Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Solutions to Written Assignment 8, due in class, April 4, 2003

Suppose that $T : \mathcal{R}^n \longrightarrow \mathcal{R}^n$ is a linear operator. We define the *kernel of* T, as $ker(T) = \{\vec{v} \in \mathcal{R}^n : T\vec{v} = \vec{0}\}$. [In case $T = T_A$, this is just the null space of A.]

1. Show that $ker(T^k) \leq ker(T^{k+1})$ for any natural number k.

   Suppose that $\vec{v} \in ker(T^k)$; then $T^k\vec{v} = \vec{0}$. So $T^{k+1}\vec{v} = T(T^k\vec{v}) = T\vec{0} = \vec{0}$. So $\vec{v} \in ker(T^{k+1})$. This shows that $ker(T^k)$ is a subset of $ker(T^{k+1})$. It is a subspace because it is closed under addition and scalar multiplication; we check this now.

   Suppose that $\vec{v}$ and $\vec{w}$ are in $ker(T^k)$, and that c is a scalar. Then $T^k\vec{v} = T^k\vec{w} = \vec{0}$, so $T^k(\vec{v} + \vec{w}) = T^k\vec{v} + T^k\vec{w} = \vec{0} + \vec{0} = \vec{0}$, so $\vec{v} + \vec{w} \in ker(T^k)$. Also $T^k(c\vec{v}) = cT^k\vec{v} = c\vec{0}$, so $c\vec{v} \in ker(T^k)$, too.

2. Show that there is an integer $0 \leq k \leq n$ such that $ker(T^k) = ker(T^{k+1})$.

   If $ker(T^k) \neq ker(T^{k+1})$, then the latter has larger dimension than the former. If this happens for every $k \leq n$, then the dimensions $0 = dim(ker(T^0)) < dim(ker(T)) < dim(ker(T^2)) < \ldots < dim(ker(T^n)) < dim(ker(T^{n+1}))$, so that $dim(ker(T^{n+1}))$ must be at least $n + 1$; but $ker(T^{n+1})$ is a subspace of $\mathcal{R}^n$ and as such cannot have dimension larger than n. So (by contradiction) we must have $ker(T^k) = ker(T^{k+1})$ for some $0 \leq k \leq n$.

3. Give examples, for $n = 4$, to show that it is possible that

   (a) $T \neq 0$, but $ker(T) = ker(T^2)$.
   (b) $ker(T) \neq ker(T^2)$, but $ker(T^2) = ker(T^3)$.
   (c) $ker(T^2) \neq ker(T^3)$, but $ker(T^3) = ker(T^4)$.
   (d) $ker(T^3) \neq ker(T^4)$, but $ker(T^4) = ker(T^5)$.

   In our examples, we specify a 4×4 matrix A such that T_A satisfies the given condition.

   (a) Any invertible A (for instance, $A = I$) will do, as will any diagonal matrix with some (but not all) zeroes on the diagonal.
   (b) How about this one? $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$