+\kappa]\beta}\|S\|_{r,\nu}^{\infty}, \tag{34}$$

with S bounded and $-(N+1)\mu_* + \kappa < 0$ by hypothesis. It follows that

$$\begin{aligned}
|H(w,z)| &= |\tilde{H}(0)| \\
&= \left| - \int_0^{\infty} \Omega(\beta)\tilde{S}(\beta)\, d\beta \right| \\
&\leq \tilde{C}\|S\|_{r,\nu}^{\infty} \int_0^{\infty} e^{[-(N+1)\mu_*+\kappa]\beta}\, d\beta \\
&\leq \frac{\tilde{C}\|S\|_{r,\nu}^{\infty}}{(N+1)\mu_* - \kappa},
\end{aligned} \tag{35}$$

and we see that H is indeed bounded. To see that H is analytic in any of $w, z_1, \ldots, z_k$ we remark that Equation (34) shows that the integrand is bounded and we know the integrand is analytic in each variable. Morera's Theorem [63] then gives the analyticity of H. To see that H is an N-tail we simply use that S is an N-tail and apply the Leibnitz rule on Equation (33) (exploiting the boundedness of the integrand in order to pass the derivative under the integral).

Since S was arbitrary we see that the desired inverse operator is defined explicitly by $\mathcal{L}^{-1}[S] = H$. Taking the supremum in Equation (35) over all S with norm one gives the bound claimed in the Equation (29). $\square$

Lemma 2.8. *Assume that the hypotheses of Theorem 2.4 are satisfied. Given $H \in \mathcal{X}$, let $R\colon \mathbb{A}_r \times \mathbb{D}_\nu^k \to \mathbb{C}^d$ be the function defined by*

$$f[P_N(w,z) + H(w,z)] = f[P_N(w,z)] + Df[P_N(w,z)]H(w,z) + R[P_N(w,z), H(w,z)]. \tag{36}$$

Then R is a T-periodic family of analytic N-tails satisfying the following bounds:

- *For any $\delta \leq \rho - \rho'$ and $H \in U_\delta$,*

$$\|R(P_N, H)\|_{r,\nu}^{\infty} \leq M_1 M_2 \delta^2. \tag{37}$$

- *For any $\delta \leq (\rho - \rho')e^{-1}$ and $H_1, H_2 \in U_\delta$ we have*

$$\|R[P_N, H_1] - R[P_N, H_2]\|_{r,\nu}^{\infty} \leq 2\pi e d M_1 M_2 \delta \|H_1 - H_2\|_{r,\nu}^{\infty}. \tag{38}$$

Proof. Let $s \stackrel{\text{def}}{=} \rho - \rho'$. For any fixed $w_0, z_0 \in \mathbb{A}_r \times \mathbb{D}_\nu^k$ and $\eta \in \mathbb{D}_s^d$ we have that

$$f[P_N(w_0,z_0) + \eta] = f[P_N(w_0,z_0)] + Df[P_N(w_0,z_0)]\eta + R[P_N(w_0,z_0),\eta],$$

where R is given by the Lagrange form of the Taylor Remainder

$$R_j[P_N(w_0,z_0),\eta] = \sum_{|\beta|=2} \frac{2}{\beta!}\eta^\beta \int_0^1 (1-t)\frac{\partial^2}{\partial \eta^\beta} f_j(P_N(w_0,z_0) + t\eta)\, dt. \tag{39}$$

Applying Morera's Theorem (e.g. see [63]) to Equation (39) shows that R is analytic as $(w_0, z_0) \in \mathbb{A}_r \times \mathbb{D}^k_\nu$ as well as $\eta \in \mathbb{D}^d_s$ vary. It is also clear that R_j is T-periodic in w_0. Moreover

$$|R_j[P_N(w_0, z_0), \eta]| \leq \sum_{|\beta|=2} \frac{2}{\beta!} \sup_{z \in U_\rho(\gamma)} \left| \frac{\partial^2}{\partial z^\beta} f_j(z) \right| \|\eta\|^2_d \leq M_1 M_2 \|\eta\|^2_d.$$

Fixing $H \in U_\delta$ with $\delta < s$ we define $R\colon \mathbb{A}_r \times \mathbb{D}^k_\nu \to \mathbb{C}^d$ by $R[P_N(w, z), H(w, z)]$ and have

$$\|R[P_N, H]\|^\infty_{r,\nu} \leq M_1 M_2 \delta^2,$$

as desired.

We now introduce a "loss of domain" parameter $\sigma \in (0, 1]$ which is used to leverage the supremum bound on R into a Lipschitz bound. So, fix $(w_0, z_0) \in \mathbb{A}_r \times \mathbb{D}^k_\nu$ and $\eta_1, \eta_2 \in \mathbb{D}^d_{e^{-\sigma}s}$. Then

$$R[P_N(w_0, z_0), \eta_1] - R[P_N(w_0, z_0), \eta_2] = D_\eta R[P_N(w_0, z_0), \tilde{\eta}](\eta_1 - \eta_2),$$

for some $\tilde{\eta} \in \mathbb{D}^d_{e^{-\sigma}s}$. The fact that R is analytic and zero to second order in η implies that $\partial/\partial\eta_j R_i$ is analytic and zero to first order in η for each $1 \leq i, j \leq d$. Then for any $0 < t < 1$ we have

$$\|D_\eta\, R[P_N(w_0, z_0), t\eta]\|_M \leq t \|D_\eta\, R[P_N(w_0, z_0), \eta]\|_M,$$

Let $t = \delta / s e^{-\sigma}$ with $0 < \sigma \leq 1$ and note that $0 < t < 1$ by the hypothesis given by Equation (25) of Theorem 2.4. It follows that

$$\begin{aligned}
\|D_\eta[P_N(w_0, z_0), t\eta]\|_M &\leq \frac{\delta}{se^{-\sigma}} \|D_\eta R[P_N(w_0, z_0), \eta]\|_M \\
&\leq \frac{\delta}{se^{-\sigma}} \sup_{|\eta| = e^{-\sigma}s} \|DR[P_N(w_0, z_0), \eta]\|_M \\
&\leq \frac{\delta e^\sigma}{s} \left(\frac{2\pi d}{s\sigma} \sup_{|\eta|=s} |R[P_N(w_0, z_0), \eta] \right) \\
&\leq \delta \; \frac{2\pi d e^\sigma}{\sigma s^2} M_1 M_2 s^2 \\
&\leq 2\pi e d M_1 M_2 \delta,
\end{aligned} \tag{40}$$

as e^σ/σ is minimized when $\sigma = 1$. Note that we have used the Cauchy Bounds of Lemma 2.2 in order to pass from the second to the third line of the estimate. The use of the Cauchy Bounds explains the loss of domain and the parameter σ appearing in the argument.

Fixing $H_1, H_2 \in U_\delta$ we now have

$$\|R[P_N, H_1] - R[P_N, H_2]\|^\infty_{r,\nu} \leq 2\pi e d M_1 M_2 \delta \|H_1 - H_2\|^\infty_{r,\nu}$$

as desired. □

2.4 Proof of Theorem 2.4

We will show that the operator $\Phi\colon \mathcal{X} \to \mathcal{X}$ defined in (16) has a unique fixed point in $U_\delta \subset \mathcal{X}$. First note that Φ is well defined as the hypothesis of Equation (24) allow us to apply Lemma 2.7 and obtain that $\mathcal{L}$ is boundedly invertible on $\mathcal{X}$.

Now for any $H \in U_\delta$ and consider

$$
\begin{aligned}
\|\Phi[H]\|_{r,\nu}^{\infty} &= \|\mathcal{L}^{-1}\left[E_N + R[P_N, H]\right]\|_{r,\nu}^{\infty} \\
&\leq \|\mathcal{L}^{-1}\|_{r,\nu}^{\infty}\|E_N\|_{r,\nu}^{\infty} + \|\mathcal{L}^{-1}\|_{r,\nu}^{\infty}\|R[P_N, H]\|_{r,\nu}^{\infty} \\
&\leq \frac{\tilde{C}}{(N+1)\mu_* - \kappa}\left(\epsilon + M_1 M_2 \delta^2\right) \\
&\leq \delta. \qquad (41)
\end{aligned}
$$

Here we have applied the bounds of Lemma 2.7, as well as the bound given by Equation (37), the the definition of ϵ from condition (vii) if the definition of validation values, as well as the hypothesis given by Equations (25) and (26) of the present theorem. This shows that in fact $\Phi : U_\delta \to U_\delta$.

What remains is to show that Φ is a contraction on U_δ. To see this let $H_1, H_2 \in U_\delta$. The inequality hypothesized by Equation (25) of the present theorem gives that $\delta \leq (\rho - \rho')/e$. It follows that we can apply the bound given by Equation (38) to the expression

$$
\begin{aligned}
\|\Phi[H_1] - \Phi[H_2]\|_{r,\nu}^{\infty} &= \|\mathcal{L}^{-1}\left(R[P_N(w,z), H_1(w,z)] - R[P_N(w,z), H_2(w,z)]\right)\|_{r,\nu}^{\infty} \\
&\leq \frac{\tilde{C}}{(N+1)\mu_* - \kappa} 2\pi e d M_1 M_2 \delta \|H_1 - H_2\|_{r,\nu}^{\infty}. \qquad (42)
\end{aligned}
$$

Here we have again used the bounds of Lemma 2.7. Moreover,

$$
\frac{\tilde{C}}{(N+1)\mu_* - \kappa} 2\pi e d M_1 M_2 \delta < 1
$$

by the inequality hypothesised in Equation (25) of the present theorem. Then $\Phi\colon U_\delta \to U_\delta$ is a contraction mapping and hence has unique fixed point $H \in U_\delta$. Since H is the truncation error we have

$$
\|P - P_N\|_{r,\nu}^{\infty} = \|H\|_{r,\nu}^{\infty} \leq \delta,
$$

as desired. □

2.5 Validation Algorithm

The parameters ν and the scalings of the basis functions ξ_j, $1 \leq j \leq k$ are free in the discussion above. In theory there is no difference between fixing the norms of the basis vectors and then choosing the size $\nu > 0$ of the validation domain, or vice versa. In practice however it is preferable to fix $\nu = 1$ and choose numerically convenient scalings for the basis functions. This is simply due to the fact that $\nu = 1$ is the most stable choice in the $\nu^{|\alpha|}$ terms appearing in the Taylor-norms.

Algorithms for optimizing the choice of the scalings are discussed in detail in [64], however we make a few heuristic comments. In many applications it is sufficient to simply compute once the parameterization P^N with any convenient choice of scalings, and then examine the growth of the resulting $\|p_\alpha\|_r^{\infty}$. If these coefficients grow either too fast or too slow then simply rescale so that the desired decay rate is achieved. The dependence of the new decay on the choice of rescaling is given explicitly in Equation (8). A good heuristic is to choose that the highest order coefficients have magnitude close to machine epsilon.

Once the scalings and ν are picked then we try the following algorithm. If the algorithm fails then we modify the scalings and try again.
Recipe: Input choice of ν from above.

1. **Choose:** any positive constant κ with

$$\|Df(\gamma)\|_r^\infty < \kappa.$$

In practice κ will be an interval arithmetic bound on $\|Df(\gamma)\|_r^\infty$ as this quantity depends only in the known Fourier coefficients of γ as well as the known decay rate bounds.

2. **Choose:** a positive constant $\tilde{C}$ with

$$\exp\left(\sum_{|\alpha|=1}^{(M-1)N} \frac{\|A_\alpha\|_r^\infty}{|\alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k|}\nu^{|\alpha|}\right) \leq \tilde{C}.$$

In practice $\tilde{C}$ is any bound on the sum obtained using interval arithmetic.

3. **Choose:** a constant ρ' with

$$\sum_{|\alpha|=1}^{N} \|a_\alpha\|_r^\infty \nu^{|\alpha|} < \rho'.$$

In practice the terms $\|a_\alpha\|_r^\infty$ are bound using the $\|a_\alpha\|_r$ norms.

4. **Choose:** $s > 0$. For polynomial vector fields the choice of s is more or less arbitrary and we often obtain good results with $s = 0.1\rho'$. Then define ρ to be any number with

$$\rho' + s \leq \rho.$$

If f is not polynomial then we have to pick a $\rho > 0$ and estimate the second derivative of f in a ρ neighborhood of γ. This could be done either by hand, or more likely with some preliminary interval arithmetic. Note that if f has poles then this imposes a theoretical limit on the size of ρ. Next we make sure that ρ' defined above is smaller than ρ. s is then the difference.

5. **Compute:** M_1 and M_2 on the "tube" $U_\rho(\gamma)$. For quadratic vector fields the second partials are of course constant.

6. **Choose:** ϵ to be any positive number with

$$\|E_N\|_{r,\nu}^\infty \leq \epsilon.$$

7. **Choose:** μ_* to be any lower bound on the absolute values of the real parts of the stable eigenvalues.

8. **Check:**

$$N + 1 > \frac{\kappa}{\mu_*}.$$

If not then the proof fails.

9. **Choose:** $\delta > 0$ so that

$$\frac{2\tilde{C}}{(N+1)\mu_* - \kappa}\epsilon \leq \delta. \tag{43}$$

10. **Check:**
$$\delta < e^{-1} \min\left(\frac{(N+1)\mu_* - \kappa}{2n\pi M_1 M_2 \tilde{C}}, \rho - \rho' \right). \tag{44}$$
If not then the proof fails.

11. **Return:** δ.

If the proof does not fail before Step 11, then it succeeds and we have the validated bound
$$\|H\|_{r,\nu}^{\infty} \leq \|H\|_{r,\nu} \leq \delta,$$
for the truncation error associated with the approximation P_N on the domain $\mathbb{A}_r \times \mathbb{D}_\nu^k$.

Remark 2.9 (Implementation details)**.** In practice we must obtain the data hypothesized in Assumptions 1, 2 and 3 of Section 1.1 by separate computer assisted arguments. Since the functions $\gamma(w)$, $\xi_1(w), \ldots, \xi_k(w)$, and $a_\alpha(w)$ for $2 \leq |\alpha| \leq N$ are analytic and periodic, it is reasonable to represent these functions as Fourier series. Then the first step is to compute numerical approximations
$$\begin{aligned}
\gamma^M(w) &= \sum_{m=-M}^{M} \gamma_m e^{\frac{2\pi i m}{T} w}, \\
\xi_j^M(w) &= \sum_{m=-M}^{M} (\xi_j)_m e^{\frac{2\pi i m}{2T} w}, \qquad 1 \leq j \leq k, \\
a_\alpha^M(w) &= \sum_{m=-M}^{M} a_{\alpha,m} e^{\frac{2\pi i m}{2T} w}, \qquad 2 \leq |\alpha| \leq N.
\end{aligned}$$
In addition to these numerical approximations we need positive constants r, ϵ_0, ϵ_1, …, ϵ_k, and ϵ_α, for $2 \leq |\alpha| \leq N$ so that
$$\|\gamma^M - \gamma\|_r^\infty \leq \epsilon_0, \qquad \|\xi_j^M - \xi_j\|_r^\infty \leq \epsilon_j, \qquad \text{and} \qquad \|a_\alpha^M - a_\alpha\|_r^\infty \leq \epsilon_\alpha, \tag{45}$$
for $1 \leq j \leq k$ and $2 \leq |\alpha| \leq N$. In the present work we compute the approximate Fourier coefficient sequences $\{\gamma_m\}_{m=-M}^M$, $\{(\xi_j)_m\}_{m=-M}^M$ for the periodic orbit/parameterization of the normal bundles, as well as the constants r and $\epsilon_1, \epsilon_1, \ldots, \epsilon_k$, using the computer aided methods of [14, 15] (for the periodic orbit) and the methods of [16, 17] (for the basis functions of the normal bundles). The approximate Fourier coefficients $\{a_{\alpha,m}\}_{m=-M}^M$ all in $\mathbb{C}^d$ for the solutions of the homological Equations are computed numerically using the methods of [1]. What remains is to compute validated constants ϵ_α for $2 \leq |\alpha| \leq N$ satisfying (45). This is topic of Section 3. We note that all of this data is assumed as input for the validation algorithm above.

3 Rigorous solution of the homological equations

The goal of this section is to obtain computer assisted error bounds satisfying (45) on the solutions $a_\alpha \colon \mathbb{A}_r \to \mathbb{C}^d$ of the homological equations for $2 \leq |\alpha| \leq N$. Or to put it another way, the goal is to obtain the data hypothesized in Assumption 3 of Section 1.1. More precisely, recall that for each $\alpha \in \mathbb{N}^k$ we seek a_α so that
$$\frac{da_\alpha(w)}{dw} + (\alpha \cdot \Lambda) a_\alpha(w) = (f \circ P)_\alpha(w),$$

where $\alpha \cdot \Lambda \stackrel{\text{def}}{=} \alpha_1\lambda_1 + \ldots + \alpha_k\lambda_k$, and with

$$f(P(w,z)) = \sum_{|\alpha|=0}^{\infty} (f \circ P)_\alpha(w) z^\alpha = \sum_{|\alpha|=0}^{\infty} \sum_{m\in\mathbb{Z}} (f \circ P)_{\alpha,m}\, e^{\frac{2\pi \mathbf{i}}{2T} mw} z^\alpha.$$

For $j = 1, \ldots, d$, denote

$$F^{(j)}_{\alpha,m} \stackrel{\text{def}}{=} \left(\frac{2\pi \mathbf{i} m}{2T} + \alpha \cdot \Lambda\right) a^{(j)}_{\alpha,m} - (f \circ P)^{(j)}_{\alpha,m}. \tag{46}$$

Denote

$$a = (a_\alpha)_{|\alpha|=2}^N = (a^{(j)}_\alpha)_{\substack{j=1,\ldots,d \\ |\alpha|=2,\ldots,N}},$$

with

$$a_\alpha = \left(a^{(1)}_\alpha, \ldots, a^{(d)}_\alpha\right)^T \in \mathbb{R}^d \quad \text{and} \quad a^{(j)}_\alpha = \left(a^{(j)}_{\alpha,m}\right)_{m\in\mathbb{Z}}, \quad \text{for } j = 1, \ldots, d.$$

Similarly, denote

$$F = (F_\alpha)_{|\alpha|=2}^N = (F^{(j)}_\alpha)_{\substack{j=1,\ldots,d \\ |\alpha|=2,\ldots,N}},$$

with

$$F_\alpha = \left(F^{(1)}_\alpha, \ldots, F^{(d)}_\alpha\right)^T \in \mathbb{R}^d \quad \text{and} \quad F^{(j)}_\alpha = \left(F^{(j)}_{\alpha,m}\right)_{m\in\mathbb{Z}}, \quad \text{for } j = 1, \ldots, d.$$

We look for a solution of

$$F(a) = 0. \tag{47}$$

Given a bi-infinite sequence $c = (c_m)_{m\in\mathbb{Z}}$ of complex numbers and given $\nu \geq 1$, let

$$\|c\|_{1,\nu} \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}} |c_m| \nu^{|m|},$$

which is used to define the Banach space

$$\ell^1_\nu \stackrel{\text{def}}{=} \left\{c = (c_m)_{m\in\mathbb{Z}} : \|c\|_{1,\nu} < \infty\right\}.$$

Denote

$$\mathcal{B}(\ell^1_\nu, \ell^1_\nu) \stackrel{\text{def}}{=} \left\{ A : \ell^1_\nu \to \ell^1_\nu \;\middle|\; A \text{ is linear and } |||A||| \stackrel{\text{def}}{=} \sup_{\|v\|_{1,\nu}=1} \|Av\|_{1,\nu} < \infty \right\}$$

the space of bounded linear operator from ℓ^1_ν to ℓ^1_ν.

For each $\alpha \in \mathbb{N}^k$ with $|\alpha| \in \{2, \ldots, N\}$ and $j = 1, \ldots, d$, we consider $a^{(j)}_\alpha \in \ell^1_\nu$. Let $n = n(m, N)$ be the cardinality

$$n = \#\left\{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \mid |\alpha| = \alpha_1 + \cdots + \alpha_k \in \{2, \ldots, N\}\right\}.$$

For each $i = 1, \ldots, n$, denote by $\alpha(i) \in \mathbb{N}^k$ the corresponding multi-index such that $|\alpha(i)| \in \{2, \ldots, N\}$. In other words, we choose an ordering for the set of multi-indices satisfying $|\alpha| \in \{2, \ldots, N\}$.

Consider the Banach space

$$X \stackrel{\text{def}}{=} \left(\ell^1_\nu\right)^{nd} \tag{48}$$

endowed with the norm

$$\|a\|_X = \max_{\substack{i=1,\dots,n \\ j=1,\dots,d}} \left\{ \|a^{(j)}_{\alpha(i)}\|_{1,\nu} \right\}. \tag{49}$$

The goal is to compute solutions of (47) within the Banach space $(X, \|\cdot\|_X)$ using a Newton-Kantorovich type theorem (the radii polynomial approach, e.g. see [65, 14]). This approach first requires computing an approximate solution using a finite dimensional projection.

Given a Fourier truncation order M, consider a finite dimension projection $\Pi^{(M)} : X = \left(\ell^1_\nu\right)^{nd} \to \mathbb{R}^{nd(2M-1)}$ defined by

$$\Pi^{(M)} : \left(\ell^1_\nu\right)^{nd} \to \mathbb{R}^{nd(2M-1)} : a \mapsto \Pi^{(M)} a = \left(\left(a^{(j)}_{\alpha(i),m} \right)_{|m|<M} \right)_{\substack{i=1,\dots,n \\ j=1,\dots,d}}.$$

Given $a \in \left(\ell^1_\nu\right)^{nd}$, denote $a_F \stackrel{\text{def}}{=} \Pi^{(M)} a$. Consider the finite dimensional projection $F^{(m)} : \mathbb{R}^{nd(2m-1)} \to \mathbb{R}^{nd(2m-1)}$ of (47) given by

$$F^{(M)}(a_F) = \left(F^{(M)}_{\alpha(i)}(a_F) \right)_{i=1}^{n} \stackrel{\text{def}}{=} \Pi^{(M)} F(a_F) \in \mathbb{R}^{nd(2M-1)}.$$

Assume that using an iterative scheme (e.g. Newton's method), a numerical approximation $\bar{a}_F$ has been computed that is $F^{(M)}(\bar{a}_F) \approx 0$. For the sake of simplicity of the presentation, we introduce the notation $\bar{a} \in \left(\ell^1_\nu\right)^{nd}$ to denote the embedding of the numerical approximation $\bar{a}_F \in \mathbb{R}^{nd(2M-1)}$ by adding zeroes in the tail.

The idea now is to introduce a Newton-like operator based at $\bar{a}$ whose fixed point correspond to a solution (near $\bar{a}$) of the infinite dimensional problem (47). A Newton-like operator is an operator of the form

$$\mathcal{T}(a) = a - AF(a), \tag{50}$$

with A a linear operator chosen carefully so that it is a good enough approximate inverse for the derivative operator $DF(\bar{a})$. Since the Newton operator $a \mapsto a - DF(a)^{-1} F(a)$ should be a contraction on a neighbourhood of an hyperbolic fixed point, there is hope that for a carefully chosen approximation A of $DF(\bar{a})^{-1}$ the perturbation of the Newton operator (50) will still be a contraction near the fixed point.

The computation of the operator A is problem dependent, and by using carefully the structure of the problem under study, one can get better approximate inverses. However, for the specific problem (47), there is natural choice that we present next.

3.1 Construction of the approximate inverse A

Consider the Jacobian matrix $DF^{(M)}(\bar{a}) \in M_{nd(2M-1)}(\mathbb{C})$ and let $A^{(M)}$ a numerically computed *pseudo-inverse* of $DF^{(M)}(\bar{a})$. Denote $A^{(M)}$ block-wise as

$$A^{(M)} = \begin{bmatrix} (A^{(M)}_{1,1})_{1,1} & \cdots & (A^{(M)}_{1,1})_{1,d} & & (A^{(M)}_{1,n})_{1,1} & \cdots & (A^{(M)}_{1,n})_{1,d} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ (A^{(M)}_{1,1})_{d,1} & \cdots & (A^{(M)}_{1,1})_{d,d} & & (A^{(M)}_{1,n})_{d,1} & \cdots & (A^{(M)}_{1,n})_{d,d} \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ (A^{(M)}_{n,1})_{1,1} & \cdots & (A^{(M)}_{n,1})_{1,d} & & (A^{(M)}_{n,n})_{1,1} & \cdots & (A^{(M)}_{n,n})_{1,d} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ (A^{(M)}_{n,1})_{d,1} & \cdots & (A^{(M)}_{n,1})_{d,d} & & (A^{(M)}_{n,n})_{d,1} & \cdots & (A^{(M)}_{n,n})_{d,d} \end{bmatrix} \in M_{nd(2M-1)}(\mathbb{C}) \tag{51}$$

with $(A^{(M)}_{n_1,n_2})_{d_1,d_2} \in M_{(2M-1)}(\mathbb{C})$ for $1 \leq n_1, n_2 \leq n$ and $1 \leq d_1, d_2 \leq d$. The operator A which acts as an approximate inverse for $Df(\bar{x})$ is given block-wise by

$$A = \begin{bmatrix} (A_{1,1})_{1,1} & \cdots & (A_{1,1})_{1,d} & & (A_{1,n})_{1,1} & \cdots & (A_{1,n})_{1,d} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ (A_{1,1})_{d,1} & \cdots & (A_{1,1})_{d,d} & & (A_{1,n})_{d,1} & \cdots & (A_{1,n})_{d,d} \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ (A_{n,1})_{1,1} & \cdots & (A_{n,1})_{1,d} & & (A_{n,n})_{1,1} & \cdots & (A_{n,n})_{1,d} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ (A_{n,1})_{d,1} & \cdots & (A_{n,1})_{d,d} & & (A_{n,n})_{d,1} & \cdots & (A_{n,n})_{d,d} \end{bmatrix} \tag{52}$$

where $(A_{n_1,n_2})_{d_1,d_2} \in \mathcal{B}\left(\ell^1_\nu, \ell^1_\nu\right)$ for $1 \leq n_1, n_2 \leq n$ and $1 \leq d_1, d_2 \leq d$. For $c = (c_m)_{m\in\mathbb{Z}} \in \ell^1_\nu$, denote by $c_F = (c_{-M+1}, \ldots, c_{M-1}) \in \mathbb{C}^{2M-1}$.

Given $b = (b_{\alpha(i)})_{i=1}^n = (b^{(j)}_{\alpha(i)})_{\substack{i=1,\ldots,n \\ j=1,\ldots,d}} \in X$, the action of the operator A on b is given by

$$(Ab)^{(j_1)}_{\alpha(i_1)} = \sum_{i_2=1}^{n} \left(A_{i_1,i_2} b_{\alpha(i_2)}\right)^{(j_1)} = \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (A_{i_1,i_2})_{j_1,j_2} b^{(j_2)}_{\alpha(i_2)},$$

where the action of $(A_{i_1,i_2})_{j_1,j_2}$ on $c \in \ell^1_\nu$ is given by

$$((A_{i_1,i_2})_{j_1,j_2} c)_m = \begin{cases} \left((A^{(M)}_{i_1,i_2})_{j_1,j_2} c_F\right)_m, & |m| < M, \\ \delta_{i_1,i_2}\delta_{j_1,j_2} \left(\dfrac{1}{\frac{2\pi \mathbf{i} m}{2T} + \alpha(i_1)\cdot\Lambda}\right) c_m, & |m| \geq M, \end{cases} \tag{53}$$

where $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise.

Lemma 3.1. *Define $\mathcal{T}$ as in* (50) *with A defined by* (52)*. Then*

$$AF : X \to X. \tag{54}$$

Proof. Let $h \in X$, $b \overset{\text{def}}{=} F(h)$ and $a \overset{\text{def}}{=} Ab$. Denote $b = (b_{\alpha(i)})_{i=1}^n = (b_{\alpha(i)}^{(j)})_{\substack{i=1,\ldots,n \\ j=1,\ldots,d}}$. The goal is to show that $a \in X$. We have

$$\begin{aligned}
a &= \left\{a_{\alpha(i)}\right\}_{i=1}^n \\
&= \left\{a_{\alpha(i)}^{(j)}\right\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,d}} \\
&= \left\{(Ab)_{\alpha(i)}^{(j)}\right\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,d}} \\
&= \left\{\sum_{i_2=1}^n \left(A_{i,i_2} b_{\alpha(i_2)}\right)^{(j)}\right\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,d}} \\
&= \left\{\sum_{i_2=1}^n \sum_{j_2=1}^d (A_{i,i_2})_{j,j_2} b_{\alpha(i_2)}^{(j_2)}\right\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,d}}.
\end{aligned}$$

Fix $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,d\}$, and let

$$c(i,j) \overset{\text{def}}{=} \sum_{i_2=1}^n \sum_{j_2=1}^d (A_{i,i_2})_{j,j_2} b_{\alpha(i_2)}^{(j_2)}.$$

Let us show that $c(i,j) \in \ell_\nu^1$. Recalling (83), (46) and that $b = F(h)$, we have $b_{\alpha(i_2)}^{(j_2)} = F_{\alpha(i_2)}^{(j_2)}(h)$, and therefore

$$\begin{aligned}
\|c(i,j)\|_{1,\nu} &= \sum_{m=-\infty}^{\infty} |c(i,j)_m| \nu^{|m|} \\
&= \sum_{m=-\infty}^{-M} \left| \left(\frac{1}{\frac{2\pi \mathbf{i} m}{2T} + \alpha(i)\cdot\Lambda}\right) \left(F_{\alpha(i)}^{(j)}(h)\right)_m \right| \nu^{|m|} \\
&+ \sum_{m=-M+1}^{M-1} \left| \left(\sum_{i_2=1}^n \sum_{j_2=1}^d (A_{i_1,i_2}^{(M)})_{j_1,j_2} \left(F_{\alpha(i_2)}^{(j_2)}(h)\right)_F \right)_m \right| \nu^{|m|} \\
&+ \sum_{m=M}^{\infty} \left| \left(\frac{1}{\frac{2\pi \mathbf{i} m}{2T} + \alpha(i)\cdot\Lambda}\right) \left(F_{\alpha(i)}^{(j)}(h)\right)_m \right| \nu^{|m|} \\
&= \sum_{m=-\infty}^{-M} \left| \left(\frac{1}{\frac{2\pi \mathbf{i} m}{2T} + \alpha(i)\cdot\Lambda}\right) \left[\left(\frac{2\pi \mathbf{i} m}{2T} + \alpha(i)\cdot\Lambda\right) h_{\alpha(i),m}^{(j)} - (f\circ P)_{\alpha(i),m}^{(j)}(h) \right] \right| \nu^{|m|} \\
&+ \sum_{m=-M+1}^{M-1} \left| \left(\sum_{i_2=1}^n \sum_{j_2=1}^d (A_{i_1,i_2}^{(M)})_{j_1,j_2} \left(F_{\alpha(i_2)}^{(j_2)}(h)\right)_F \right)_m \right| \nu^{|m|} \\
&+ \sum_{m=M}^{\infty} \left| \left(\frac{1}{\frac{2\pi \mathbf{i} m}{2T} + \alpha(i)\cdot\Lambda}\right) \left[\left(\frac{2\pi \mathbf{i} m}{2T} + \alpha(i)\cdot\Lambda\right) h_{\alpha(i),m}^{(j)} - (f\circ P)_{\alpha(i),m}^{(j)}(h) \right] \right| \nu^{|m|},
\end{aligned}$$

where the middle sum is finite, and the first and the third sums are bounded since $h_{\alpha(i)}^{(j)} \in \ell_\nu^1$, and since by the Banach algebra property of ℓ_ν^1, we have that $(f\circ P)_{\alpha(i)}^{(j)}(h) \in \ell_\nu^1$. Therefore, $\|c(i,j)\|_{1,\nu} < \infty$ for all $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,d\}$. Hence, $a = AF(h) \in X$. ☐

We introduce the notation $\mathrm{conj}(z)$ to denote the complex conjugate of a complex number $z \in \mathbb{C}$. Since we are interested in real solutions, we impose the following symmetries on the blocks of A. Assume that $A^{(M)}$ satisfies

$$((A^{(M)}_{i_1,i_2})_{j_1,j_2})_{-k_1,-k_2} = \mathrm{conj}\left((A^{(M)}_{i_1,i_2})_{j_1,j_2})_{k_1,k_2}\right), \quad \forall\, k_1,k_2 = -M+1,\dots,M-1. \tag{55}$$

Define the space

$$X_{sym} \stackrel{\text{def}}{=} \left(\widetilde{\ell^1_\nu}\right)^{nd}, \tag{56}$$

where

$$\widetilde{\ell^1_\nu} \stackrel{\text{def}}{=} \left\{c \in \ell^1_\nu \,|\, c_{-m} = \mathrm{conj}(c_m) \;\; \forall\, m \in \mathbb{Z}\right\}. \tag{57}$$

Denote by $B_r(\bar a)$ the closed ball of radius $r > 0$ centered at $\bar a$ in the Banach space X, that is

$$B_r(\bar a) = \{a \in X : \|a - \bar a\|_X \le r\}.$$

Lemma 3.2. *Assume that $\bar a \in X_{sym}$ and set $r > 0$. Define $\mathcal{T}$ as in* (50) *and let A defined by* (52) *such that the assumption* (55) *holds. Then*

$$AF : X_{sym} \to X_{sym}. \tag{58}$$

Assume moreover that $\mathcal{T} : B_r(\bar a) \to B_r(\bar a)$ is a contraction, and let $\tilde a \in X$ the unique fixed point of $\mathcal{T}$ in $B_r(\bar a)$ which exists by the contraction mapping theorem. Then, $\tilde a \in X_{sym}$.

Proof. We begin by showing that (54) holds. Let $h \in X_{sym}$, $b \stackrel{\text{def}}{=} F(h)$ and $a \stackrel{\text{def}}{=} Ab$. As in the proof of Lemma 3.1, we have that

$$a = \left\{ \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (A_{i,i_2})_{j,j_2} b^{(j_2)}_{\alpha(i_2)} \right\}_{\substack{i=1,\dots,n\\ j=1,\dots,d}}.$$

By Lemma 3.1, we know that $\|a\|_X < \infty$. It remains to show that each component $a^{(j)}_{\alpha(i)}$ is in $\widetilde{\ell^1_\nu}$. Now, for each $i, i_2 = 1,\dots,n$ and $j, j_2 = 1,\dots,d$, let $c \stackrel{\text{def}}{=} (A_{i,i_2})_{j,j_2} b^{(j_2)}_{\alpha(i_2)}$. Then, for each $m \in \mathbb{Z}$, and using the symmetry assumption (55),

$$\begin{aligned}
c_{-m} &= \sum_{m_1=-\infty}^{\infty} ((A_{i,i_2})_{j,j_2})_{-m,k_1} \left(b^{(j_2)}_{\alpha(i_2)}\right)_{k_1} \\
&= \sum_{k_1=-\infty}^{\infty} ((A_{i,i_2})_{j,j_2})_{-m,-k_1} \left(b^{(j_2)}_{\alpha(i_2)}\right)_{-k_1} \\
&= \sum_{k_1=-\infty}^{\infty} \mathrm{conj}\left(((A_{i,i_2})_{j,j_2})_{m,k_1}\right) \mathrm{conj}\left(\left(b^{(j_2)}_{\alpha(i_2)}\right)_{k_1}\right) \\
&= \mathrm{conj}\left(\sum_{k_1=-\infty}^{\infty} ((A_{i,i_2})_{j,j_2})_{m,k_1} \left(b^{(j_2)}_{\alpha(i_2)}\right)_{k_1}\right) \\
&= \mathrm{conj}\left(c_{-m}\right).
\end{aligned}$$

Finally, by (54), $\mathcal{T} : X_{sym} \to X_{sym}$. Using that $\bar a \in X_{sym} \cap B_r(\bar a)$, and that X_{sym} is a closed subset of X, we obtain that

$$\tilde a = \lim_{n\to\infty} \mathcal{T}^n(\bar a) \in X_{sym}.$$

□

3.2 The radii polynomial approach

The goal of the present section is to determine an efficient way of determining a ball of the form $B_r(\bar{a})$ on which the operator $\mathcal{T}: X \to X$ as defined in (50) is a contraction.

For each $i = 1, \ldots, n$ and $j = 1, \ldots, d$, consider bounds $Y^{(j)}_{\alpha(i)}$ and $Z^{(j)}_{\alpha(i)}$ satisfying

$$\left\| \left(\mathcal{T}(\bar{a}) - \bar{a}\right)^{(j)}_{\alpha(i)} \right\|_{1,\nu} \leq Y^{(j)}_{\alpha(i)} \quad \text{and} \quad \sup_{b,c \in B_r(0)} \left\| \left(D\mathcal{T}(\bar{a}+b)c\right)^{(j)}_{\alpha(i)} \right\|_{1,\nu} \leq Z^{(j)}_{\alpha(i)}. \tag{59}$$

Remark 3.3 (**The bound $Z^{(j)}_{\alpha(i)}$ as a polynomial in r**). *To compute $Z^{(j)}_{\alpha(i)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, d$, one estimates $(D\mathcal{T}(\bar{a}+b)c)^{(j)}_{\alpha(i)}$ for all $b, c \in B_r(0)$. This is equivalent to estimating $(D\mathcal{T}(\bar{a}+ur)vr)^{(j)}_{\alpha(i)}$ for all $u, v \in B_1(0)$. If the nonlinearities of the vector field f are polynomials, then F will consists of Cauchy products of discrete convolutions. Since $\mathcal{T}(x) = x - AF(x)$ and $D\mathcal{T}(\bar{a}+ur)vr \in X$, then each component of $D\mathcal{T}(\bar{a}+ur)vr$ can be expanded as a polynomial in r with the coefficients being in ℓ^1_ν.*

Hence, from now on we assume that the bound $Z^{(j)}_{\alpha(i)} = Z^{(j)}_{\alpha(i)}(r)$ is polynomial in r.

Definition 3.4. For each $i = 1, \ldots, n$ and $j = 1, \ldots, d$, the radii polynomial $p^{(j)}_{\alpha(i)}(r)$ is given by

$$p^{(j)}_{\alpha(i)}(r) \stackrel{\text{def}}{=} Y^{(j)}_{\alpha(i)} + Z^{(j)}_{\alpha(i)}(r) - r. \tag{60}$$

Lemma 3.5. *Define*

$$\mathcal{I} = \bigcap_{i=1}^{n} \bigcap_{j=1}^{d} \left\{ r > 0 : p^{(j)}_{\alpha(i)}(r) < 0 \right\}. \tag{61}$$

If $\mathcal{I} \neq \emptyset$, then $\mathcal{I} = (I_-, I_+)$ is an open interval, and for any fixed $\bar{r} \in \mathcal{I}$, the ball $B_{\bar{r}}(\bar{a})$ contains a unique solution of (47).

The proof of this result is standard (e.g. see [14, 65, 66]) and will be omitted.

Before proceeding with the explicit computation of the bounds satisfying (59), we present a remark describing how a successful application of Lemma 3.5 can be used to obtain the rigorous error bounds required in (45).

Remark 3.6. *Consider the radii polynomials* (60) *constructed with a fixed decay rate $\nu > 1$. Let $\mathcal{I}$ the set defined in* (61) *and assume it is non empty. Let $\bar{r} \in \mathcal{I}$ and let $a \in B_{\bar{r}}(\bar{a})$ such that $F(a) = 0$. Then*

$$\|a - \bar{a}\|_X = \max_{\substack{i=1,\ldots,n \\ j=1,\ldots,d}} \left\{ \|a^{(j)}_{\alpha(i)} - \bar{a}^{(j)}_{\alpha(i)}\|_{1,\nu} \right\} \leq \bar{r}. \tag{62}$$

Let α be a multi-index such that $2 \leq |\alpha| \leq N$. In the context of the error bounds (45), *set $a^M_\alpha = \bar{a}_\alpha$. Also, set*

$$r = \frac{T}{\pi} \log(\nu). \tag{63}$$

the width of the strip $\mathbb{A}_r$ in the complex plane.Then, $2 \le |\alpha| \le N$, using (62),

$$\begin{aligned}
\|a_\alpha^M - a_\alpha\|_r^\infty &= \|\bar{a}_\alpha - a_\alpha\|_r^\infty \\
&= \max_{1\le j\le d} \|\bar{a}_\alpha^{(j)} - a_\alpha^{(j)}\|_{\mathbb{A}_r}^\infty \\
&= \max_{1\le j\le d} \sup_{w\in\mathbb{A}_r} |\bar{a}_\alpha^{(j)}(w) - a_\alpha^{(j)}(w)| \\
&= \max_{1\le j\le d} \sup_{w\in\mathbb{A}_r} \left| \sum_{m=-\infty}^{\infty} \bar{a}_{\alpha,m}^{(j)} e^{\frac{2\pi i m}{2T} w} - a_{\alpha,m}^{(j)} e^{\frac{2\pi i m}{2T} w} \right| \\
&\le \max_{1\le j\le d} \sup_{w\in\mathbb{A}_r} \sum_{m=-\infty}^{\infty} \left| \bar{a}_{\alpha,m}^{(j)} - a_{\alpha,m}^{(j)} \right| \left| e^{\frac{\pi i m}{T} w} \right| \\
&\le \max_{1\le j\le d} \sum_{m=-\infty}^{\infty} \left| \bar{a}_{\alpha,m}^{(j)} - a_{\alpha,m}^{(j)} \right| e^{\frac{\pi |m|}{T} r} \\
&\le \max_{1\le j\le d} \sum_{m=-\infty}^{\infty} \left| \bar{a}_{\alpha,m}^{(j)} - a_{\alpha,m}^{(j)} \right| \nu^{|m|} \\
&= \max_{1\le j\le d} \left\| \bar{a}_\alpha^{(j)} - a_\alpha^{(j)} \right\|_\nu \\
&\le \epsilon_\alpha.
\end{aligned}$$

Hence, the rigorous error bound required in (45) can be set to be $\epsilon_\alpha \stackrel{\text{def}}{=} \bar{r}$ for all $2 \le |\alpha| \le N$.

Here is our general strategy: we use Lemma 3.5 to solve (47) and Remark 3.6 shows how the rigorous error bounds required in (45) can be obtained. Next, we attempt to provide general formulas for the construction of the radii polynomials valid for any vector field, any periodic orbit, no matter what the dimension of the manifold and the dimension of the phase space are. However, some of the formula would be significantly hard to write in the full generality and almost impossible to read, hence we prefer to combine the exposition of the general case with a concrete example.

To begin with, we remind that the unknowns of the problem are the sequences $a_\alpha^{(j)} \in \ell_\nu^1$ of Fourier coefficients of the functions $a_\alpha^{(j)}(w)$ where $|\alpha| \in \{2, \dots, N\}$ while the Fourier coefficients of the function $a_\alpha^{(j)}(w)$ with $|\alpha| \in \{0, 1\}$ are data for the problem. More precisely $a_0^{(j)}$ is the sequence of Fourier coefficients on the basis of $2T$-periodic functions of the periodic orbit $\gamma(t)$ while $a_\alpha^{(j)}$ with $|\alpha| = 1$ are sequences of the Fourier coefficients forming the normal bundle. We assume that both the periodic orbit and the linear bundle have been previously computed and that the data are given in the form

$$a_\alpha^{(j)} = \bar{a}_\alpha^{(j)} + \mathcal{E}a_\alpha^{(j)}, \quad |\alpha| \in \{0, 1\} \tag{64}$$

where $(\bar{a}_\alpha^{(j)})_m \equiv 0$ for any $|m| > M$ and the remainder $\mathcal{E}a_\alpha^{(j)}$ is known only in norm, $\|\mathcal{E}a_\alpha^{(j)}\|_{1,\nu} \le \epsilon_\alpha^{(j)}$.

For any $a \in X$, the function $F(a)$ depends on the data a_α with $|\alpha| \in \{0, 1\}$. Let us write explicitly such a dependence as $F(a) = F(\{a_\alpha\}_{\|\alpha|=0}^1, a) = F(\{\bar{a}_\alpha + \mathcal{E}a_\alpha\}_{\|\alpha|=0}^1, a)$. Denote $\bar{F}(a) = F(\{\bar{a}_\alpha\}_{\|\alpha|=0}^1, a)$ and introduce $\mathcal{E}F(a)$ so that the equality holds

$$F(a) = \bar{F}(a) + \mathcal{E}F(a). \tag{65}$$

In practice, $\mathcal{E}F(a)$ is a correction that can only be estimated in norm, since the contributions $\mathcal{E}a_\alpha$, $|\alpha| \in \{0,1\}$ are only known in norm. Afterwards, the notation $\mathcal{E}\cdot$ always stands for a quantity the knowledge of which is only given in norm.

Similarly, the derivative $DF(a)$ splits into the sum

$$DF(a) = \overline{DF}(a) + \mathcal{E}DF(a) \tag{66}$$

where $\overline{DF}(a) = DF(\{\bar{a}_\alpha\}_{|\alpha|=0}^{1}, a)$.

<<COMMENT>> What are ϵ_0 and ϵ_1? Why the dependency is only on these terms?
JP
SOLVED !!

3.2.1 Bound Y

Formally,

$$\begin{aligned}\left\| \left(AF(\bar{a})\right)^{(j)}_{\alpha(i)} \right\|_{1,\nu} &= \left\| \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (A_{i,i_2})_{j,j_2} \left[\bar{F}(\bar{a})^{(j_2)}_{\alpha(i_2)} + \mathcal{E}F^{(j_2)}_{\alpha(i_2)}(\bar{a}) \right] \right\|_{1,\nu} \\ &\leq \left\| \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (A_{i,i_2})_{j,j_2} \bar{F}(\bar{a})^{(j_2)}_{\alpha(i_2)} \right\|_{1,\nu} + \left\| \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (A_{i,i_2})_{j,j_2} \mathcal{E}F^{(j_2)}_{\alpha(i_2)}(\bar{a}) \right\|_{1,\nu}.\end{aligned}$$

Any $\bar{F}(\bar{a})^{(j)}_\alpha$ is a finite dimensional sequence, the length of which depends on the degree of the nonlinearity of the vector field. Therefore the first sum is a finite computations which can be rigorously bounded using interval arithmetic. For the second sum we estimate

<<COMMENT>> Is it really what is done here? Don't we loose the fact that somehow only the tail of A is involved? JP
Why it should be like that? All the $\mathcal{E}F^{(j_2)}_{\alpha(i_2)}(\bar{a})$ are sequences different than zero also in the finite part !!

$$\left\| \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (A_{i,i_2})_{j,j_2} \mathcal{E}F^{(j_2)}_{\alpha(i_2)}(\bar{a}) \right\|_{1,\nu} \leq \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} |||(A_{i,i_2})_{j,j_2}||| \, \|\mathcal{E}F^{(j_2)}_{\alpha(i_2)}(\bar{a})\|_{1,\nu}$$

and we take advantage from the algebra property $\|a*b\|_{1,\nu} \leq \|a\|_{1,\nu}\|b\|_{1,\nu}$ in order to bound $\|\mathcal{E}F^{(j_2)}_{\alpha(i_2)}\|_{1,\nu}$.

3.2.2 The bounds $Z^{(j)}_{\alpha(i)}$, $i = 1, \ldots, n$, $j = 1, \ldots, d$

For each $i = 1, \ldots, n$ and $j = 1, \ldots, d$ we seek a bound $Z^{(j)}_{\alpha(i)}$, a polynomial in the variable r satisfying

$$\sup_{b,c \in B_r(0)} \left\| \left(D\mathcal{T}(\bar{a}+b)c\right)^{(j)}_{\alpha(i)} \right\|_{1,\nu} \leq Z^{(j)}_{\alpha(i)}(r).$$

Recalling that $DF^{(M)}(\bar{a}) \in M_{nd(2M-1)}(\mathbb{C})(\bar{a})$ is the Jacobian of $F^{(M)}$ at $\bar{a}$, denote as $DF^{(M)} = \{(DF^{(M)}_{n_1,n_2})_{d_1,d_2}\}$ the component-wise representation of $DF^{(M)}(\bar{a})$ similar to (51). Define the linear operator $A^\dagger$ so that, for any $b \in (\ell^1_\nu)^{nd}$

$$(A^\dagger b)^{(j_1)}_{\alpha(i_1)} = \sum_{i_2=1}^{n} \left(A^\dagger_{i_1,i_2} b_{\alpha(i_2)}\right)^{(j_1)} = \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (A^\dagger_{i_1,i_2})_{j_1,j_2} b^{(j_2)}_{\alpha(i_2)},$$

where the action of $(A^\dagger_{i_1,i_2})_{j_1,j_2}$ on $h \in \ell^1_\nu$ is given by

$$\Big((A^\dagger_{i_1,i_2})_{j_1,j_2} h\Big)_m = \begin{cases} \Big((DF^{(M)}_{i_1,i_2})_{j_1,j_2} h_F\Big)_m, & |m| < M, \\ \delta_{i_1,i_2}\delta_{j_1,j_2}\left(\dfrac{2\pi \mathbf{i} m}{2T} + \alpha(i_1)\cdot\Lambda\right) h_m, & |m| \geq M, \end{cases} \tag{67}$$

where $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise.

Consider now the splitting

$$D\mathcal{T}(\bar a + b)c = (I - ADF(\bar a + b))c = (I - AA^\dagger)c - A(DF(\bar a) - A^\dagger)c - A(DF(\bar a + b) - DF(\bar a))c$$

and

$$\begin{aligned} \left\| \Big(D\mathcal{T}(\bar a + b)c\Big)^{(j)}_{\alpha(i)} \right\|_{1,\nu} \leq{}& \left\| \Big((I - AA^\dagger)c\Big)^{(j)}_{\alpha(i)} \right\|_{1,\nu} + \left\| \Big(A(DF(\bar a) - A^\dagger)c\Big)^{(j)}_{\alpha(i)} \right\|_{1,\nu} \\ &+ \left\| \Big(A(DF(\bar a + b) - DF(\bar a))c\Big)^{(j)}_{\alpha(i)} \right\|_{1,\nu}. \end{aligned}$$

Since $b, c \in B_r(0)$, we can factor out r and write $b = ru$ and $c = rv$ where $u, v \in B_1(0)$. This means that $u = \{u^{(j)}_{\alpha(i)}\}_{\substack{i=1,\dots,n \\ j=1,\dots,d}}$ satisfies $\|u^{(j)}_{\alpha(i)}\|_{1,\nu} \leq 1$ for any $i = 1, \dots, n$ and $j = 1, \dots, d$. The same holds for v.

The bound is constructed as

$$Z^{(j)}_{\alpha(i)}(r) = (Z^0)^{(j)}_{\alpha(i)} r + (Z^1)^{(j)}_{\alpha(i)} r + (Z^2)^{(j)}_{\alpha(i)} r^2 + \cdots + (Z^p)^{(j)}_{\alpha(i)} r^p$$

where

$$\begin{aligned} \left\| \Big((I - AA^\dagger)v\Big)^{(j)}_{\alpha(i)} \right\|_{1,\nu} &\leq (Z^0)^{(j)}_{\alpha(i)}, \quad \|v\|_X \leq 1 \\ \left\| \Big(A(DF(\bar a) - A^\dagger)v\Big)^{(j)}_{\alpha(i)} \right\|_{1,\nu} &\leq (Z^1)^{(j)}_{\alpha(i)}, \quad \|v\|_X \leq 1 \\ \left\| \Big(A(DF(\bar a + ru) - DF(\bar a))rv\Big)^{(j)}_{\alpha(i)} \right\|_{1,\nu} &\leq (Z^2)^{(j)}_{\alpha(i)} r^2 + \cdots + (Z^p)^{(j)}_{\alpha(i)} r^p, \quad \|u\|_X \leq 1, \|v\|_X \leq 1. \end{aligned}$$

The exponent p refers to the maximal power of r that appears on the last relation and equals the degree of the polynomial nonlinearity of the vector field $f(x)$.

3.2.3 The bound Z^0

Denote by $B \stackrel{\text{def}}{=} I - AA^\dagger$. Since the tail diagonal components of $(A_{i,i})_{j,j}$ are defined as the inverse of the tail diagonal components of $(A^\dagger_{i,i})_{j,j}$, the operator B results in a finite dimensional operator acting on v_F. Hence

$$\Big((I - AA^\dagger)v\Big)^{(j)}_{\alpha(i)} = (Bv)^{(j)}_{\alpha(i)} = \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} (B_{i,i_2})_{j,j_2} (v_F)^{(j_2)}_{\alpha(i_2)}.$$

The operator $DF^{(M)}$ used in the definition of $A^\dagger$ represents the genuine (finite-dimensional) Jacobian of F. Therefore $DF^{(M)}$ and $A^\dagger$ inherit the same splitting already discussed for DF, that is $A^\dagger = \bar A^\dagger + \mathcal{E}A^\dagger$.

<<COMMENT>> Don't we have that $\mathcal{E}B = 0$? JP
$DF^{(M)}$ and thus $A^\dagger$ depend on a_0 and a_1 both given within bounds. So when computing $AA^\dagger$ one splits $A(\bar{A}^\dagger + \mathcal{E}A^\dagger)$. !!

Thus, we set $B = \bar{B} + \mathcal{E}B$ where $\bar{B} = I - A\bar{A}^\dagger$ and $\mathcal{E}B = -A\mathcal{E}A^\dagger$. It follows

$$\left\| \left((I - AA^\dagger)v\right)^{(j)}_{\alpha(i)} \right\|_{1,\nu} \leq \|(\bar{B}v)^{(j)}_{\alpha(i)}\|_{1,\nu} + \left\| \left((A\mathcal{E}A^\dagger)v\right)^{(j)}_{\alpha(i)} \right\|_{1,\nu}$$
$$\leq \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} |||(\bar{B}_{i,i_2})_{j,j_2}||| + \sum_{i_2=1}^{n} \sum_{j_2=1}^{d} |||(A_{i,i_2})_{j,j_2}||| \left\| (\mathcal{E}A^\dagger(v))^{(j_2)}_{\alpha(i_2)} \right\|_{1,\nu}.$$

3.2.4 The bound Z^1

$$\left\| \left(A(DF(\bar{a}) - A^\dagger)v\right)^{(j)}_{\alpha(i)} \right\|_{1,\nu} \leq (Z^1)^{(j)}_{\alpha(i)}, \quad \|v\|_X \leq 1.$$

In order to properly compute a bound of the above norm, we need to recast the quantity $A(DF(\bar{a}) - A^\dagger)v$ as a linear operator acting on the components of v. It is convenient to treat the linear operator $DF(\bar{a}) - A^\dagger$ as a matrix of operators. We remind that the index $i \in \{1, \dots, n\}$ and j runs between 1 and d. Without separating further the different role of the two indexes, we can consider a unique index ranging from 1 to nd labelling the elements of v.

Suppose that the operator $DF(\bar{a}) - A^\dagger$ is represented by the matrix $\Gamma = \Gamma(s,t)$, where $s, t \in \{1, \dots, nd\}$ and $\Gamma(s,t) \in \mathcal{B}(\ell^1_\nu, \ell^1_\nu)$ for any s, t. That is, $\left[\left(DF(\bar{a}) - A^\dagger\right)v\right]_s = \sum_t \Gamma(s,t)v_t$. Suppose also that A is represented by $A(p,q)$, $p, q \in \{1, \dots nd\}$. Then the operator $A(DF(\bar{a}) - A^\dagger)$ is represented by the matrix $A\Gamma$ with $A\Gamma(q,t) = \sum_s A(q,s)\Gamma(s,t)$, $A\Gamma(q,t) \in \mathcal{B}(\ell^1_\nu, \ell^1_\nu)$ and $\left(A(DF(\bar{a}) - A^\dagger)v\right)_q = \sum_t A\Gamma(q,t)v_t$. Therefore

$$\left\| \left(A(DF(\bar{a}) - A^\dagger)v\right)_q \right\|_{1,\nu} \leq \sum_t \|A\Gamma(q,t)v_t\|_{1,\nu} \leq \sum_t \sum_s \|A(q,s)\Gamma(s,t)v_t\|_{1,\nu}. \tag{68}$$

Also, we separate the ϵ-contribution and write $\Gamma(s,t) = \overline{\Gamma}(s,t) + \mathcal{E}\Gamma(s,t)$. Accordingly, the general receipt for the definition of Z^1_q is

$$\begin{aligned} \left\| \left(A(DF(\bar{a}) - A^\dagger)v\right)_q \right\|_{1,\nu} &\leq \sum_t \sum_s \|A(q,s)(\overline{\Gamma}(s,t) + \mathcal{E}\Gamma(s,t))v_t\|_{1,\nu} \\ &\leq \sum_t \sum_s |||A(q,s)\overline{\Gamma}(s,t)||| + \sum_t \sum_s |||A(q,s)|||\, \|\mathcal{E}\Gamma(s,t)v_t\|_{1,\nu} \\ &=: Z^1_q\,. \end{aligned} \tag{69}$$

The previous formula provides the scheme for the computation of the bound Z^1 for any given vector field. However, the definition of Γ is problem dependent, hence we prefer to present more details about the construction of the bound Z^1 having a concrete case in mind. That will be done in the subsequent section, where the Lorenz system is considered as toy model.

3.2.5 Bound Z^i, $2 \le i \le p$

For the construction of the bound Z^i, $2 \le i \le p$ we proceed as follows: define a vector of sequences $\{\mathcal{V}_q\}_{q=1}^{nd}$, $\mathcal{V}_q = \{(\mathcal{V}_q)_m\}_{m\in\mathbb{Z}}$ so that $\mathcal{V}_q$ contains the r^i contributions of $\left[\left(DF(\bar{a}+ru) - A^\dagger\right) rv\right]_q$. Then introduce $V = \{V_q\}_{q=1}^{nd}$ so that $V_q \le \sup_{\|u\|_X\le 1, \|v\|_X \le 1} \|\mathcal{V}_q\|_{1,\nu}$. Finally, define

$$Z_q^i = \sum_{1\le t\le nq} |||A(q,t)|||V_t.$$

3.3 Enclosure of a_α in case of large N

The first condition required by Theorem 2.4 is that $(N+1) > A/\mu_*$, where A only depends on the jacobian of the vector field along γ and μ_* on the Floquet exponents. If the degree of the polynomial nonlinearity of the vector field is large, or the magnitude of orbit γ is large in phase space, the bound A can be considerably large. Also, if the real part of the Floquet exponents are small, μ_* is small. The combination of these factors may lead to a choice of large N. From the theoretical point of view, the procedure so far exposed is well working for any N. On the other hand, the number of equation in the nonlinear system $F_{\alpha,m}^{(j)} = 0$ to be rigorously solved increases with N and the effective computation can be problematic.

In this section we present a modification of the technique that allows to compute the coefficients a_α when $|\alpha|$ is large. The key observation is that $\|a_\alpha\|_r^\infty \to 0$ for $|\alpha|$ large enough, thus it is reasonable to compute the enclosure of a_α around $\bar{a}_\alpha = 0$.

Suppose that for a choice of $\tilde{N} < N$ the method of the previous sections returns the enclosure of the functions $a_\alpha(w)$ for $|\alpha| \le \tilde{N}$. We are now concerned with the computation of $a_\alpha(w)$ for $\tilde{N} < |\alpha| \le N$, that is, we need to solve $F_{\alpha,m}^{(j)} = 0$, in the unknowns $a_\alpha^{(j)}(w)$, $\tilde{N} < |\alpha| \le N$. Instead, a_α with $|\alpha| \le \tilde{N}$ are data of the problem and given within rigorous bounds computed before.

Since F_α depends on a_α and on a_β with $|\beta| \le |\alpha|$ we can solve *layer-by-layer*. That means, we solve $\{F_\alpha = 0\}_{|\alpha|=\tilde{N}+1}$ in the unknown $\{a_\alpha\}_{|\alpha|=\tilde{N}+1}$. Once the $\tilde{N}+1$ layer is rigorously enclosed, we move to the next $\tilde{N}+2$ layer, treating the previous computed coefficients as data for the new problem.

We apply the same technique as before, with the following modifications.

- The numerical approximate solution $\bar{a}$ is set to zero. For a choice of M, that might be different than the previous, define $\bar{a}_{\alpha,m}^{(j)} = 0$ for any $|m| < M$, $j = 1,\dots,d$, $\tilde{N} < |\alpha| \le N$.
- The operators A and $A^\dagger$, see (83), (67) are now defined as diagonal operators in α, that is $A_{i_1,i_2} \equiv 0$ if $i_1 \ne i_2$. The same for $A^\dagger$. In practice only the Jacobian of F_α with respect to a_α is considered in constructing A and $A^\dagger$, even in the finite dimensional subspace. Use the notation $A_{\alpha,\alpha}$ in place of $A_{i,i}$ when $\alpha = \alpha(i)$.

The definition of the Z bound is the same as before. However, since $A^\dagger$ and A are diagonal in α, the operators $\Gamma(s,t)$ are slightly different and the sum (68) is taken over those s's that refer to the same α as q. The meaning of these statements will be clearer in the next section, where the computational technique is applied to the Lorenz system.

4 A study case: the Lorenz system

The Lorenz vector field is given by

$$f(x,y,z) = \begin{pmatrix} -\sigma x + \sigma y \\ \rho x - y - xz \\ -\beta z + xy \end{pmatrix}, \qquad (\sigma, \beta, \rho \in \mathbb{R}). \tag{70}$$

The composition of f and the parameterisation P reads as

$$(f \circ P)_\alpha(w) = \begin{pmatrix} -\sigma a_\alpha^{(1)}(w) + \sigma a_\alpha^{(2)}(w) \\ \rho a_\alpha^{(1)}(w) - a_\alpha^{(2)}(w) - \left(a^{(1)}a^{(3)}\right)_\alpha(w) \\ -\beta a_\alpha^{(3)}(w) + \left(a^{(1)}a^{(2)}\right)_\alpha(w) \end{pmatrix},$$

where

$$\left(a^{(1)}a^{(3)}\right)_\alpha(w) \stackrel{\text{def}}{=} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 0}} a_{\alpha_1}^{(1)}(w) a_{\alpha_2}^{(3)}(w) \quad \text{and} \quad \left(a^{(1)}a^{(2)}\right)_\alpha(w) \stackrel{\text{def}}{=} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 0}} a_{\alpha_1}^{(1)}(w) a_{\alpha_2}^{(2)}(w).$$

Moreover,

$$(f \circ P)_{\alpha,m} = \begin{pmatrix} -\sigma a_{\alpha,m}^{(1)} + \sigma a_{\alpha,m}^{(2)} \\ \rho a_{\alpha,m}^{(1)} - a_{\alpha,m}^{(2)} - \left(a^{(1)}a^{(3)}\right)_{\alpha,m} \\ -\beta a_{\alpha,m}^{(3)} + \left(a^{(1)}a^{(2)}\right)_{\alpha,m} \end{pmatrix},$$

where

$$\left(a^{(1)}a^{(3)}\right)_{\alpha,m} = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 0}} \left(a_{\alpha_1}^{(1)} * a_{\alpha_2}^{(3)}\right)_m \quad \text{and} \quad \left(a^{(1)}a^{(2)}\right)_{\alpha,m} = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 0}} \left(a_{\alpha_1}^{(1)} * a_{\alpha_2}^{(2)}\right)_m,$$

with

$$\left(a_{\alpha_1}^{(1)} * a_{\alpha_2}^{(3)}\right)_m = \sum_{\substack{k_1+k_2=m \\ k_i \in \mathbb{Z}}} a_{\alpha_1,k_1}^{(1)} a_{\alpha_2,k_2}^{(3)} \quad \text{and} \quad \left(a_{\alpha_1}^{(1)} * a_{\alpha_2}^{(2)}\right)_m = \sum_{\substack{k_1+k_2=m \\ k_i \in \mathbb{Z}}} a_{\alpha_1,k_1}^{(1)} a_{\alpha_2,k_2}^{(2)}.$$

Therefore, in the context of the Lorenz system, (46) is given by

$$\begin{pmatrix} F_{\alpha,m}^{(1)} \\ F_{\alpha,m}^{(2)} \\ F_{\alpha,m}^{(3)} \end{pmatrix} = \begin{pmatrix} \left(\frac{2\pi \mathbf{i} m}{2T} + \alpha\lambda\right) a_{\alpha,m}^{(1)} + \sigma a_{\alpha,m}^{(1)} - \sigma a_{\alpha,m}^{(2)} \\ \left(\frac{2\pi \mathbf{i} m}{2T} + \alpha\lambda\right) a_{\alpha,m}^{(2)} - \rho a_{\alpha,m}^{(1)} + a_{\alpha,m}^{(2)} + \sum\limits_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 0}} \left(a_{\alpha_1}^{(1)} * a_{\alpha_2}^{(3)}\right)_m \\ \left(\frac{2\pi \mathbf{i} m}{2T} + \alpha\lambda\right) a_{\alpha,m}^{(3)} + \beta a_{\alpha,m}^{(3)} - \sum\limits_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 0}} \left(a_{\alpha_1}^{(1)} * a_{\alpha_2}^{(2)}\right)_m \end{pmatrix}, \tag{71}$$

with $2 \leq \alpha \leq N$ and $m \in \mathbb{Z}$.

Concerning the separation (65), by direct computation, it results

$$(\mathcal{E}F(a))_2 = \begin{pmatrix} 0 \\ \mathcal{E}a_0^{(1)} * a_2^{(3)} + \mathcal{E}a_0^{(3)} * a_2^{(1)} + \mathcal{E}a_1^{(1)} * a_1^{(3)} + \mathcal{E}a_1^{(3)} * a_1^{(1)} + \mathcal{E}a_1^{(1)} * \mathcal{E}a_1^{(3)} \\ \mathcal{E}a_0^{(1)} * a_2^{(2)} + \mathcal{E}a_0^{(2)} * a_2^{(1)} + \mathcal{E}a_1^{(1)} * a_1^{(2)} + \mathcal{E}a_1^{(2)} * a_1^{(1)} + \mathcal{E}a_1^{(1)} * \mathcal{E}a_1^{(2)} \end{pmatrix}$$

$$
(\mathcal{E}F(a))_\alpha = \begin{pmatrix} 0 \\ \mathcal{E}a_0^{(1)} * a_\alpha^{(3)} + \mathcal{E}a_0^{(3)} * a_\alpha^{(1)} + \mathcal{E}a_1^{(1)} * a_{\alpha-1}^{(3)} + \mathcal{E}a_1^{(3)} * a_{\alpha-1}^{(1)} \\ \mathcal{E}a_0^{(1)} * a_\alpha^{(2)} + \mathcal{E}a_0^{(2)} * a_\alpha^{(1)} + \mathcal{E}a_1^{(1)} * a_{\alpha-1}^{(2)} + \mathcal{E}a_1^{(2)} * a_{\alpha-1}^{(1)} \end{pmatrix}, \quad \forall \alpha > 2.
$$

Since ℓ_ν^1 is an algebra with the convolution product, it follows that

$$
\|(\mathcal{E}F(a))_\alpha\|_{1,\nu} \le \begin{pmatrix} 0 \\ \epsilon_0^{(1)}\|a_\alpha^{(3)}\|_{1,\nu} + \epsilon_0^{(3)}\|a_\alpha^{(1)}\|_{1,\nu} + \epsilon_1^{(1)}\|a_{\alpha-1}^{(3)}\|_{1,\nu} + \epsilon_1^{(3)}\|a_{\alpha-1}^{(1)}\|_{1,\nu} \\ \epsilon_0^{(1)}\|a_\alpha^{(2)}\|_{1,\nu} + \epsilon_0^{(2)}\|a_\alpha^{(1)}\|_{1,\nu} + \epsilon_1^{(1)}\|a_{\alpha-1}^{(2)}\|_{1,\nu} + \epsilon_1^{(2)}\|a_{\alpha-1}^{(1)}\|_{1,\nu} \end{pmatrix} + \begin{pmatrix} 0 \\ \epsilon_1^{(1)}\epsilon_1^{(3)} \\ \epsilon_1^{(1)}\epsilon_1^{(2)} \end{pmatrix} \delta_{\alpha,2}\,.
$$

The derivative of F at any point a is a linear operator $DF(a) : (\ell_\nu^1)^{3n} \to (\ell_\nu^1)^{3n}$. The action of $DF(a)$ on $v \in (\ell_\nu^1)^{3n}$ is explicitly given by

$$
\left(\left(DF(a)(v)\right)_\alpha\right)_m = \left(\frac{2\pi \mathbf{i} m}{2T} + \alpha\lambda\right) v_{\alpha,m} + \begin{pmatrix} \sigma v_{\alpha,m}^{(1)} - \sigma v_{\alpha,m}^{(2)} \\ -\rho v_{\alpha,m}^{(1)} + v_{\alpha,m}^{(2)} \\ +\beta v_{\alpha,m}^{(3)} \end{pmatrix} \tag{72}
$$

$$
+ \begin{pmatrix} 0 \\ \displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\ge 0, \alpha_1 \ge 2}} \left(v_{\alpha_1}^{(1)} * a_{\alpha_2}^{(3)}\right)_m + \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\ge 0, \alpha_2 \ge 2}} \left(a_{\alpha_1}^{(1)} * v_{\alpha_2}^{(3)}\right)_m \\ -\displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\ge 0, \alpha_1 \ge 2}} \left(v_{\alpha_1}^{(1)} * a_{\alpha_2}^{(2)}\right)_m - \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\ge 0, \alpha_2 \ge 2}} \left(a_{\alpha_1}^{(1)} * v_{\alpha_2}^{(2)}\right)_m \end{pmatrix}
$$

For any a, the derivative $DF(a)$ splits into two parts (66). The data a_0 and a_1 contribute only to the nonlinearity, therefore we have

$$
\left(\mathcal{E}DF(a)(v)\right)_\alpha = \begin{pmatrix} 0 \\ \mathcal{E}a_0^{(1)} * v_\alpha^{(3)} + \mathcal{E}a_0^{(3)} * v_\alpha^{(1)} + \mathcal{E}a_1^{(1)} * v_{\alpha-1}^{(3)} + \mathcal{E}a_1^{(3)} * v_{\alpha-1}^{(1)} \\ \mathcal{E}a_0^{(1)} * v_\alpha^{(2)} + \mathcal{E}a_0^{(2)} * v_\alpha^{(1)} + \mathcal{E}a_1^{(1)} * v_{\alpha-1}^{(2)} + \mathcal{E}a_1^{(2)} * v_{\alpha-1}^{(1)} \end{pmatrix}.
$$

From where, we infer that the action of $\mathcal{E}A^\dagger$ on any $v \in B_1(0)$ is such that

$$
\left\|(\mathcal{E}A^\dagger(v))_2\right\|_{1,\nu} \le \begin{pmatrix} 0 \\ \epsilon_0^{(1)} + \epsilon_0^{(3)} \\ \epsilon_0^{(1)} + \epsilon_0^{(2)} \end{pmatrix}, \quad \left\|(\mathcal{E}A^\dagger(v))_\alpha\right\|_{1,\nu} \le \begin{pmatrix} 0 \\ \epsilon_0^{(1)} + \epsilon_0^{(3)} + \epsilon_1^{(1)} + \epsilon_1^{(3)} \\ \epsilon_0^{(1)} + \epsilon_0^{(2)} + \epsilon_1^{(1)} + \epsilon_1^{(2)} \end{pmatrix}, \quad \forall \alpha > 2.
$$

Now we depict in more details the definition of Z^1.

For $x = \{x_m\}_{m\in\mathbb{Z}}$ we define $x^I = (I - \Pi^{(M)})x$, that is the sequence so that $x_m^I = 0$ for $|m| < M$, $x_m^I = x_m$ for $|m| \ge M$. Our goal is to write the action of $DF(\bar{a}) - A^\dagger$. Recalling formula (72), the definition of $A^\dagger$ and the assumption (64), it turns out that

$$
\left[\left(DF(\bar{a}) - A^\dagger\right) v\right]_{\alpha,m} =
$$

$$
\left\{
\begin{array}{ll}
\left[\begin{array}{c}
0 \\
\displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\geq 0,\alpha_1\geq 2}} \left(\bar{a}^{(3)}_{\alpha_2} * v^{(1)I}_{\alpha_1}\right)_m + \left(\bar{a}^{(1)}_{\alpha_2} * v^{(3)I}_{\alpha_1}\right)_m \\
-\displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\geq 0,\alpha_1\geq 2}} \left(\bar{a}^{(2)}_{\alpha_2} * v^{(1)I}_{\alpha_1}\right)_m + \left(\bar{a}^{(1)}_{\alpha_2} * v^{(2)I}_{\alpha_1}\right)_m
\end{array}\right] + (\mathcal{E}v)_{\alpha,m}, & |m| < M \\
\left[\begin{array}{c}
\sigma(v^{(1)}_{\alpha,m} - v^{(2)}_{\alpha,m}) \\
-\rho v^{(1)}_{\alpha,m} + v^{(2)}_{\alpha,m} + \displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\geq 0,\alpha_1\geq 2}} \left(\bar{a}^{(3)}_{\alpha_2} * v^{(1)}_{\alpha_1}\right)_m + \left(\bar{a}^{(1)}_{\alpha_2} * v^{(3)}_{\alpha_1}\right)_m \\
\beta v^{(3)}_{\alpha,m} - \displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\geq 0,\alpha_1\geq 2}} \left(\bar{a}^{(2)}_{\alpha_2} * v^{(1)}_{\alpha_1}\right)_m + \left(\bar{a}^{(1)}_{\alpha_2} * v^{(2)}_{\alpha_1}\right)_m
\end{array}\right] + (\mathcal{E}v)_{\alpha,m}, & |m| \geq M.
\end{array}
\right.
$$

In the term $(\mathcal{E}v)_{\alpha,m}$ all the contributions due to $\mathcal{E}a_0$ and $\mathcal{E}a_1$ are collected.

In order to define the matrix of operators Γ, it is convenient to introduce further notation. Let $\tilde{I}$ the infinite dimensional matrix given component wise by

$$
\tilde{I}(j,m) = \left\{ \begin{array}{ll} \delta_{j,m}, & |j|,|m| \geq M \\ 0, & \text{otherwise} \end{array} \right. .
$$

To any $\bar{a}^{(j)}_\alpha$ let be associated the matrix $\mathcal{A}^{(j)}_\alpha$ and $\tilde{\mathcal{A}}^{(j)}_\alpha$ as

$$
\mathcal{A}^{(j)}_\alpha(m,n) = (\bar{a}^{(j)}_\alpha)_{m-n}, \quad \tilde{\mathcal{A}}^{(j)}_\alpha(m,n) = \left\{ \begin{array}{cc} 0, & |n|,|m| < M \\ (\bar{a}^{(j)}_\alpha)_{m-n}, & \text{otherwise} \end{array} \right. . \tag{73}
$$

The action of $\tilde{I}$ on a sequence $w = \{w_m\}_{m\in Z}$ is $(\tilde{I}w)_m = 0$ for $|m| < M$ and $(\tilde{I}w)_m = w_m$ for $|m| \geq M$. The action of $\mathcal{A}^{(j)}_\alpha$ on w is such that $(\mathcal{A}^{(j)}_\alpha w)_m = (\bar{a}^{(j)}_\alpha * w)_m$ while $(\tilde{\mathcal{A}}^{(j)}_\alpha w)_m = (\bar{a}^{(j)}_\alpha * w^I)_m$ for $|m| < M$ and $(\tilde{\mathcal{A}}^{(j)}_\alpha w)_m = (\bar{a}^{(j)}_\alpha * w)_m$ for $|m| \geq M$.

We are now in the position of writing

$$
\left[\left(DF(\bar{a}) - A^\dagger\right) v\right]_\alpha = \left[\begin{array}{c}
\sigma\tilde{I}(v^{(1)}_\alpha - v^{(2)}_\alpha) \\
-\rho\tilde{I}v^{(1)}_\alpha + \tilde{I}v^{(2)}_\alpha + \displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\geq 0,\alpha_1\geq 2}} \tilde{\mathcal{A}}^{(3)}_{\alpha_2} v^{(1)}_{\alpha_1} + \tilde{\mathcal{A}}^{(1)}_{\alpha_2} v^{(3)}_{\alpha_1} \\
\beta\tilde{I}v^{(3)}_\alpha - \displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i\geq 0,\alpha_1\geq 2}} \tilde{\mathcal{A}}^{(2)}_{\alpha_2} v^{(1)}_{\alpha_1} + \tilde{\mathcal{A}}^{(1)}_{\alpha_2} v^{(2)}_{\alpha_1}
\end{array}\right] + (\mathcal{E}v)_\alpha.
$$

Referring to the label set $\{1,\ldots,nd\}$ previously introduced, we can now define the operators $\overline{\Gamma}(s,t)$ associated to $DF(\bar{a}) - A^\dagger$, for any $s,t \in \{1,\ldots,nd\}$.

Suppose the couples of labels (i,j), $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,d\}$ are one-to-one related to the set $\{1,\ldots,nd\}$ through the bijection ϕ. Given $q = \phi(i,j)$ denote by $q_\alpha = \alpha(i)$ and $q_j = j$. For instance $\left[\left(DF(\bar{a}) - A^\dagger\right) v\right]_s = \left[\left(DF(\bar{a}) - A^\dagger\right) v\right]^{s_j}_{s_\alpha}$. Also, given $q = \phi(i_1,j_1)$ and $s = \phi(i_2,j_2)$, according to the notation in (52), $A(q,s) = (A_{i_1,i_2})_{j_1,j_2}$.

The non zero $\overline{\Gamma}(s,t)$ are the following:

$s_j = 1$	$s_\alpha = t_\alpha$	$t_j = 1$ $t_j = 2$	$\overline{\Gamma}(s,t) = \sigma \tilde{I}$ $\overline{\Gamma}(s,t) = -\sigma \tilde{I}$			
$s_j = 2$	$s_\alpha = t_\alpha$	$t_j = 1$ $t_j = 2$ $t_j = 3$	$\overline{\Gamma}(s,t) = -\rho \tilde{I} + \tilde{\mathcal{A}}_0^{(3)}$ $\overline{\Gamma}(s,t) = \tilde{I}$ $\overline{\Gamma}(s,t) = \tilde{\mathcal{A}}_0^{(1)}$,	$s_\alpha > t_\alpha$	$t_j = 1$ $t_j = 3$	$\overline{\Gamma}(s,t) = \tilde{\mathcal{A}}_{s_\alpha - t_\alpha}^{(3)}$ $\overline{\Gamma}(s,t) = \tilde{\mathcal{A}}_{s_\alpha - t_\alpha}^{(1)}$
$s_j = 3$	$s_\alpha = t_\alpha$	$t_j = 1$ $t_j = 2$ $t_j = 3$	$\overline{\Gamma}(s,t) = -\tilde{\mathcal{A}}_0^{(2)}$ $\overline{\Gamma}(s,t) = -\tilde{\mathcal{A}}_0^{(1)}$, $\overline{\Gamma}(s,t) = \beta \tilde{I}$	$s_\alpha > t_\alpha$	$t_j = 1$ $t_j = 2$	$\overline{\Gamma}(s,t) = -\tilde{\mathcal{A}}_{s_\alpha - t_\alpha}^{(2)}$ $\overline{\Gamma}(s,t) = -\tilde{\mathcal{A}}_{s_\alpha - t_\alpha}^{(1)}$

Concerning the contribution of $\mathcal{E}\Gamma$, as said before, we are interested in the the image of $\mathcal{E}\Gamma(s,t)$ applied to a sequence $w \in \ell^1_\nu$.

$s_j = 2$	$s_\alpha = t_\alpha$	$t_j = 1$ $t_j = 3$	$\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_0^{(3)} * w$ $\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_0^{(1)} * w$,	$s_\alpha = t_\alpha + 1$	$t_j = 1$ $t_j = 3$	$\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_1^{(3)} * w$ $\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_1^{(1)} * w$
$s_j = 3$	$s_\alpha = t_\alpha$	$t_j = 1$ $t_j = 2$	$\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_0^{(2)} * w$ $\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_0^{(1)} * w$,	$s_\alpha = t_\alpha + 1$	$t_j = 1$ $t_j = 2$	$\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_1^{(2)} * w$ $\mathcal{E}\Gamma(s,t)w = \mathcal{E}a_1^{(1)} * w$

Therefore, the various terms $\|\mathcal{E}\Gamma(s,t)v_t\|_{1,\nu}$ appearing in (69) are easily bounded by either $\|\mathcal{E}a_0^{(i)}\|_{1,\nu} = \epsilon_0^{(i)}$ or $\|\mathcal{E}a_1^{(i)}\|_{1,\nu} = \epsilon_1^{(i)}$.

It remains to provide a mean to compute $|||A(q,s)\overline{\Gamma}(s,t)|||$.

Looking at the definition of A, we realise that $A(q,s)$ is either a square finite dimensional matrix of dimension $2M-1$ or a infinite dimensional operator with a diagonal action out of the central block. The latter case holds when $q = s$. We depict the two cases

$$A(q,s) = \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \mathcal{A} & 0 \\ \hline 0 & 0 & 0 \end{array} \quad q \neq s, \qquad A(q,q) = \begin{array}{c|c|c} \ddots & 0 & 0 \\ \hline 0 & \mathcal{A} & 0 \\ \hline 0 & 0 & \ddots \end{array} \quad . \tag{74}$$

Suppose $\overline{\Gamma}(s,t) = \tilde{\mathcal{A}}_\alpha^{(j)}$ (the case $\tilde{I}$ is immediate). Since $(\bar{a}_\alpha^{(j)})_m = 0$ for $|m| \geq M$, the infinite dimensional matrix $\overline{\Gamma}(s,t)$ is zero out of a strip $2M-1$ thick around the main diagonal. For the proceeding, it is useful to introduce the matrix

$$E = \begin{bmatrix} \bar{a}_0 & \bar{a}_{-1} & \dots & \dots & \bar{a}_{-M+1} & 0 & \dots & \dots & 0 \\ \bar{a}_1 & \bar{a}_0 & \bar{a}_{-1} & \dots & \bar{a}_{-M+2} & \bar{a}_{-M+1} & 0 & \dots & 0 \\ & & \ddots & & \vdots & & \ddots & & \\ & & & \ddots & \vdots & & & \ddots & 0 \\ \bar{a}_{M-1} & \dots & \dots & & \bar{a}_0 & \dots & & \dots & \bar{a}_{-M+1} \\ 0 & \bar{a}_{M-1} & \dots & \dots & 0 & \bar{a}_0 & \dots & \dots & \bar{a}_{-M+2} \\ & 0 & \ddots & & \vdots & & \ddots & & \\ & & & \ddots & \vdots & & & \ddots & \\ 0 & \dots & & 0 & \bar{a}_{M-1} & \dots & \dots & \dots & \bar{a}_0 \end{bmatrix}$$

and the $(M-1) \times (M-1)$ sub matrixes $\hat{B}$, $\hat{C}$ of E as

$$E = \left[\begin{array}{c|c} * & \hat{C} \\ \hline \hat{B} & * \end{array} \right] .$$

Finally, define $\mathcal{B}$ and $\mathcal{C}$ the $(2M-1)\times(2M-1)$ matrices given by

$$\mathcal{B}=\left[\begin{array}{c|c} 0 & \hat{B} \\ \hline 0 & 0 \end{array}\right], \quad \mathcal{C}=\left[\begin{array}{c|c} 0 & 0 \\ \hline \hat{C} & 0 \end{array}\right].$$

Using these matrices, we can see the infinity dimensional operator $\overline{\Gamma}(s,t)$ as tri-diagonal concatenation of $\mathcal{B}, E, \mathcal{C}$ with a empty block in the centre.

$$\overline{\Gamma}(s,t)=\begin{array}{ccc|c|cccc} \ddots & \ddots & & & & & & \\ \ddots & E & \mathcal{C} & 0 & & & & \\ 0 & \mathcal{B} & E & \mathcal{C} & 0 & & & \\ \hline & 0 & \mathcal{B} & 0 & \mathcal{C} & 0 & & \\ \hline & & 0 & \mathcal{B} & E & \mathcal{C} & 0 & \\ & & & 0 & \mathcal{B} & E & \mathcal{C} & \\ & & & & & \ddots & \ddots & \end{array} \tag{75}$$

In case $\overline{\Gamma}(s,t)$ is $\tilde{I}$, then $E = Id$ and $\mathcal{B}=\mathcal{C}=0$.

Let us now compute the multiplication $A(q,s)\overline{\Gamma}(s,t)$. If $p \neq q$ the operator $A(q,s)$ is of the first of the two forms depicted in figure (74). It results

$$A(q,s)\overline{\Gamma}(s,t)=\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \mathcal{AB} & 0 & \mathcal{AC} \\ \hline 0 & 0 & 0 \end{array}.$$

It is a finite dimensional operator and the operator norm can be easily computed. Note that if $\overline{\Gamma}(s,t)=c\tilde{I}$ then $A(q,s)\overline{\Gamma}(s,t)=0$.

Consider now the case $s=q$. The operator A is of the form

$$A(q,q)=\begin{array}{cccc|c|cccc} \ddots & & & & & & & & \\ & D_3 & & & & & & & \\ & & D_2 & & & & & & \\ & & & D_1 & & & & & \\ \hline & & & & \mathcal{A} & & & & \\ \hline & & & & & H_1 & & & \\ & & & & & & H_2 & & \\ & & & & & & & H_3 & \\ & & & & & & & & \ddots \end{array}$$

where $dim(D_i)=dim(\mathcal{A})=dim(H_i)=(2M-1)\times(2M-1)$. Remember that the elements on the diagonal are strictly decreasing, hence $\max_m |D_i(m,m)| < \inf_m |D_{i-1}(m,m)|$ and the same for H_i.

The multiplication with $\overline{\Gamma}(s,t)$ produces

$$
A(s,s)\overline{\Gamma}(s,t) = \begin{array}{ccccc|c|cccc}
\ddots & \ddots & & & & & & & \\
D_3\mathcal{B} & D_3E & D_3\mathcal{C} & & & & & & \\
 & D_2\mathcal{B} & D_2E & D_2\mathcal{C} & 0 & 0 & & & \\
 & & D_1\mathcal{B} & D_1E & D_1C & 0 & & & \\
\hline
 & & 0 & \mathcal{A}\mathcal{B} & 0 & \mathcal{A}\mathcal{C} & 0 & & \\
\hline
 & & & 0 & H_1\mathcal{B} & H_1E & H_1\mathcal{C} & & \\
 & & & 0 & 0 & H_2\mathcal{B} & H_2E & H_2\mathcal{C} & \\
 & & & & & & H_3\mathcal{B} & H_3E & H_3\mathcal{C} \\
 & & & & & & & \ddots & \ddots \\
\underbrace{\qquad}_{V_3} & \underbrace{\qquad}_{V_2} & \underbrace{\qquad}_{V_1} & \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{M}} & \underbrace{\qquad}_{W_1} & \underbrace{\qquad}_{W_2} & \underbrace{\qquad}_{W_3}
\end{array}
$$

The operator norm of the above infinite dimensional operator is given by

$$
|||A(s,s)\overline{\Gamma}(s,t)||| = \max\{|||M|||, \max\{|||V_i|||\}, \max\{|||W_i|||\}\}
$$

Since $|D_i| < |D_{i-1}|$ and $|H_i| < |H_{i-1}|$, it follows that $|||V_i||| \leq |||V_{i-1}|||$ and $|||W_i||| \leq |||W_{i-1}|||$. Hence we conclude that

$$
|||A(s,s)\overline{\Gamma}(s,t)||| \leq \max\{|||M|||, |||V_1|||, |||W_1|||\}.
$$

Such a quantity can be rigorously computed because it involves only a finite number of operations.

Concerning the Z^2 bound, in the case of the Lorenz system, the r^2 contribution is

$$
\left[\left(DF(\bar{a}+ru) - A^{\dagger}\right) rv\right]_{\alpha} = r^2 \begin{bmatrix} 0 \\ \displaystyle\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 2}} u^{(1)}_{\alpha_1} * v^{(3)}_{\alpha_2} + u^{(3)}_{\alpha_1} * v^{(1)}_{\alpha_2} \\ \displaystyle -\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i \geq 2}} u^{(1)}_{\alpha_1} * v^{(2)}_{\alpha_2} + u^{(2)}_{\alpha_1} * v^{(1)}_{\alpha_2} \end{bmatrix} \tag{76}
$$

Since any convolution is bounded by $\|u^{(j)}_{\alpha_1} * v^{(l)}_{\alpha_2}\|_{1,\nu} \leq \|u^{(j)}_{\alpha_1}\|_{1,\nu}\|v^{(l)}_{\alpha_2}\|_{1,\nu} \leq 1$, we define

$$
V_q = \begin{cases} 0 & \text{if } q_j = 1 \\ 2(q_\alpha - 3) & \text{if } q_j = 2 \text{ or } q_j = 3, \text{ and } q_\alpha \geq 4. \end{cases}
$$

4.1 Higher order terms

Now we apply the variant described in Section 3.3 to the case of the Lorenz equation. Suppose the enclosure of the functions $a^{(j)}_\alpha$ has been computed for any $\alpha \leq \tilde{N}$ and it results $a^{(j)}_\alpha = \bar{a}^{(j)}_\alpha + \mathcal{E}a^{(j)}_\alpha$ with $\|\mathcal{E}a^{(j)}_\alpha\| \leq \epsilon^{(j)}_\alpha$, for any $\alpha \leq \tilde{N}$ and $j = 1,2,3$.

We are now concerned with the computation of $a_{\tilde{N}+1}$ as solution of $F_{\tilde{N}+1} = 0$. Let $\alpha = \tilde{N}+1$. The system F_α is formally the same as (71). It is however convenient to rewrite the nonlinearity in the form

$$
\sum_{\substack{\alpha^1+\alpha^2=\alpha \\ \alpha^i \geq 0}} a^{(1)}_{\alpha^1} * a^{(3)}_{\alpha^2} = \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 0 \leq \alpha^i \leq \tilde{N}}} a^{(1)}_{\alpha^1} * a^{(3)}_{\alpha^2} + (a^{(1)}_0 * a^{(3)}_\alpha + a^{(1)}_\alpha * a^{(3)}_0)
$$

where the dependence on the unknown a_α is highlighted.

Setting $\bar{a}_\alpha = 0$, it results

$$F_\alpha(\bar{a}) = \left[0, \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 0\le\alpha^i\le\tilde{N}}} a^{(1)}_{\alpha^1} * a^{(3)}_{\alpha^2}, \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 0\le\alpha^i\le\tilde{N}}} a^{(1)}_{\alpha^1} * a^{(2)}_{\alpha^2}\right].$$

A bound for $\|F_\alpha(\bar{a})\|_{1,\nu}$ is provided by

$$\|F_\alpha(\bar{a})\|_{1,\nu} \le \begin{bmatrix} 0 \\ \left\| \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 0\le\alpha^i\le\tilde{N}}} \bar{a}^{(1)}_{\alpha^1} * \bar{a}^{(3)}_{\alpha^2} \right\|_{1,\nu} \\ \left\| \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 0\le\alpha^i\le\tilde{N}}} \bar{a}^{(1)}_{\alpha^1} * \bar{a}^{(2)}_{\alpha^2} \right\|_{1,\nu} \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 0\le\alpha^i\le\tilde{N}}} \|\bar{a}^{(1)}_{\alpha^1}\|_{1,\nu}\epsilon^{(3)}_{\alpha^2} + \|\bar{a}^{(3)}_{\alpha^1}\|_{1,\nu}\epsilon^{(1)}_{\alpha^2} + \epsilon^{(1)}_{\alpha^1}\epsilon^{(3)}_{\alpha^2} \\ \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ 0\le\alpha^i\le\tilde{N}}} \|\bar{a}^{(1)}_{\alpha^1}\|_{1,\nu}\epsilon^{(2)}_{\alpha^2} + \|\bar{a}^{(2)}_{\alpha^1}\|_{1,\nu}\epsilon^{(1)}_{\alpha^2} + \epsilon^{(1)}_{\alpha^1}\epsilon^{(2)}_{\alpha^2} \end{bmatrix}.$$

The definition of $A_{\alpha,\alpha}$ and $A^\dagger_{\alpha,\alpha}$ are based on the Jacobian of F_α with respect to a_α, say $DF_{\alpha,\alpha}$. Recalling the definition (73) and denoting by μ_α the diagonal matrix with $\mu_\alpha(m,m) = \frac{2\pi \mathbf{i} m}{2T} + \alpha\lambda \ \forall m$, the operator $DF_{\alpha,\alpha} = \overline{DF}_{\alpha,\alpha} + \mathcal{E}DF_{\alpha,\alpha}$, where

$$\overline{DF}_{\alpha,\alpha} = \begin{bmatrix} \mu_\alpha + \sigma I & -\sigma I & 0 \\ -\rho I + \mathcal{A}^{(3)}_0 & \mu_\alpha + I & \mathcal{A}^{(1)}_0 \\ -\mathcal{A}^{(2)}_0 & -\mathcal{A}^{(1)}_0 & \mu_\alpha + \beta I \end{bmatrix}$$

$$\mathcal{E}DF_{\alpha,\alpha} \begin{bmatrix} v^{(1)} \\ v^{(2)} \\ v^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{E}a^{(3)}_0 * v^{(1)} + \mathcal{E}a^{(1)}_0 v^{(3)} \\ -\mathcal{E}a^{(2)}_0 * v^{(1)} - \mathcal{E}a^{(1)}_0 v^{(2)} \end{bmatrix}$$

The finite dimensional operator $(A_{\alpha,\alpha})^{(M)}$ is defined as approximate inverse of $(\overline{DF}_{\alpha,\alpha})^{(M)}$, while $(A^\dagger_{\alpha,\alpha})^{(M)}$ is given by $DF^{(M)}_{\alpha,\alpha}$.

The operators $\Gamma(s,t)$ are similar to those reported in the previous section. The difference is that any $\tilde{\mathcal{A}}^{(j)}_\alpha$ with $\alpha > 0$ is replaced by zero. The construction of Z^1_q follows form (68). However, since the only unknown is a_α, the sum in (68) restricts to those s such that $s(\alpha) = \alpha$.

Explicitly, for $j = 1,2,3$,

$$\begin{aligned} Z^1_{\alpha,j} \ge & 2|||(A_{\alpha,\alpha})_{j,1}\sigma\tilde{I}||| \\ & + |||(A_{\alpha,\alpha})_{j,2}(-\rho\tilde{I} + \tilde{\mathcal{A}}^{(3)}_0)||| + |||(A_{\alpha,\alpha})_{j,2}\tilde{I}||| + |||(A_{\alpha,\alpha})_{j,2}\tilde{\mathcal{A}}^{(1)}_0||| \\ & + |||(A_{\alpha,\alpha})_{j,2}|||(\epsilon^{(1)}_0 + \epsilon^{(3)}_0) \\ & + |||(A_{\alpha,\alpha})_{j,3}(-\tilde{\mathcal{A}}^{(2)}_0)||| + |||(A_{\alpha,\alpha})_{j,3}(-\tilde{\mathcal{A}}^{(1)}_0)||| + |||(A_{\alpha,\alpha})_{j,3}\beta\tilde{I}||| \\ & + |||(A_{\alpha,\alpha})_{j,3}|||(\epsilon^{(1)}_0 + \epsilon^{(2)}_0) \end{aligned}$$

We remark that some of the above terms vanish, for instance $|||(A_{\alpha,\alpha})_{j,i}\tilde{I}||| = 0$ whenever $j \neq i$.

4.2 Some remarks

1. According to (45), the computation of the parameterisation begins with the rigorous enclosure of the periodic orbit $\gamma(t)$. Denoted by T the period of the orbit, the function $\gamma(t)$ is expanded on the T-periodic exponential Fourier basis. For a choice of ν_γ, the sequence of Fourier coefficients $\gamma = \{\gamma_m\}_{m\in\mathbb{Z}}$ is proved to be in a ball of radius r_γ, with respect to the ν_γ-norm, around the numerical approximation γ^M. Define $\nu < \sqrt{\nu_\gamma}$ and r as in (63). Arguing as in Remark 3.6, it follows that $\|\gamma^M - \gamma\|_r^\infty \le \epsilon_0$, where $\epsilon_0 \stackrel{\text{def}}{=} r_\gamma$.

2. The computation of the normal bundle is performed following the method described in [16]. The functions $\xi_j(w)$ are defined as $\xi_j(w) = Q(w)v_j$, where v_j is an eigenvector of R and $(Q(w), R)$ is the Floquet normal form decomposition of the fundamental matrix solution of the linearised system around $\gamma(t)$. The matrix function $Q(t)$ is expanded in Fourier series on the $2T$-periodic exponential basis. Denoted by $\mathcal{Q}$ the sequence of Fourier coefficients of $Q(w)$, the computation returns a radius r_{Fl} so that

$$\|\mathcal{Q}(i,j) - \bar{\mathcal{Q}}(i,j)\|_{1,\nu} \le r_{Fl}, \qquad |R(i,j) - \bar{R}(i,j)|_\infty \le r_{Fl}, \quad 1 \le i,j \le d$$

   where $(\bar{\mathcal{Q}}, \bar{R})$ is a finite dimensional approximate solution.

   Without loss of generality, let us choose $j = 1$ and denote $v = v_1$, so that $\xi_1 = Q(w)v$. The eigenvector v of R is computed rigorously, for instance following [67], so that $|v(j) - \bar{v}(j)| \le r_v$, for any $1 \le j \le d$. For convenience, let us write $\mathcal{Q} = \bar{\mathcal{Q}} + \epsilon_Q$ and $v = \bar{v} + \epsilon_v$, with $\|\epsilon_Q\|_{1,\nu} \le r_{Fl}$, $|\epsilon_v| \le r_v$ both component wise.

   Set $\bar{\xi}_1 = \bar{\mathcal{Q}}\bar{v}$. Thus

$$\xi_1 - \bar{\xi}_1 = (\bar{\mathcal{Q}} + \epsilon_Q)(\bar{v} + \epsilon_v) - \bar{\mathcal{Q}}\bar{v} = \epsilon_Q\bar{v} + \bar{\mathcal{Q}}\epsilon_v + \epsilon_Q\epsilon_v$$

   and

$$\begin{aligned}\|\xi_1(i) - \bar{\xi}_1(i)\|_{1,\nu} &\le \|\sum_j \epsilon_Q(i,j)\bar{v}(j)\|_{1,\nu} + \|\sum_j \bar{\mathcal{Q}}(i,j)\epsilon_v(j)\|_{1,\nu} + \|\sum_j \epsilon_Q(i,j)\epsilon_v(j)\|_{1,\nu} \\ &\le r_{Fl}\sum_j |\bar{v}(j)| + r_v \sum_j \|\bar{\mathcal{Q}}(i,j)\|_{1,\nu} + d r_{Fl} r_v.\end{aligned}$$

   Hence

$$\max_{i=1,\dots,d} \|\xi_1(i) - \bar{\xi}_1(i)\|_{1,\nu} \le r_{Fl}\sum_j |\bar{v}(j)| + d r_{Fl} r_v + r_v \max_i \sum_j \|\bar{\mathcal{Q}}(i,j)\|_{1,\nu}.$$

   Again, according to remark 3.6, the bound ϵ_1 appearing in (45) is provided by the right hand side of the above relation. The same applies for all the normal bundles ξ_i, $1 \le i \le k$. In the case of the Lorenz system $d = 3$.

3. The eigenvalues λ of R are also given within some bounds, precisely $|\lambda - \bar{\lambda}| \le r_v$.

4. The parameterisation of the normal bundle, i.e $a_\alpha(w)$ with $|\alpha| = 1$ is defined in terms of the eigenvectors v_j of the matrix R. Therefore the scaling of the eigenvectors is a further free parameter. From one side a larger eigenvectors will result in a larger image of the parameterisation. On the other, the rigorous enclosure of the coefficients might be problematic. Besides the scaling of the eigenvectors, the size of the image of the validated parameterisation is affected by the choice of the Taylor norm decay parameter ν. A large value of ν produces large image but it also leads to a larger a posteriori error bound.

4.3 Validation values for the Lorenz system

We now report the validation values for the stable manifold of a periodic orbit of the Lorenz system, with $\sigma = 10, \beta = 8/3$.

Fix the parameter $\rho = 22$ and choose the decay rate $\nu_\gamma = 1.3$ and $\nu = 1.14017$. The periodic orbit is computed with period $T = 0.764386427$ and enclosure radius $r_\gamma = 8.8448 \cdot 10^{-12}$ around a numerical approximate solution. Hence define

$$\epsilon_0 = 8.8448 \cdot 10^{-12}.$$

The computation for the Floquet normal form decomposition $(\mathcal{Q}, R)$ returns the enclosure radius $r_{Fl} = 3.0955 \cdot 10^{-11}$. The periodic orbit is hyperbolic has one positive and one negative Floquet exponent. Here we are concerned with the stable manifold. The stable eigenvalue λ and associated eigenvector v of R are proved to be in a ball of radius $r_v = 8.0431 \cdot 10^{-11}$ around the numerical approximation

$$\bar{\lambda} = -13.861695717713566,$$

$$\bar{v} = [0.793402810447226, 0.025855938972115, 0.608147556760951]^T$$

The procedure explained in section 4.2 point 2, provides

$$\epsilon_1 = 5.0003 \cdot 10^{-10}.$$

Now choose $N = 10$ and $M = 40$, respectively the order of the parameterization P_N and the finite dimensional projection coefficient. The rigorous enclosure of the functions $a_\alpha^{(j)}$ returns the bound

$$\epsilon_\alpha = 1.3389 \cdot 10^{-10}.$$

Following the algorithm proposed in section 2.5, the validation values can be defined as follow. First define $r = 0.031918155042942$ and $\nu = 1$. Note that the latter ν refers to the norm in the Taylor space.

We need to bound the norm $\|Df(\gamma)\|_r^\infty$. In the case of Lorenz we have

$$Df(\gamma) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - \gamma^{(3)} & -1 & -\gamma^{(1)} \\ \gamma^{(2)} & \gamma^{(1)} & -\beta \end{bmatrix}$$

and $\gamma = \overline{\Gamma} + \mathcal{E}_\gamma$ where $\overline{\Gamma}(w) = \sum_{m=-M}^{M} \overline{\Gamma}_m e^{i\frac{2\pi}{T}mw}$ and $\|\mathcal{E}_\gamma\|_r^\infty \le \epsilon_0$. Explicitly, from (17), $\|Df(\gamma)\|_r^\infty = \max_{1\le i\le 3} \sum_{j=1}^{3} \|Df(\gamma)(i,j)\|_{\mathbb{A}_r}^\infty$.

In practice, one computes

$$\|Df(\gamma)\|_{\mathbb{A}_r}^\infty = \sum_{|m|<M} |Df_m(\overline{\Gamma})| e^{\frac{2\pi r}{T}|m|} + \begin{bmatrix} 0 & 0 & 0 \\ \epsilon_0 & 0 & \epsilon_0 \\ \epsilon_0 & \epsilon_0 & 0 \end{bmatrix}$$

and then takes the maximum over sum along the rows. In our example

$$\kappa = 32.9823362305.$$

For the computation of $\tilde{C}$, let us first compute the norm $\|a_\alpha\|_{\mathbb{A}_r}^\infty$. For $\alpha = 1, \dots, N$, $a_\alpha(w) = \bar{a}_\alpha + \mathcal{E}a_\alpha$ with $\|\mathcal{E}a_\alpha\|_r^\infty \le \epsilon_\alpha$. Hence

$$\|a_\alpha\|_{\mathbb{A}_r}^\infty \le \sum_m |(\bar{a}_\alpha)_m| e^{\frac{2\pi r}{\hat{T}}|m|} + \epsilon_\alpha.$$

The matrices A_α have the form

$$A_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ -a_\alpha^{(3)} & 0 & -a_\alpha^{(1)} \\ a_\alpha^{(2)} & a_\alpha^{(1)} & 0 \end{bmatrix}.$$

Plugging in the r-norms above computed, and afterwards taking the maximum of the sums along the rows, we obtain

α	1	2	3	4	5
$\|A_\alpha\|_r^\infty$	3.450453	0.092396	$4.4772 \cdot 10^{-3}$	$1.4108 \cdot 10^{-4}$	$5.0794 \cdot 10^{-6}$
α	6	7	8	9	10
$\|A_\alpha\|_r^\infty$	$1.4752 \cdot 10^{-7}$	$4.8646 \cdot 10^{-9}$	$4.0205 \cdot 10^{-10}$	$2.7729 \cdot 10^{-10}$	$2.7367 \cdot 10^{-10}$

Inserting into formula (27), having $\lambda = \lambda_1 \in -13.861695 \pm r_v$, $\tilde{N} = N$ because the nonlinearity of the vector field is quadratic, it follows

$$\tilde{C} = 1.28706319.$$

Fix $M_1 = 2$, $M_2 = 1$, $\mu_* = 13.8616957$. The definition of ρ' and ρ is useless in this example, because the second derivatives of the vector field are constants and do not depend on ρ.

It remains to compute ϵ so that $\|E_N\|_{r,\nu}^\infty \leq \epsilon$. By definition, E_N is given by

$$E_N(w,z) = \sum_{|\alpha| \geq N+1} \boldsymbol{R}_\alpha(w) z^\alpha.$$

For the Lorenz equation

$$\boldsymbol{R}_\alpha(w) = \begin{bmatrix} 0 \\ -\sum\limits_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i>0}} a_{\alpha_1}^{(1)}(w) a_{\alpha_2}^{(3)}(w) \\ \sum\limits_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_i>0}} a_{\alpha_1}^{(1)}(w) a_{\alpha_2}^{(2)}(w) \end{bmatrix} \tag{77}$$

Since $a_\alpha(w) = 0$ for any $\alpha > N$, it holds $\boldsymbol{R}_\alpha(w) = 0$ for any $\alpha > 2N$. Thus

$$E_N(w,z) = \sum_{\alpha=N+1}^{2N} \boldsymbol{R}_\alpha(w) z^\alpha$$

and

$$\|E_N(w,z)\|_{r,\nu}^\infty \leq \max_{1\leq i\leq 3} \sum_{\alpha=N+1}^{2N} \|\boldsymbol{R}_\alpha^{(i)}\|_r^\infty \nu^\alpha$$

The functions $a_\alpha^{(i)}$ are known within bounds, that is $\|a_\alpha^{(i)}\|_r^\infty \leq \|\bar{a}_\alpha^{(i)}\|_r^\infty + \epsilon_\alpha$. The norm of the components of (77) can be bounded by

$$\|\boldsymbol{R}_\alpha\|_r^\infty = \begin{bmatrix} 0 \\ \sum\limits_{\alpha_1=\alpha-N}^{N} \Big((\|\bar{a}_{\alpha_1}^{(1)}\|_r^\infty + \epsilon_{\alpha_1})(\|\bar{a}_{\alpha-\alpha_1}^{(3)}\|_r^\infty + \epsilon_{\alpha-\alpha_1}) \Big) \\ \sum\limits_{\alpha_1=\alpha-N}^{N} \Big((\|\bar{a}_{\alpha_1}^{(1)}\|_r^\infty + \epsilon_{\alpha_1})(\|\bar{a}_{\alpha-\alpha_1}^{(2)}\|_r^\infty + \epsilon_{\alpha-\alpha_1}) \Big) \end{bmatrix} \tag{78}$$

For the example under investigation, it holds

$$\epsilon = 4.75196887496 \cdot 10^{-10}.$$

Then, check that $(N+1) > \kappa/\mu^*$ and define $\delta = 1.0236439709 \cdot 10^{-11}$ so that the relation (43) is satisfied. Since also the last check (44) is satisfied, we conclude that the finite order parameterisation $P_N(w,z)$ is rigorously validated and the N-tail is such that $\|H\|^{\infty}_{r,\nu} \leq \delta$.

Figure 2 (left) shows the image of the parameterisation $P_N(w,z)$, $|z| < \nu$ above discussed.

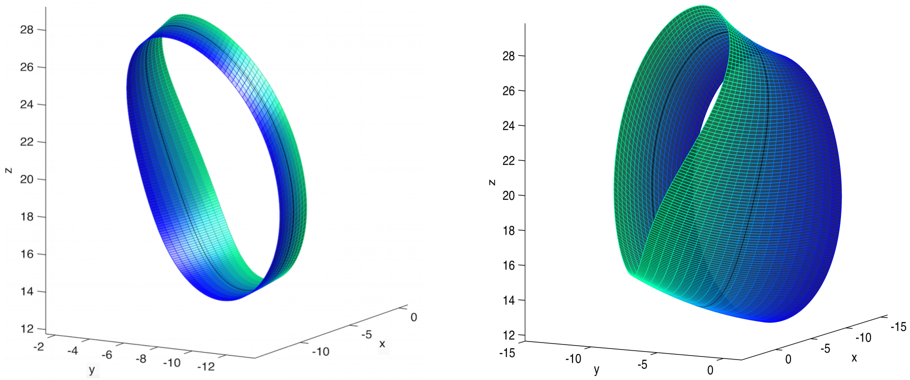

Figure 2: Two images of the parameterisation $P_N(w,z)$, $|z| \leq \nu$, $N = 10$, of the local stable manifold associated to the periodic orbit of the Lorenz system at parameter $\rho = 22$. In the left case $\nu = 1$ and the N-tail norm is $\|H\|^{\infty}_{r,\nu} \leq 1.024 \cdot 10^{-11}$. On the right $\nu = 4$ and $\|H\|^{\infty}_{r,\nu} \leq 9.77 \cdot 10^{-5}$.

In Table 1 we summerize the data and results for other examples. Each line reports the value of the parameter ρ of the Lorenz system, the period T of the orbit, the finite order/dimension parameters N and M, the Fourier/Taylor norm parameters r, ν and the resulting δ so that $\|H\|_{r,\nu} \leq \delta$. All the computations concern the stable manifold. Figure 2, on the right, shows the image of the parameterisation computed in the third example.

We also compute the parameterisation for the local unstable manifold for the periodic orbit with $\rho = 28$. The unstable Floquet exponents is $\bar{\lambda} = 0.99465$ and a preliminary analysis shows that a value for N larger than 50 is required. We choose $N = 70$. Thus, setting $\tilde{N} = 10$, $M = 60$, the computation of a_α, $\alpha \leq \tilde{N}$ is performed according to the general technique. Rather, the remaining coefficients a_α, $\tilde{N} < \alpha \leq N$ are enclosed layer-by-layer as discussed in section 4.1. In this case, we scale the eigenvector to be of length 2 and we set the Taylor decay rate $\nu = 1$. The N-order parameterisation is validated with N-tail norm $\|H\|_{r,\nu} \leq 9.0789 \cdot 10^{-9}$, being $r = 0.023644$. In Fig. 3, right, the image of the unstable local manifold is plot.

5 The bridge problem

Consider the vector field

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -120x_1 - x_1^3 - 154x_2 - 71x_3 - 14x_4 \end{pmatrix} \tag{79}$$

Table 1: default

ρ	T	N	M	r	ν	δ
17.32	1.1084183	6	40	0.08291	1	$2.6769{\cdot}10^{-5}$
22	0.7643864	10	40	0.03191	2	$2.8987{\cdot}10^{-8}$
22	0.7643864	10	40	0.01915	4	$9.7698{\cdot}10^{-5}$
20	0.8765522	10	40	0.03660	1	$1.2643{\cdot}10^{-10}$
28	1.5586522	10	40	0.02364	1	$4.6806{\cdot}10^{-5}$
28	1.5586522	10	60	0.02364	1.2	$1.4591{\cdot}10^{-6}$

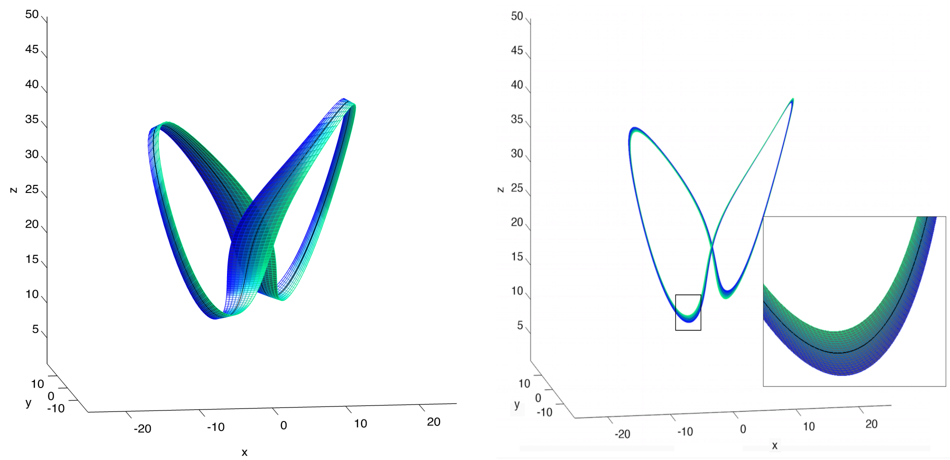

Figure 3: Image of the local stable (left) and unstable (right) manifold associated to a periodic orbit of the Lorenz system at parameter $\rho = 28$.

equivalent to the $4-th$ order differential equation

$$w'''' + 14w''' + 71w'' + 154w' + 120w + w^2 = 0$$

considered in [68].

In the work [68] it is rigorously computed a T-periodic orbit $\Gamma(t)$

$$\Gamma(t) = \begin{pmatrix} \gamma(t) \\ \Gamma_2(t) \\ \Gamma_3(t) \\ \Gamma_4(t) \end{pmatrix}$$

together with the Floquet exponents. The orbit Γ has two negative Floquet exponents

$$\lambda = (\lambda_1, \lambda_2), \quad \lambda_i < 0$$

hence the stable manifold is a 3-dimensional manifold in 4-dimensional space. We now apply the procedure explained before to compute the parameterisation of the stable manifold.

Note that $\gamma(t)$, the first component of $\Gamma(t)$, represents the periodic solution of the 4-th order ODE.

The validation of the coefficients of the high order parameterisation

$$P_N(w,z) = \sum_{|\alpha|=0}^{N} a_\alpha(w) z^\alpha, \quad \alpha \in \mathbb{N}^2, a_\alpha(w) \in \mathbb{R}^4$$

is split into three steps: for a choice of ν,

1. For $\tilde{N} < N$ compute the enclosure in ℓ^1_ν of each component of $a_\alpha(w)$ with $2 \le |\alpha| \le \tilde{N}$ around a numerical approximation $\bar{a}_\alpha(w)$.
2. Let N^* be such that $3\tilde{N} < N^* < N$. Compute layer-by-layer the ℓ^1_ν-norm of each component of $a_\alpha(w)$ for any α with $\tilde{N} < |\alpha| \le N^*$.
3. Compute an uniform bound for the ℓ^1_ν-norm of all the $a_\alpha(w)$ for $N^* < |\alpha| \le N$.

Details about the rigorous computation of the enclosure of the coefficients $a_\alpha(w)$ are provided in Section A.

5.1 Computational results

The rigorous computation performed in [68] provides the enclosure of the periodic solution $\gamma(t)$, see Fig. 4, and the enclosure of the Floquet exponents together with the normal bundles, that is the coefficients $a_{(1,0)}(w)$ and $a_{(0,1)}(w)$ of the parameterisation.

The period T and the sequence of Fourier coefficients of γ are proved to be in a ball of radius $r_\gamma = 7.5955 \cdot 10^{-13}$ around the numerical solution in the space $\ell^1_{\nu_\gamma}$ with $\nu_\gamma = 1.3$. That is

$$|T - \bar{T}| \le r_\gamma, \quad \|\gamma - \bar{\gamma}\|_{\nu_\gamma} \le r_\gamma,$$

where $\bar{T} = 1.908097232051104$.

The Floquet exponents are computed so that

$$|\lambda_1 - (-7)| \le 4.4942 \cdot 10^{-12}, \qquad |\lambda_2 - (-8)| \le 4.9807 \cdot 10^{-13}$$

and the normal bundles, rescaled so that $|a_{(1,0)}(0)| = 5$, $|a_{(0,1)}(0)| = 20$, are proved within bounds

$$\|a^{(j)}_{(1,0)} - \bar{a}^{(j)}_{(1,0)}\|_{\nu_{Fl}} \le 2.24709 \cdot 10^{-11} \quad \|a^{(j)}_{(0,1)} - \bar{a}^{(j)}_{(0,1)}\|_{\nu_{Fl}} \le 9.96140 \cdot 10^{-12}, \quad \forall j = 1, \dots, 4,$$

where $\nu_{Fl} = 1.001$. Set

$$\tilde{N} = 3, \qquad N^* = 15, \qquad N = 380, \quad \nu = 1.0005.$$

The Fourier coefficients of the parameterisation of the stable manifold are numerically computed up to order $|\alpha| = 3$ with finite dimensional parameter $M = 60$. For $|\alpha| > \tilde{N}$ the numerical approximation $\bar{a}_\alpha(w)$ is set to zero. The rigorous computation returns the enclosure of the sequence of Fourier coefficients $a^{(j)}_\alpha$ of the functions $a^{(j)}_\alpha(w)$ as $\|a^{(j)}_\alpha - \bar{a}^{(j)}_\alpha\|_\nu \le \epsilon_\alpha$, where

- for any $2 \le |\alpha| \le \tilde{N}$, $\epsilon_\alpha = 2.413502 \cdot 10^{-10}$
- for each $\tilde{N} < |\alpha| \le N^*$, $2.20396 \cdot 10^{-18} < \epsilon_\alpha < 2.5209 \cdot 10^{-8}$
- for any $N^* < |\alpha| \le N$, $\epsilon_\alpha = \bar{\epsilon} = 9.208077 \cdot 10^{-13}$.

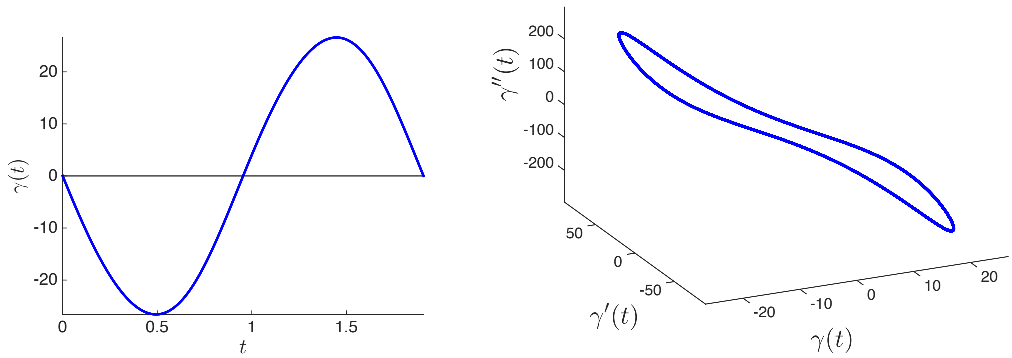

Figure 4: Left: image of the periodic solution $\gamma(t)$. Right: image of the first 3 components of $\Gamma(t)$, periodic solution of the vector field (79).

5.2 Validation values

According to the scheme explained in section 2.5, the validation values can be defined as follow. Fix the Taylor decay rate $\nu = 1$ and choose $r = 3.03607 \cdot 10^{-4}$.

A positive constant κ such that $\|Df(\Gamma)\|_r^\infty \leq \kappa$ is computed as

$$\kappa = 2.520751 \cdot 10^3.$$

The bound $\tilde{C}$ is given as

$$\tilde{C} = 4.834293.$$

The a-posteriori error bound is estimated by (see next section)

$$\|E\|_{r,\nu}^\infty \leq 1.412547 \cdot 10^{-6}.$$

Compute

$$\rho' = 72.23678$$

and choose

$$\rho = 79.46046.$$

Set

$$\mu_\star = 7, \quad M_1 = 1$$

and compute

$$M_2 = 6(\sup|\gamma(t)| + \rho) = 639.2028.$$

The validation of the parameterisation is successful with

$$\|H\|_{r,\nu} \leq 9.338422 \cdot 10^{-8}.$$

Figure 5 represents the projection on the first 3 components of two sub manifolds of the stable manifold. More precisely, writing the parameterization in the form

$$P(w,z) = \sum_{(\alpha^1,\alpha^2)} a_{(\alpha^1,\alpha^2)}(w) z_1^{\alpha^1} z_2^{\alpha^2},$$

the red-to-yellow and the green-to-blue plot of Fig. 5 are the image of $P(w, z_1, z_2 = 0)$ and $P(w, z_1 = 0, z_2)$ respectively.

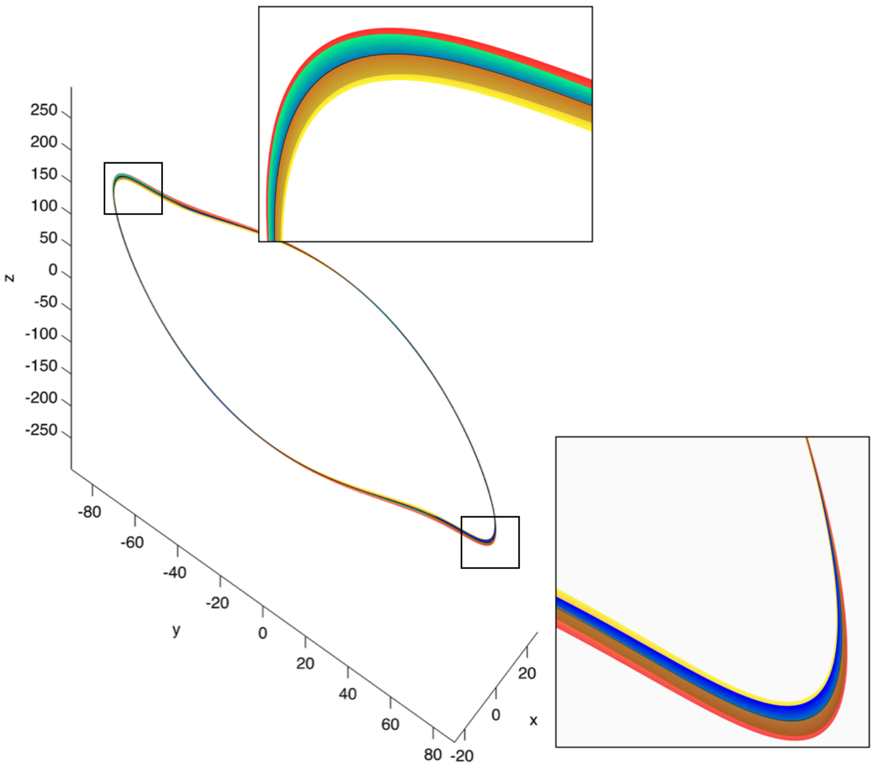

Figure 5: Image of the parameterisation of two sub manifolds of the local stable manifold associated to the periodic orbit $\Gamma(t)$. Only the first three components of $P_N(w, z)$ are drawn. The red-to-yellow surface is the image of the sub manifold tangent to the eigenspace $a_{(1,0)}(w)$, associated to the Floquet exponent $\lambda_1 = -7$. The green-to-blue surface is the sub manifold tangent to the eigenspace $a_{(0,1)}(w)$, associated to $\lambda_2 = -8$.

5.3 A-posteriori error bound

For a choice of ν, the a-posteriori error bound is given by

$$\|E_N\|^{\infty}_{r,\nu} = \sum_{|\alpha|=N+1}^{3N} \left\| \sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha \\ |\alpha^i|\leq N}} a^{(1)}_{\alpha^1}(w) a^{(1)}_{\alpha^2}(w) a^{(1)}_{\alpha^3}(w) \right\|_r \nu^{|\alpha|}$$

We know bounds for the r-norm of $a^{(1)}_\alpha$ for any α up to $|\alpha| = N$. In particular $\|a^{(1)}_\alpha\|_r = \bar{\epsilon}$ for any $|\alpha| > N^*$. It is not advised to run over all possible α's of the above sum and compute precisely all the terms in the inner sum. Rather, by means of some combinatorial calculations, we can provide an upper bound for the error.

Denote by

$$R_\alpha = \left\| \sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha \\ |\alpha^i|\leq N}} a^{(1)}_{\alpha^1}(w) a^{(1)}_{\alpha^2}(w) a^{(1)}_{\alpha^3}(w) \right\|_r .$$

First compute the quantities

$$a^2_\alpha \geq \left\| \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ |\alpha^i|\leq N}} a^{(1)}_{\alpha^1}(w) a^{(1)}_{\alpha^2}(w) \right\|_r, \qquad \forall\ 0 \leq |\alpha| \leq 2N$$

and consider the splitting

$$\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha \\ |\alpha^i|\leq N}} a^{(1)}_{\alpha^1}(w) a^{(1)}_{\alpha^2}(w) a^{(1)}_{\alpha^3}(w) = \sum_{|\alpha^1|\leq N} a^{(1)}_{\alpha^1} \sum_{\substack{\alpha^2+\alpha^3=\alpha-\alpha^1 \\ |\alpha^i|\leq N}} a^{(1)}_{\alpha^2}(w) a^{(1)}_{\alpha^3}(w).$$

For any α such that $N+1 \leq |\alpha| \leq 2N + N^*$ we have

$$\begin{aligned} R_\alpha &\leq \sum_{0<|\alpha^1|\leq N^*} \|a^{(1)}_{\alpha^1}\|_r \max_{|\alpha^2|\geq N+1-N^*} a^2_{\alpha^2} + \sum_{N^*<|\alpha^1|\leq N} \bar{\epsilon} \max_{|\alpha^2|\geq N} a^2_{\alpha^2} \\ &\leq \sum_{0<|\alpha^1|\leq N^*} \|a^{(1)}_{\alpha^1}\|_r \max_{|\alpha^2|\geq N+1-N^*} a^2_{\alpha^2} + \bar{\epsilon}\Big(\frac{(N+1)(N+2)}{2} - \frac{(N^*+1)(N^*+2)}{2}\Big) \max_{|\alpha^2|\geq N} a^2_{\alpha^2}. \end{aligned}$$

For all the remaining $2N + N^* < |\alpha| \leq 3N$ we bound

$$R_\alpha \leq \bar{\epsilon}\Big(\frac{(N+1)(N+2)}{2} - \frac{(N^*+1)(N^*+2)}{2}\Big) \max_{|\alpha^2|\geq N+N^*} a^2_{\alpha^2}.$$

A Rigorous enclosure of the coefficients for the bridge problem

The multi-indeces $\alpha \in \mathbb{N}^2$ and are denote by $\alpha = (\alpha_1, \alpha_2)$, whereas superscripts (α^i) label different α's.

Following the scheme proposed in section 3, the rigorous enclosure of the coefficients $a_\alpha(w)$ of the parametersation with $2 \leq |\alpha| \leq \tilde{N}$ are computed.

The system (46) is given by

$$\begin{pmatrix} F^{(1)}_{\alpha,m} \\ F^{(2)}_{\alpha,m} \\ F^{(3)}_{\alpha,m} \\ F^{(4)}_{\alpha,m} \end{pmatrix} = \begin{pmatrix} \left(\dfrac{2\pi\mathbf{i}m}{2T} + \alpha\cdot\lambda\right) a^{(1)}_{\alpha,m} - a^{(2)}_{\alpha,m} \\ \left(\dfrac{2\pi\mathbf{i}m}{2T} + \alpha\cdot\lambda\right) a^{(2)}_{\alpha,m} - a^{(3)}_{\alpha,m} \\ \left(\dfrac{2\pi\mathbf{i}m}{2T} + \alpha\cdot\lambda\right) a^{(3)}_{\alpha,m} - a^{(4)}_{\alpha,m} \\ \left(\dfrac{2\pi\mathbf{i}m}{2T} + \alpha\cdot\lambda\right) a^{(4)}_{\alpha,m} + 120 a^{(1)}_{\alpha,m} + \displaystyle\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha \\ \alpha^i\geq 0}} \left(a^{(1)}_{\alpha^1} * a^{(1)}_{\alpha^2} * a^{(1)}_{\alpha^3}\right)_m \\ +154 a^{(2)}_{\alpha,m} + 71 a^{(3)}_{\alpha,m} + 14 a^{(4)}_{\alpha,m} \end{pmatrix}, \tag{80}$$

with $2 \leq |\alpha| \leq \tilde{N}$ and $m \in \mathbb{Z}$.

The derivative of F (with respect to $a_{\alpha,m}$, $|\alpha| \geq 2$) at the point a acts on an element v

as

$$\left(\left(DF(a)(v)\right)_\alpha\right)_m = \left(\frac{2\pi\mathbf{i}m}{2T}+\alpha\cdot\lambda\right)v_{\alpha,m} + \begin{pmatrix} -v^{(2)}_{\alpha,m} \\ -v^{(3)}_{\alpha,m} \\ -v^{(4)}_{\alpha,m} \\ +120v^{(1)}_{\alpha,m}+154v^{(2)}_{\alpha,m}+71v^{(3)}_{\alpha,m}+14v^{(4)}_{\alpha,m} \end{pmatrix} \tag{81}$$

$$+\begin{pmatrix} 0 \\ 0 \\ 0 \\ 3\displaystyle\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha \\ \alpha^i\geq 0, |\alpha^3|\geq 2}} \left(a^{(1)}_{\alpha^1}*a^{(1)}_{\alpha^2}*v^{(1)}_{\alpha^3}\right)_m \end{pmatrix}.$$

Denoting by

$$a^2_\alpha = \sum_{\substack{\alpha^1+\alpha^2=\alpha \\ |\alpha^i|\geq 0}} a^{(1)}_{\alpha^1}*a^{(1)}_{\alpha^2}$$

the last term of the above derivative is rewritten as $3\displaystyle\sum_{\substack{\alpha^1+\alpha^2=\alpha \\ |\alpha^1|\geq 0, |\alpha^2|\geq 2}} \left(a^2_{\alpha^1}*v^{(1)}_{\alpha^2}\right)_m$.

We now choose the operator A and $A^\dagger$. We remind that $A^\dagger$ approximates the derivative $DF(\bar{a})$ (where $\bar{a}$ is an approximate solution of $F(a)=0$) while A approximates the inverse of $DF(\bar{a})$. We have some freedom in defining these operators but it is advisable that the composition $AA^\dagger : X\to X$ acts as the identity out of a certain finite dimensional subspace of X. Arguing as in (67), define a finite dimensional part of $A^\dagger$ as the exact derivative of $DF^{(M)}$, while the action of the derivative on the infinite dimensional complement is only approximated. In (67) we chose to approximate the derivative with the diagonal action of $\left(\frac{2\pi\mathbf{i}m}{2T}+\alpha\cdot\Lambda\right)$ because this term is growing with m, hence, it is asymptotically dominant. However, besides those terms that are growing, it is advisable to consider in $A^\dagger$ also those terms that are *big*. In our case the vector field has a cubic nonlinearity and the first few Fourier coefficients γ_m of the $2T$-periodic orbit $\gamma(t)$ are

$$\gamma_0=0,\ \gamma_{\pm 2}\approx 0.178\pm 13.34\mathbf{i},\ \gamma_{\pm 6}=-0.17\pm 0.064\mathbf{i},\ \gamma_{\pm 1,\pm 3,\pm 4\pm 5}\equiv 0.$$

It follows that the first few Fourier coefficients of $a^2_{\mathbf{0}}$ are the following ($\mathbf{0}=(0,0)$):

$$(a^2_{\mathbf{0}})_0\approx 356.31,\quad (a^2_{\mathbf{0}})_{\pm 4}\approx -176.40\pm 9.52\mathbf{i}\quad (a^2_{\mathbf{0}})_{\pm 1,2,3}\approx 0.$$

Therefore, the cubic term alone produces contributions in the derivative as big as multiplication by 1100.

We decide to include these contributions (3 times the multiplication by $(a^2_{\mathbf{0}})_0$, $(a^2_{\mathbf{0}})_{\pm 4}$), together with the linear term $120v^{(1)}_{\alpha,m}, 154v^{(2)}_{\alpha,m}, 71v^{(3)}_{\alpha,m}$ in the definition of $A^\dagger$. In practice, define $A^\dagger$ so that

$$\left((A^\dagger_{i_1,i_2})_{j_1,j_2}d\right)_m = \begin{cases} \left((DF^{(M)}_{i_1,i_2})_{j_1,j_2}d_F\right)_m, & |m|<M, \\ \delta_{i_1,i_2}\delta_{j_1,j_2}\left(\dfrac{2\pi\mathbf{i}m}{2T}+\alpha(i_1)\cdot\Lambda\right)d_m, & |m|\geq M, \end{cases} \tag{82}$$

and for any i ranging on the set of possible α's, we augment the action of the operators $(A^\dagger_{i,i})_{4,j}$, $j=1,2,3$ with the multiplication by the infinite dimensional tri-diagonal matrix or the diagonal matrix as depicted in Figure 6 where

$$d_0 = 120+3(a^2_{\mathbf{0}})_0,\quad d_4=3(a^2_{\mathbf{0}})_4,\quad d_{-4}=3(a^2_{\mathbf{0}})_{-4},$$

$$f_0 = 154, \quad g_0 = 71.$$

Similarly, we define the operator A as

$$((A_{i_1,i_2})_{j_1,j_2} c)_m = \begin{cases} \left((A^{(M)}_{i_1,i_2})_{j_1,j_2} c_F\right)_m, & |m| < M, \\ \delta_{i_1,i_2}\delta_{j_1,j_2} \left(\dfrac{1}{\frac{2\pi \mathbf{i} m}{2T} + \alpha(i_1)\cdot\Lambda}\right) c_m, & |m| \geq M, \end{cases} \tag{83}$$

and, at first, we augment the action of $(A_{i,i})_{4,j}$, $j = 1, 2, 3$, for any i ranging on the set of possible α's, with the multiplication by the infinite dimensional tri-diagonal matrix as depicted in Figure 7 where

$$e_0^{(m)} = -\frac{d_0}{\mu_m^2}, \quad e_{-4}^{(m)} = -\frac{d_{-4}}{\mu_{m+4}\mu_m}, \quad e_4^{(m)} = -\frac{d_4}{\mu_{m-4}\mu_m},$$

$$h_0^{(m)} = -\frac{f_0}{\mu_m^2}, \quad \ell_0^{(m)} = -\frac{g_0}{\mu_m^2}$$

and

$$\mu_m = \frac{2\mathbf{i}\pi}{2T} m + \alpha(i)\cdot\Lambda.$$

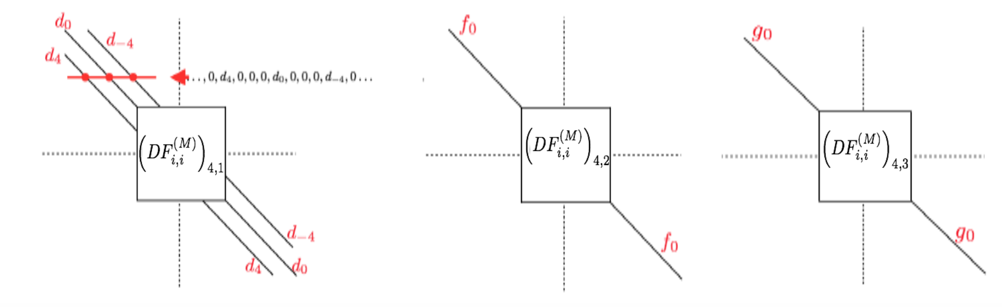

Figure 6: Structure of the components of the operator $A^\dagger_{i,i}$.

Let us now compute the action of $A^\dagger$ on $v \in X$. The finite dimensional part results

$$\left((A^\dagger(v))_\alpha\right)_F = \begin{bmatrix} (DF^{(M)}v)_F^{(1)} \\ (DF^{(M)}v)_F^{(2)} \\ (DF^{(M)}v)_F^{(3)} \\ (DF^{(M)}v)_F^{(4)} + \epsilon \end{bmatrix}.$$

where ϵ denotes the multiplication of d_4 and d_{-4} times $v^{(1)}_{\alpha,m}$ where $m = -M-3, \ldots, -M$ and $m = M, \ldots, M+3$ respectively.

Instead, for $|m| \geq M$

$$\left((A^\dagger(v))_\alpha\right)_m = \begin{bmatrix} \mu_m v^{(1)}_{\alpha,m} \\ \mu_m v^{(2)}_{\alpha,m} \\ \mu_m v^{(3)}_{\alpha,m} \\ d_4 v^{(1)}_{\alpha,m-4} + d_0 v^{(1)}_{\alpha,m} + d_{-4} v^{(1)}_{\alpha,m+4} + f_0 v^{(2)}_{\alpha,m} + g_0 v^{(3)}_{\alpha,m} + \mu_m v^{(4)}_{\alpha,m} \end{bmatrix}.$$

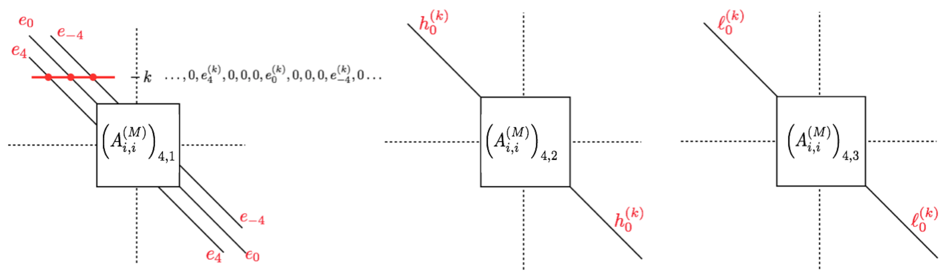

Figure 7: Structure of the components of the operator $A_{i,i}$. The infinite dimensional diagonal and ti-diagonal terms are defined so that the composition $AA^\dagger$ acts as the identity out of $\Pi^{(M+4)}X$.

Then, let us apply the operator A to $A^\dagger(v)$. For any α and $|m| \geq M+4$, it holds

$$[A(A^\dagger(v))]_{\alpha,m} = \begin{bmatrix} \frac{1}{\mu_m}(\mu_m v^{(1)}_{\alpha,m}) \\ \frac{1}{\mu_m}(\mu_m v^{(2)}_{\alpha,m}) \\ \frac{1}{\mu_m}(\mu_m v^{(3)}_{\alpha,m}) \\ e_4^{(m)}(\mu_{m-4}v^{(1)}_{\alpha,m-4}) + e_0^{(m)}(\mu_m v^{(1)}_{\alpha,m}) + e_{-4}^{(m)}(\mu_{m+4}v^{(1)}_{\alpha,m+4}) + h_0^{(m)}\mu_m v^{(2)}_{\alpha,m} + \ell_0^{(m)}\mu_m v^{(3)}_{\alpha,m} \\ \frac{1}{\mu_m}\left(d_4 v^{(1)}_{\alpha,m-4} + d_0 v^{(1)}_{\alpha,m} + d_{-4}v^1_{\alpha,m+4} + f_0 v^{(2)}_{\alpha,m} + g_0 v^{(3)}_{\alpha,m} + \mu_m v^{(4)}_{\alpha,m}\right) \end{bmatrix}.$$

By definition of $e_0, e_4, e_{-4}, h_0, \ell_0$, we have $[A(A^\dagger(v))]_{\alpha,m} = v_{\alpha,m}$, that is $AA^\dagger$ acts as the identity on the infinite dimensional subspace $(I - \Pi^{(M+4)})X$. On the contrary, we can not guarantee that $AA^\dagger$ is close to the identity in the finite dimensional space $\Pi^{(M+4)}(X)$, because of the out of diagonal terms. Thus, we compute a numerical inverse of the restriction of $A^\dagger$ on $\Pi^{(M+4)}(X)$, and we append the result in the construction of A. In practice, the matrices $(A^{(M)}_{i_1,i_2})_{j_1,j_2}$ are replaced by slightly larger matrices $(A^{(M+4)}_{i_1,i_2})_{j_1,j_2}$. In conclusion, the structure of the operator A is the same as the one depicted in Fig. 7 with $M+4$ in place of M.

Bound $Z^{(1)}$

As in (73), let $\tilde{\mathcal{A}}^2_\alpha$ be the matrix with components $(\tilde{\mathcal{A}}^2_\alpha)(m,n) = (\bar{a}^2_\alpha)_{m-n}$. The matrices $\overline{\Gamma}(s,t)$ used in the computation of the $Z^{(1)}$ bound, are of the following form

$$\begin{array}{llll} s_j = 1 & s_\alpha = t_\alpha & t_j = 2 & \overline{\Gamma}(s,t) = -\tilde{I} \\ s_j = 2 & s_\alpha = t_\alpha & t_j = 3 & \overline{\Gamma}(s,t) = -\tilde{I} \\ s_j = 3 & s_\alpha = t_\alpha & t_j = 4 & \overline{\Gamma}(s,t) = -\tilde{I} \\ s_j = 4 & s_\alpha = t_\alpha & t_j = 1 & \overline{\Gamma}(s,t) = 3(\tilde{\mathcal{A}}^2_{\mathbf{0}})^* \\ & & t_j = 4 & \overline{\Gamma}(s,t) = 14\tilde{I} \\ & s_\alpha > t_\alpha & t_j = 1 & \overline{\Gamma}(s,t) = 3\tilde{\mathcal{A}}^2_{s_\alpha - t_\alpha} \end{array}$$

where $(\tilde{\mathcal{A}}^2_{\mathbf{0}})^*$ is the same as $\tilde{\mathcal{A}}^2_{\mathbf{0}}$ after replacing $(\bar{a}^2_{\mathbf{0}})_0 = (\bar{a}^2_{\mathbf{0}})_{\pm 4} = 0$. That is one of the consequences of considering the tridiagonal action in $(A^\dagger_{i,i})_{4,1}$. The other main consequence is that there are no the linear terms $120\tilde{I}$, $154\tilde{I}$, $71\tilde{I}$.

Bound $Z^{(2)}$ and $Z^{(3)}$

Because of the cubic nonlinearity, besides the $Z^{(2)}$ bound we also have the $Z^{(3)}$ bound. Indeed

$$\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\|\alpha^1|\geq 0,|\alpha^2|\geq 0,|\alpha^3|\geq 2}} (\bar{a}^{(1)}_{\alpha^1}+ru_{\alpha^1})*(\bar{a}^{(1)}_{\alpha^2}+ru_{\alpha^2})*(rv^{(1)}_{\alpha^3}) =$$

$$r\sum_{\substack{\alpha^1+\alpha^2=\alpha\\|\alpha^1|\geq 0,|\alpha^2|\geq 2}} (\bar{a}^2)_{\alpha^1}*v^{(1)}_{\alpha^2}+r^2\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\|\alpha^1|,|\alpha^2|\geq 0\\|\alpha^3|\geq 2}}(\bar{a}^{(1)}_{\alpha^1}*u_{\alpha^2}+\bar{a}^{(1)}_{\alpha^2}*u_{\alpha^1})*v_{\alpha^3}+r^3\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\|\alpha^i|\geq 2}} u_{\alpha^1}*u_{\alpha^2}*v_{\alpha^3}$$

Hence $(Z^{(2)})^{(j)}_\alpha = (Z^{(3)})^{(j)}_\alpha = 0$, for $j=1,2,3$, and

$$(Z^{(2)})^{(4)}_\alpha = 2\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\|\alpha^1|\geq 0,|\alpha^2|,|\alpha^3|\geq 2}} \|\bar{a}^{(1)}_{\alpha^1}\|_{1,\nu}, \quad (Z^{(3)})^{(4)}_\alpha = \sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\|\alpha^i|\geq 2}} 1$$

A.1 Extra coefficients, $\tilde{N} < |\alpha| \leq N$

Once the enclosure of the function $a_\alpha(w)$ for $|\alpha| < \tilde{N}$ is computed, following the approach of section 3.3, the coefficients $a_\alpha(w)$ for $\tilde{N} < |\alpha| \leq N$ can be addressed layer by layer. In the case under analysis, the value of N required by the proof is pretty big. As already stated, we do not compute all the coefficients one-by-one for any $|\alpha|$ up to N, rather for $|\alpha|$ big enough a uniform bound is employed. More precisely, for a choice of $N^\star$, $3\tilde{N} < N^\star < N$, the functions a_α are one-by-one enclosed for any $\tilde{N} < |\alpha| \leq N^\star$. Then uniform bounds provide the enclosure for all the remaining a_α.

The case $|\alpha| \leq N^\star$.

In the unknown a_α, the function F_α is the same as in (80). The nonlinearity is decomposed into

$$\sum_{\alpha^1+\alpha^2+\alpha^3=\alpha} a^{(1)}_{\alpha^1}*a^{(1)}_{\alpha^2}*a^{(1)}_{\alpha^3} = \sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\|\alpha^i|<|\alpha|}} a^{(1)}_{\alpha^1}*a^{(1)}_{\alpha^2}*a^{(1)}_{\alpha^3} + 3a^{(1)}_\alpha*(a^2_{\mathbf{0}}).$$

Since $\bar{a} = \bar{a}_\alpha = 0$, it follows that

$$(F(\bar{a}))_\alpha = \left[0,0,0,\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\|\alpha^i|<|\alpha|}}\left(a^{(1)}_{\alpha^1}*a^{(1)}_{\alpha^2}*a^{(1)}_{\alpha^3}\right)\right]^T$$

Definition of $A_{\alpha,\alpha}$

Let us first write explicitly $A^\dagger_{\alpha,\alpha}$

$$\left((A^\dagger_{\alpha,\alpha})_{j_1,j_2}d\right)_m = \begin{cases} \left((DF^{(M)}_{\alpha,\alpha})_{j_1,j_2}d_F\right)_m, & |m|<M,\\ \delta_{j_1,j_2}\left(\dfrac{2\pi\mathbf{i}m}{2T}+\alpha\cdot\Lambda\right)d_m, & |m|\geq M,\end{cases} \tag{84}$$

and, as done before, we augment the operators $(A^\dagger_{\alpha,\alpha})_{4,j_2}$, $j_2=1,2,3$ with the tridiagonal and diagonal operators depicted in figure 6. Here $DF^{(M)}_{\alpha,\alpha}$ is the derivative of $F^{(M)}_\alpha$ with respect to $a^{(M)}_\alpha$ and it is given by

$$DF^{(M)}_{\alpha,\alpha} = \begin{bmatrix} \mu^{(M)}_\alpha & -I^{(M)} & 0 & 0\\ 0 & \mu^{(M)}_\alpha & -I^{(M)} & 0\\ 0 & 0 & \mu^{(M)}_\alpha & -I^{(M)}\\ 120I^{(m)}+3(\mathcal{A}^2_{\mathbf{0}})^{(M)} & 154I^{(M)} & 71I^{(M)} & \mu^{(M)}_\alpha+14I^{(M)}\end{bmatrix}$$

where $\mu_\alpha^{(M)}$ is the $(2M-1)\times(2M-1)$ diagonal matrix with $\frac{2\pi\mathbf{i}m}{2T}+\alpha\cdot\Lambda$ on the diagonal, $|m|<M$, $I^{(M)}$ is the $(2M-1)\times(2M-1)$ identity matrix and $(\mathcal{A}^2_{\mathbf{0}})^{(M)}$ is the $(2M-1)\times(2M-1)$ matrix representing the action of the convolution $a^2_{\mathbf{0}}*x$ on $X^{(M)}$.

The operator $A_{\alpha,\alpha}$ is of the same shape as done previously

$$((A_{\alpha,\alpha})_{j_1,j_2}c)_m = \begin{cases} \left((A^{(M+4)}_{\alpha,\alpha})_{j_1,j_2}c_F\right)_m, & |m|<M+4, \\ \delta_{j_1,j_2}\left(\dfrac{1}{\frac{2\pi\mathbf{i}m}{2T}+\alpha\cdot\Lambda}\right)c_m, & |m|\geq M+4, \end{cases} \tag{85}$$

with the tridiagonal and diagonal elements of Figure 7 appended to $(A_{\alpha,\alpha})_{4,j2}$, $j_2=1,2,3$. Again, as before, we moved M to $M+4$ to ensure that $AA^\dagger=I$ on $I-\Pi^{(M+4)}X$ and $A^{(M+4)}_{\alpha,\alpha}$ is a numerical inverse of $(A^\dagger_{\alpha,\alpha})^{(M+4)}$.

Construction of the bounds Y and $Z(r)$

$$Y_\alpha \geq |||A_{\alpha,\alpha}||| \left[0,0,0,\left\|\sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha\\ |\alpha^i|<|\alpha|}}\left(a^{(1)}_{\alpha^1}*a^{(1)}_{\alpha^2}*a^{(1)}_{\alpha^3}\right)\right\|_\nu\right]^T.$$

For any α, let $R_\alpha = I-A^{(M+4)}_{\alpha,\alpha}(A^\dagger_{\alpha,\alpha})^{(M+4)}$ the residual that occurs when multiplying $A^\dagger$ with the approximative inverse A and define

$$(Z^{(0)}_\alpha)^{(j)} \geq \sum_{j_1}|||(R_\alpha)_{j,j_1}|||.$$

Explicitly, for any $j=1,\ldots,4$ the $Z^{(1)}$ bound is as follows:

$$\begin{aligned} Z^{(1)}_{\alpha,j} = \quad & |||(A_{\alpha,\alpha})_{j,1}\tilde{I}|||+|||(A_{\alpha,\alpha})_{j,2}\tilde{I}|||+|||(A_{\alpha,\alpha})_{j,3}\tilde{I}|||+|||(A_{\alpha,\alpha})_{j,4}14\tilde{I}|||+ \\ & 3|||(A_{\alpha,\alpha})_{j,4}(\tilde{\mathcal{A}}^2_{\mathbf{0}})^*|||+3|||(A_{\alpha,\alpha})_{j,4}|||\epsilon_{\mathbf{0},\mathbf{2}} \end{aligned} \tag{86}$$

where $\|(\bar{a}^2)_0-(a^2)_0\|_\nu \leq \epsilon_{\mathbf{0},\mathbf{2}}$.

Uniform bound for $N^*<|\alpha|\leq N$

It remains to compute rigorous enclosure for a_α for any $N^*<|\alpha|\leq N$. The idea is to solve the system $\{F_\alpha=0\}_{N^*<|\alpha|\leq N}$ in the unknowns $\{a_\alpha\}_{N^*<|\alpha|\leq N}$. The operators $A^\dagger$ and A are constructed as diagonal operator in α so that any polynomial p_α depends on the operator $A_{\alpha,\alpha}$. Also, the numerical approximate solution is taken to be zero for any α. In order to define an unique polynomial that provides the enclosure for any α, uniform bound on Y_α, Z_α are sought, together with uniform bound of $|||A_{\alpha,\alpha}|||$. Let us briefly discuss how to define uniform bound for $|||A_{\alpha,\alpha}|||$. The crucial point is to bound $|||A^{M+4}_{\alpha,\alpha}|||$, where $A^{(M+4)}_{\alpha,\alpha}$ is an approximate inverse of $(A^\dagger_{\alpha,\alpha})^{(M+4)}$.For the system under analysis, we have

$$(A^\dagger_{\alpha,\alpha})^{(M+4)} = \begin{bmatrix} \mu_\alpha^{(M+4)} & -I^{(M)} & 0 & 0 \\ 0 & \mu_\alpha^{(M+4)} & -I^{(M)} & 0 \\ 0 & 0 & \mu_\alpha^{(M+4)} & -I^{(M)} \\ C_{4,1} & C_{4,2} & C_{4,3} & \mu_\alpha^{(M+4)}+14I^{(M)} \end{bmatrix}$$

where $C_{4,1}$ is the $2(M+4)-1\times 2(M+4)-1$ matrix obtained by enlarging $120I^{(M)}+3(\mathcal{A}^2_{\mathbf{0}})^{(M)}$ with the terms on the 3 diagonals d_0,d_4,d_{-4}, $C_{4,2}=154I^{(M+4)}$, $C_{4,3}=71I^{(M+4)}$.

Write $(A^\dagger_{\alpha,\alpha})^{(M+4)} = P + B_\alpha$ where

$$P = \begin{bmatrix} 0 & -I^{(M)} & 0 & 0 \\ 0 & 0 & -I^{(M)} & 0 \\ 0 & 0 & 0 & -I^{(M)} \\ 0 & 0 & 0 & 14I^{(M)} \end{bmatrix}, \quad B_\alpha = \begin{bmatrix} \mu_\alpha^{(M+4)} & 0 & 0 & 0 \\ 0 & \mu_\alpha^{(M+4)} & 0 & 0 \\ 0 & 0 & \mu_\alpha^{(M+4)} & 0 \\ C_{4,1} & C_{4,2} & C_{4,3} & \mu_\alpha^{(M+4)} \end{bmatrix}$$

Now, we define $A^{(M+4)}_{\alpha,\alpha}$ as

$$A^{(M+4)}_{\alpha,\alpha} = \tilde{B}_\alpha - \tilde{B}_\alpha P \tilde{B}_\alpha.$$

where

$$\tilde{B}_\alpha = \begin{bmatrix} \mu_\alpha^{-1} & 0 & 0 & 0 \\ 0 & \mu_\alpha^{-1} & 0 & 0 \\ 0 & 0 & \mu_\alpha^{-1} & 0 \\ -\mu_\alpha^{-1}C_{4,1}\mu_\alpha^{-1} & -\mu_\alpha^{-1}C_{4,2}\mu_\alpha^{-1} & -\mu_\alpha^{-1}C_{4,3}\mu_\alpha^{-1} & \mu_\alpha^{-1} \end{bmatrix}, \quad \mu_\alpha^{-1} = (\mu_\alpha^{(M+4)})^{-1}.$$

It follows $B_\alpha \tilde{B}_\alpha = I$ and

$$I - A^{(M+4)}_{\alpha,\alpha}(A^\dagger_{\alpha,\alpha})^{(M+4)} = \tilde{B}_\alpha P \tilde{B}_\alpha P.$$

For the definition of Z^0, an uniform bound of the latest product is needed for any $|\alpha| > N^*$. Since $\tilde{B}_\alpha$ is component wise decreasing in $|\alpha|$, that is $|\tilde{B}_{\alpha'}| < \max|\tilde{B}_\alpha|$ if $|\alpha'| = |\alpha| + 1$, a bound is obtained by computing the expression for all the α's with $|\alpha| = N^*$.

Similarly, a bound for $|||A^{(M+4)}_{\alpha,\alpha}|||$ is computed, as explained in the next remark.

Remark A.1. *Direct computation provides*

$$A^{(M+4)}_{\alpha,\alpha} = \tilde{B}_\alpha - \tilde{B}_\alpha P \tilde{B}_\alpha =$$

$$\begin{bmatrix} \mu_\alpha^{-1} & \mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1} & 0 & 0 \\ 0 & \mu_\alpha^{-1} & \mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1} & 0 \\ -\mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1}C_{4,1}\mu_\alpha^{-1} & -\mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1}C_{4,2}\mu_\alpha^{-1} & \mu_\alpha^{-1} - \mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1}C_{4,3}\mu_\alpha^{-1} & \mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1} \\ X_1 & X_2 & X_3 & X_4 \end{bmatrix}$$

where

$$\begin{aligned} X_1 &= -(I - \mu_\alpha^{-1}C_{4,3}\mu_\alpha^{-1}I^{(M)} - 14\mu_\alpha^{-1}I^{(M)})\mu_\alpha^{-1}C_{4,1}\mu_\alpha^{-1} \\ X_2 &= -\mu_\alpha^{-1}C_{4,1}\mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1} - (I - \mu_\alpha^{-1}C_{4,3}\mu_\alpha^{-1}I^{(M)} - 14\mu_\alpha^{-1}I^{(M)})\mu_\alpha^{-1}C_{4,2}\mu_\alpha^{-1} \\ X_3 &= -\mu_\alpha^{-1}C_{4,2}\mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1} - (I - \mu_\alpha^{-1}C_{4,3}\mu_\alpha^{-1}I^{(M)} - 14\mu_\alpha^{-1}I^{(M)})\mu_\alpha^{-1}C_{4,3}\mu_\alpha^{-1} \\ X_4 &= -\mu_\alpha^{-1}C_{4,3}\mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1} + \mu_\alpha^{-1} - 14\mu_\alpha^{-1}I^{(M)}\mu_\alpha^{-1} \end{aligned}$$

For any α the operator norm of the operator μ_α^{-1} is given by

$$|||\mu_\alpha^{-1}||| = \max_m \left|\frac{1}{2\pi i m/2T + \alpha \cdot \Lambda}\right| = \left|\frac{1}{\alpha \cdot \Lambda}\right| = \frac{1}{\alpha \cdot |\Lambda|}$$

Clearly, if $|\alpha'| = N + 1$, $\alpha' \cdot |\Lambda| > \min_{|\alpha|=N} \alpha \cdot |\Lambda|$ therefore

$$|||\mu_{\alpha'}^{-1}||| < \max_{|\alpha|=N} |||\mu_\alpha^{-1}|||.$$

According to this remark, the knowledge of $|||\mu_\alpha^{-1}|||$ with $|\alpha| = N^\star$ allows to uniformly bound $|||\mu_{\alpha'}^{-1}|||$ for any $|\alpha'| > N^\star$. From where it follows uniform bound for the operator norm of each of the entries of $A^{(M+4)}_{\alpha,\alpha}$ that is valid for any α with $|\alpha| > N^\star$.

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