Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Solutions to written Assignment 4, due in class, Friday, February 21, 2003

Let $p(x) = \sum_{j=0}^{k} a_j x^j = a_0 + a_1 x + \cdots + a_k x^k$ be a polynomial with real coefficients, and A an $n \times n$ matrix. We define $p(A) = \sum_{j=0}^{k} a_j A^j = a_0 I + a_1 A + \cdots + a_k A^k$.

1. Show that if there is an invertible matrix P such that $B = P^{-1}AP$, then $p(B) = P^{-1}p(A)P$. [Hint: first show that $B^j = P^{-1}A^j A$.]

   Proof: $B^j = (P^{-1}AP)^j = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$ $= P^{-1}A(PP^{-1})A(PP^{-1})A\cdots(PP^{-1})AP$ by associativity of matrix multiplication. $PP^{-1} = I$, so this is $P^{-1}AIA\cdots IAP = P^{-1}A^jP$.

   So $p(B) = \sum_{j=0}^{k} a_j B^j = \sum_{j=0}^{k} a_j (P^{-1}AP)^j = \sum_{j=0}^{k} a_j P^{-1}A^jP = P^{-1}(\sum_{j=0}^{k} a_j A^j P)$ by distributivity (and the fact that each a_j is a scalar), and this equals $P^{-1}(\sum_{j=0}^{k} a_j A^j)P$ by distributivity again; i.e., $p(B) = P^{-1}p(A)P$.

2. Show that, if $p(A) = 0$, but $a_0 \neq 0$, then A is invertible. In fact, show that there is a polynomial $q(x)$ such that $q(A)A = I$.

   Proof: If $p(A) = 0$, then $a_0 I + a_1 A + \cdots + a_k A^k = 0$, so $a_0 I = -a_1 A - a_2 A^2 - \cdots - a_k A^k$, and since $a_0 \neq 0$, this gives $I = -\frac{a_1}{a_0}A - \cdots - \frac{a_k}{a_0}A^k$. By distributivity, this equals $A(-\frac{a_1}{a_0}I - \cdots - \frac{a_k}{a_0}A^{k-1})$, so if $q(x) = \sum_{j=0}^{k-1} -\frac{a_j}{a_0}x^j$, then $Aq(A) = I$, as advertized.