Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Written Assignment 7, due in class, March 28, 2003

In these problems, we assume that $T : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ is a linear operator, and we let T^k be the composite of T with itself k times. (So $T^0 = I$, $T^1 = T$ and $T^2\vec{v} = T(T\vec{v})$ for any $\vec{v} \in \mathcal{R}^n$, etc.) For any polynomial $p(x) = \sum_{k=0}^m a_k x^k$, we define $p(T) : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ to be the operator such that $p(T)\vec{v} = \sum_{k=0}^m a_k(T^k\vec{v})$ for $\vec{v} \in \mathcal{R}^n$.

1. Show that $p(T)$ is linear. Show that if $p(x) = p_1(x)p_2(x)$, then $p(T)$ is the composition $p_1(T)p_2(T)$.

   Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathcal{R}^n$ and c be a scalar. Then $p(T)(\vec{v}+\vec{w}) = \sum_{k=0}^m a_k(T^k(\vec{v}+\vec{w})) = \sum_{k=0}^m a_k(T^k\vec{v}+T^k\vec{w}) = (\sum_{k=0}^m a_k(T^k\vec{v}))+(\sum_{k=0}^m a_k(T^k\vec{w})) = p(T)\vec{v} + p(T)\vec{w}$. Also $p(T)(c\vec{v}) = \sum_{k=0}^m a_k(T^k(c\vec{v})) = \sum_{k=0}^m a_k c(T^k\vec{v}) = c(\sum_{k=0}^m a_k(T^k\vec{v}))$. This verifies that $p(T)$ is linear. (We use the linearilty of T^k in both parts.

   Now suppose that $p_1(x) = b_0 + b_1x + \cdots + b_{m_1}x^{m_1}$ and $p_2(x) = c_0 + c_1x + \cdots + c_{m_2}x^{m_2}$. Then $p(x) = p_1(x)p_2(x) = b_0p_2(x) + b_1xp_2(x) + \cdots + b_{m_1}x_1^m p_2(x)$.

   So $p_1(T) = b_0 + b_1T + \cdots + b_{m_1}T^{m_1}$ and $p_2(T) = c_0 + c_1T + \cdots + c_{m_2}T^{m_2}$. Then

$$p(T) = b_0(c_0 + c_1T + \cdots + c_{m_2}T^{m_2})+$$
$$b_1(c_0T + c_1T^2 + \cdots + c_{m_2}T^{m_2+1})+$$
$$\cdots+$$
$$b_{m_1}(c_0T^{m_1} + c_1T^{m_1+1} + \cdots + c_{m_2}T^{m_2+m_1}).$$

   We need to show that, for each $\vec{v} \in \mathcal{R}^n$, we have that $p(T)\vec{v} = p_1(T)p_2(T)\vec{v}$.

   We first consider $T^j(p_2(T)\vec{v}) = T^j(c_0\vec{v} + c_1T\vec{v} + \cdots + c_{m_2}T^{m_2}\vec{v})$; because T^j is linear, this equals $c_0T^j\vec{v} + c_1T^j(T\vec{v}) + \cdots + c_{m_2}T^j(T^{m_2}\vec{v}) = c_0T^j\vec{v} + c_1T^{j+1}\vec{v} + \cdots + c_{m_2}T^{m_2+j}\vec{v}$. This holds for any j from 0 to m_1 (and beyond, for that matter). Now $p_1(T)p_2(T)\vec{v} = \sum_{j=0}^{m_1} b_jT^j(p_2(T)\vec{v})$ by definition. Applying the above observation for each j tells us this is equal to

$$p(T)\vec{v} = b_0(c_0\vec{v} + c_1T\vec{v} + \cdots + c_{m_2}T^{m_2}\vec{v})+$$
$$b_1(c_0T\vec{v} + c_1T^2\vec{v} + \cdots + c_{m_2}T^{m_2+1}\vec{v})+$$
$$\cdots+$$
$$b_{m_1}(c_0T^{m_1}\vec{v} + c_1T^{m_1+1}\vec{v} + \cdots + c_{m_2}T^{m_2+m_1}\vec{v}).$$

2. Show that, for any linear T and fixed $\vec{v} \in \mathcal{R}^n$, there is a nonzero polynomial $p(x)$ of degree at most n such that $p(T)\vec{v} = \vec{0}$. Suppose that (for fixed T and fixed $\vec{v} \neq \vec{0}$) we pick such a p of smallest possible degree and λ is a real number such that $p(\lambda) = 0$; prove that there is a nonzero vector $\vec{w}$ such that $T\vec{w} = \lambda\vec{w}$. [Hint: $p(\lambda) = 0$ if and only if there is a polynomial $q(x)$ such that $p(x) = (x - \lambda)q(x)$.]

   The set of $n + 1$ vectors $\{\vec{v}, T\vec{v}, \ldots, T^n\vec{v}\}$ must be dependent, since they live in n-dimensional space. Thus there are scalars $a_0,\ldots,a_n$, not all zero such that $\sum_{j=0}^{n} a_j T^j \vec{v} = \vec{0}$. (Actually, it is possible that $T^r\vec{v} = T^s\vec{v}$ for some $0 \leq r < s \leq n$, but in that case we could just let $a_r = 1$ and $a_s = -1$ and all the other a_j's be zero.) Let $p(x) = \sum_{j=0}^{n} a_j x^j$; the degree of p is the largest j such that $a_j \neq 0$. It is $\leq n$.

   Now suppose p has been chosen with smallest possible degree such that $p(T)\vec{v} = \vec{0}$, and that $p(\lambda) = 0$. Let $p(x) = (x - \lambda)q(x)$ where of course q has degree one smaller than that of p. Then by the previous question, $p(T)\vec{v} = (T - \lambda I)q(T)\vec{v}$. Our degree assumption tells us that $q(T)\vec{v} \neq \vec{0}$, so we call this vector $\vec{w}$; we have $(T - \lambda I)\vec{w} = \vec{0}$, or $T\vec{w} = \lambda I\vec{w} = \lambda\vec{w}$. (In this case, we naturally call λ and eigenvalue for T and $\vec{w}$ a corresponding eigenvector.)