Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Solutions to Written Assignment 3, due in class, Friday, February 14 (♡), 2003

1. Let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a set of vectors in $\mathcal{R}^n$, and $\vec{w}$ any vector in $\mathcal{R}^n$. Show that $\vec{w} \in span(S)$ if and only if $span(S \cup \{\vec{w}\}) = span(S)$.

   Prof (no wait, that's me), uh Proof: Clearly $span(S)$ is a subset of $span(S \cup \vec{w})$, since $b_1\vec{v}_1 + \cdots b_k\vec{v}_k = b_1\vec{v}_1 + \cdots b_k\vec{v}_k + 0\vec{w}$. To show ($\Rightarrow$) we have to see that any vector in $span(S \cup \{\vec{w}\})$ is already in $span(S)$ — we are given that $\vec{w} \in span(S)$. So there are scalars $a_1, \dots, a_k$ such that $\vec{w} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$. Any vector in $span(S \cup \{\vec{w}\})$ is $b_1\vec{v}_1 + \cdots + b_k\vec{v}_k + c\vec{w}$ for some choice of scalars $b_1, \dots b_k$ and c. This vector is $b_1\vec{v}_1 + \cdots + b_k\vec{v}_k + c(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k) = (b_1 + ca_1)\vec{v}_1 + \cdots + (b_k + ca_k)\vec{v}_k$ so it's in $span(S)$. This does it.

   ($\Leftarrow$) Now assume $span(S) = span(S \cup \{\vec{w}\})$. Clearly $\vec{w} = 0\vec{v}_1 + \cdots 0\vec{v}_k + 1\vec{w}$ is in $span(S \cup \{\vec{w}\})$, so $\vec{w} \in span(S)$.

2. Suppose that $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a set of vectors in $\mathcal{R}^n$, $\vec{w} \in span(S)$, but $\vec{w} \notin span\{\vec{v}_1, \dots, v_{k-1}\}$ Show that $\vec{v}_k \in span\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{w}\}$.

   Proof: Since $\vec{w} \in span(S)$, there are scalars $a_1, \dots, a_k$ such that $\vec{w} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$. Now must have $a_k \neq 0$, because otherwise we could leave it out and $\vec{w}$ would a linear combination of vectors in $\{\vec{v}_1, \dots, \vec{v}_{k-1}\}$; our second assumption says that this doesn't happen. So we can solve for $\vec{v}_k$; $\vec{v}_k = \frac{1}{a_k}\vec{w} - \frac{a_1}{a_k}\vec{v}_1 - \cdots - \frac{a_{k-1}}{a_k}\vec{v}_{k-1}$. This shows that $\vec{v}_k \in span\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{w}\}$, as per our request.

   A word on the statements. In both, the notation-setting assumption $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ could have been omitted (from the statements, not the proofs) with no adjustment at all in problem 1, and only very slight modification to problem 2's statement. If it had been left out of the statement, probably your first step in each proof should have been "Let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$". (Actually, this notation does assume the set is finite, which can be dropped from the assumptions, and the statements remain true; why?)