Mathematics 189-133B, Winter 2003
Vectors, Matrices and Geometry
Solution to Written Assignment 2, due in class, Friday, January 31, 2003

The *convex hull* of a set $\{P_1, P_2, \ldots, P_k\}$ of points in R^3 is the set of all points Q such that $\vec{q} = c_1\vec{p}_1 + c_2\vec{p}_2 + \cdots + c_k\vec{p}_k$ for some scalars $c_1, \ldots c_k$ that are ≥ 0 and add up to 1. So $(2, 0, 4) = \frac{1}{2}(2, -2, 2) + \frac{1}{3}(3, 0, 6) + \frac{1}{6}(0, 6, 6)$ is in the convex hull of $\{(2, -2, 2), (3, 0, 6,), (0, 6, 6)\}$, since $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$. (Here $\vec{p}_j$ is the vector $\vec{OP_j}$ and $\vec{q}$ is $\vec{OQ}$.)

1. Show that the convex hull of a pair $\{P_1, P_2\}$ of points is the line segment $\overline{P_1P_2}$.

2. Show that, if P_1, P_2 and P_3 are not collinear, then the convex hull of $\{P_1, P_2, P_3\}$ is the set of all points Q on or inside the triangle $\triangle P_1P_2P_3$.

[Hint for (b). Notice that Q is on or in $\triangle P_1P_2P_3$ if and only if there is a point A on $\overline{P_1P_2}$ such that Q is on $\overline{AP_3}$.]

Solution:

1. Q is on the segment $\overline{P_1P_2}$ if and only if the vector $\vec{P_1Q}$ is a multiple $c\vec{P_1P_2}$ for some scalar $0 \leq c \leq 1$.

   Suppose there is such a c. Then $\vec{q} = \vec{p}_1 + \vec{P_1Q} = \vec{p}_1 + c\vec{P_1P_2} = \vec{p}_1 + c(\vec{p}_2 - \vec{p}_1) = (1 - c)\vec{p}_1 + c\vec{p}_2$. Taking $c_1 = 1 - c$ and $c_2 = c$ we witness that Q is in the convex hull of $\{P_1, P_2\}$.

   If Q is in the convex hull of $\{P_1, P_2\}$, then $\vec{q} = c_1\vec{p}_1 + c_2\vec{p}_2$ for some scalars c_1 and c_2 with $c_1 + c_2 = 1$. So $\vec{P_1Q} = \vec{q} - \vec{p}_1 = (c_1 - 1)\vec{p}_1 + c_2\vec{p}_2 = c_2(\vec{p}_2 - \vec{p}_1)$ [since $c_1 + c_2 = 1$], which is $c_2\vec{P_1P_2}$. Since $0 \leq c_2 \leq 1$, Q is on the segment $\overline{P_1P_2}$.

2. Suppose that Q is on or in the triangle, and A is as described. If $\vec{a} = \vec{OA}$, there are nonnegative scalars b_1 and b_2 such that $\vec{a} = b_1\vec{p}_1 + b_2\vec{p}_2$ and $b_1 + b_2 = 1$. Since Q is on $\overline{AP_3}$, there are nonnegative scalars d_1, d_2 such that $d_1 + d_2 = 1$ and $\vec{q} = d_1\vec{a} + d_2\vec{p}_3$. [We have used part (1) twice here.] So $\vec{q} = d_1(b_1\vec{p}_1 + b_2\vec{p}_2) + d_2\vec{p}_3 = (d_1b_1)\vec{p}_1 + (d_1b_2)\vec{p}_2 + d_2\vec{p}_3$. As $d_1b_1 + d_1b_2 + d_2 = d_1 + d_2 = 1$, we see that Q is in the convex hull of $\{P_1, P_2, P_3\}$.

   Conversely, if Q is in the convex hull of $\{P_1, P_2, P_3\}$, then there are $c_1, c_2, c_3 \geq 0$ such that $c_1 + c_2 + c_3 = 1$ and $\vec{q} = c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3$. If $c_3 = 1$, then Q is P_3, so suppose that $c_3 < 1$. Let $\vec{a} = \vec{OA} = \frac{1}{1-c_3}(c_1\vec{p}_1 + c_2\vec{p}_2)$. Then A is on $\overline{P_1P_2}$ since $\vec{a} = \frac{c_1}{1-c_3}\vec{p}_1 + \frac{c_2}{1-c_3}\vec{p}_2$ and $\frac{c_1}{1-c_3} + \frac{c_2}{1-c_3} = 1$. Since $\vec{q} = (1 - c_3)\vec{a} + c_3\vec{p}_3$, Q is on $\overline{AP_3}$. So Q is in (or on) $\triangle P_1P_2P_3$.

Geometrically, the convex hull of a set $\{P_1, \ldots, P_k\}$ is the smallest set C containing those points and is convex, i.e., any line segment joining two points of C lies entirely inside C.