Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Written Assignment 7, due in class, March 21, 2003

In these problems, we assume that $T : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ is a linear operator, and we let T^k be the composite of T with itself k times. (So $T^0 = I$, $T^1 = T$ and $T^2\vec{v} = T(T\vec{v})$ for any $\vec{v} \in \mathcal{R}^n$, etc.) For any polynomial $p(x) = \sum_{k=0}^{m} a_k x^k$, we define $p(T) : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ to be the operator such that $p(T)\vec{v} = \sum_{k=0}^{m} a_k(T^k\vec{v})$ for $\vec{v} \in \mathcal{R}^n$.

1. Show that $p(T)$ is linear. Show that if $p(x) = p_1(x)p_2(x)$, then $p(T)$ is the composition $p_1(T)p_2(T)$.

2. Show that, for any linear T and fixed $\vec{v} \in \mathcal{R}^n$, there is a nonzero polynomial $p(x)$ of degree at most n such that $p(T)\vec{v} = \vec{0}$. Suppose that (for fixed T and fixed $\vec{v} \neq \vec{0}$) we pick such a p of smallest possible degree and λ is a real number such that $p(\lambda) = 0$; prove that there is a nonzero vector $\vec{w}$ such that $T\vec{w} = \lambda\vec{w}$. [Hint: $p(\lambda) = 0$ if and only if there is a polynomial $q(x)$ such that $p(x) = (x - \lambda)q(x)$.]