Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Written Assignment 6, due in class, March 21, 2003

Let W_1 and W_2 be subspaces of $\mathcal{R}^n$.

1. Show that the intersection $W_1 \cap W_2$ is a subspace of $\mathcal{R}^n$.
2. Show that, if neither W_1 nor W_2 is a subspace of the other, then the union $W_1 \cup W_2$ is *not* a subspace of $\mathcal{R}^n$.
3. We define the *sum* of the susbspaces as $W_1 + W_2 = \{\vec{w}_1 + \vec{w}_2 : \vec{w}_1 \in W_1, \vec{w}_2 \in W_2\}$. Show that $W_1 + W_2$ is a subspace of $\mathcal{R}^n$.
4. Show that $dim(W_1 + W_2) + dim(W_1 \cap W_2) = dim(W_1) + dim(W_2)$. This result is known as the *modular law*, or *lunch in Chinatown*.

1. Since $\vec{0} \in W_1$ and $\vec{0} \in W_2$ (as both are subspaces), $\vec{0} \in W_1 \cap W_2$.

   To show that $W_1 \cap W_2$ is closed under $+$, suppose that $\vec{v}_1 \in W_1 \cap W_2$ and $\vec{v}_2 \in W_1 \cap W_2$. Then $\vec{v}_1$ and $\vec{v}_2$ are both in W_1, so $\vec{v}_1 + \vec{v}_2 \in W_1$ (as W_1 is a subspace); for the same reason $\vec{v}_1 + \vec{v}_2 \in W_2$. Hence $\vec{v}_1 + \vec{v}_2 \in W_1 \cap W_2$.

   To show that $W_1 \cap W_2$ is closed under scalar multiplication, suppose that $\vec{v} \in W_1 \cap W_2$ and c is a scalar. Then $\vec{v} \in W_1$ and as W_1 is a subspace, $c\vec{v} \in W_1$; for the same reason $c\vec{v} \in W_2$. So $c\vec{v} \in W_1 \cap W_2$. This does it.

2. Let $vecw_1$ be a vector in W_1 which is not in W_2; there is such a monster by our assumptions. Similarly, there is $\vec{w}_2 \in W_2$, $\vec{w}_2 \notin W_1$. Now both $\vec{w}_1$ and $\vec{w}_2$ are in $W_1 \cup W_2$, but their sum $\vec{w}_1 + \vec{w}_2$ is not.

   To see this, suppose that it is; then it's either in W_1 or it's in W_2. If it were in W_1, then as W_1 is a subspace, $(\vec{w}_1 + \vec{w}_2) - \vec{w}_1 \in W_1$. But that contradicts our choice of $\vec{w}_2 \notin W_1$. Similarly (great proof word!) $\vec{w}_1 + \vec{w}_2 \notin W_2$. So $W_1 \cup W_2$ is not closed under $+$ and is not a subspace.

   It may be worth noting that $W_1 \cup W_2$ will have the zero vector and it will be closed under scalar multiplication.

3. $\vec{0} \in W_1 + W_2$ since $\vec{0} = \vec{0} + \vec{0}$ and $\vec{0}$ is in both W_1 and W_2. (You might note that by taking one side zero, we get $W_1 \cup W_2 \subseteq W_1 + W_2$.)

   To show $W_1 + W_2$ is closed under $+$, pick any two vectors in there, which are $\vec{w}_1 + \vec{w}_2$ and $\vec{w}'_1 + \vec{w}'_2$ for some $\vec{w}_1$, $\vec{w}'_1$ in W_1 and $\vec{w}_2$, $\vec{w}'_2$ in W_2. Now $(\vec{w}_1 + \vec{w}_2) + (\vec{w}'_1 + \vec{w}'_2) = (\vec{w}_1 + \vec{w}'_1) + (\vec{w}_2 + \vec{w}'_2) \in W_1 + W_2$ since $\vec{w}_1 + \vec{w}'_1 \in W_1$ and $\vec{w}_2 + \vec{w}'_2 \in W_2$.

   Finally, for closure under scalar multiplication, suppose that $\vec{w}_1 + \vec{w}_2 \in W_1 + W_2$ and that c is a scalar. Then $c(\vec{w}_1 + \vec{w}_2) = c\vec{w}_1 + c\vec{w}_2 \in W_1 + W_2$. This does it.

4. Suppose that $dim(W_1 \cap W_2) = k$, $dim(W_1) = k + \ell$ and $dim(W_2) = k + m$; we wish to show that $dim(W_1 + W_2) = k + \ell + m$. To this end, choose a basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ of $W_1 \cap W_2$ and extend it first to a basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{u}_1, \ldots, \vec{u}_\ell\}$ of W_1 and (separately, of course) to a basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{z}_1, \ldots, \vec{z}_m\}$ of W_2. We claim that $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{u}_1, \ldots, \vec{u}_\ell, \vec{z}_1, \ldots, \vec{z}_m\}$ (the union of all three bases) is a basis for $W_1 + W_2$, which will give what we want.

   First, all those vectors ($\vec{v}$'s, $\vec{u}$'s and $\vec{z}$'s) are in $W_1 + W_2$. Next, we show that any vector in $W_1 + W_2$ is in $span\{\vec{v}_1, \ldots, \vec{v}_k, \vec{u}_1, \ldots, \vec{u}_\ell, \vec{z}_1, \ldots, \vec{z}_m\}$. For if $\vec{w}_1 \in W_1$ and $\vec{w}_2 \in W_2$, then $\vec{w}_1 = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k + b_1\vec{u}_1 + \cdots + b_1\vec{u}_\ell$ and $\vec{w}_2 = c_1\vec{v} + 1 + \cdots + c_k\vec{v}_k + d_1\vec{z}_1 + \cdots + d_m\vec{z}_m$ for some constants labelled a through d with subscripts. Then $\vec{w}_1 + \vec{w}_2 = (a_1 + c_1)\vec{v}_1 + \cdots + (a_k + c_k)\vec{v}_k + b_1\vec{u}_1 + \cdots + b_1\vec{u}_\ell + d_1\vec{z}_1 + \cdots + d_m\vec{z}_m$. This is a linear combo of the supposed basis, so all we need to finish is the independence of the set.

   Suppose then

   $$\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k + \beta_1 u_1 + \cdots + \beta_\ell u_\ell + \gamma_1\vec{z}_1 + \cdots + \gamma_m\vec{z}_m = \vec{0}$$

   where the α's β's and γ's are scalars. We need to show that all these scalars are zero.

   $$\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k + \beta_1 u_1 + \cdots + \beta_\ell u_\ell = -\gamma_1\vec{z}_1 - \cdots - \gamma_m\vec{z}_m$$

   (shifting the stuff not from W_1 to the right-hand side). The left-hand side reveals this to be in W_1 and the left-hand side says it's in W_2. So it's in $W_1 \cap W_2$. Using the right-hand side, we see that

   $$-\gamma_1\vec{z}_1 - \cdots - \gamma_m\vec{z}_m = \delta_1\vec{v}_1 + \cdots + \delta_k\vec{v}_k$$

   for some scalars $\delta_1, \ldots, \delta_k$. Now $\delta_1\vec{v}_1 + \cdots + \delta_k\vec{v}_k + \gamma_1\vec{z}_1 + \cdots + \gamma_m\vec{z}_m = \vec{0}$ and the independence of our basis for W_2 shows that all the γ's and δ's must be zero. So $\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k + \beta_1 u_1 + \cdots + \beta_\ell u_\ell = \vec{0}$, too. By the indepence of our basis for W_1, all the α's and β's are zero into the bargain; this is just what we need.