MATH 222: Calculus III

Solution Sheet for Written Assignment 1

1. (a) We have $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, an alternating series whose n-th term in absolute value is $1/\sqrt{n+1}$ which decreases to 0 as $n \to \infty$. Thus the series converges by the alternating series test. The second series $\sum_{n=1}^{\infty} \frac{n(n+6)}{(n+1)(n+3)(n+5)}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$. The limit of the ratio of nth terms of these two series is 1.

   (b) Let $a_n = \frac{\ln(n)}{n^p}$. For $p \leq 0$, $a_n \to \infty$ as $n \to \infty$ and $\sum a_n$ diverges. If $0 < p \leq 1$ then

$$\frac{a_n}{\frac{1}{n}} = \ln(n) \to \infty \text{ as } n \to \infty$$

   which gives the divergence of $\sum a_n$ by limit comparison with the harmonic series $\sum \frac{1}{n}$, a divergent series. If $p > 1$, let $1 < q < p$ and let $\alpha = p - q$. Then

$$\frac{a_n}{\frac{1}{n^q}} = \frac{\ln(n)}{n^\alpha} \to 0 \text{ as } n \to \infty$$

   which gives the convergence of $\sum a_n$ by limit comparison with $\sum \frac{1}{n^q}$, a convergent series since $q > 1$.
   Second Solution: For $p > 0$, the function $f(x) = \frac{\ln(x)}{x^p}$ is decreasing for x sufficiently large since

$$f'(x) = \frac{1 - px}{t} < 0 \text{ for } x > 1/p.$$

   Applying the integral test, using the change of variable $t = \ln(x)$, we have

$$\int_1^\infty f(x)\,dx = \int_0^\infty e^{(1-p)t}\,dt$$

   which converges if and only if $p > 1$.

2. (a) Let $a_n = \frac{(x-3)^n}{5^n\sqrt{n^3}}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x-3|^{n+1}}{5^{n+1}\sqrt{(n+1)^3}}\frac{5^n\sqrt{n^3}}{|x-3|^n} = \frac{|x-3|}{5\sqrt{(1+1/n)^3}} \to \frac{|x-3|}{5} \text{ as } n \to \infty.$$

   Hence, by the ratio test $\sum a_n$ converges for $|x-3| < 5$ and diverges for $|x-3| > 5$. Thus the radius of convergence is 5 and the interval of convergence is $-2 < x < 8$ with possibly one or more of the endpoints $-2, 8$. When $x = -2$

$$\sum a_n = \sum \frac{(-1)^n}{n^{3/2}},$$

   an absolutely convergent series since the p-series $\sum 1/n^p$ converges when $p > 1$. When $x = 8$

$$\sum a_n = \sum \frac{1}{n^{3/2}},$$

   a convergent p-series. Thus the interval of convergence $I = [-2, 8]$.

   (b) Since a power series can be differentiated term by term in the interior of the interval of convergence, we have

$$f'(x) = \sum_{n=1}^{\infty} \frac{n(x-3)^{n-1}}{5^n\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{5^n\sqrt{n}}$$

for $-2 < x < 8$. Since the radius of convergence does not change on differentiation of series term by term, the interval of convergence of this series is $(-3, 8)$ plus possibly one or more of the endpoints. At the endpoint $x = -2$ the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/2}},$$

a convergent alternating series. At the endpoint $x = 8$ the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}},$$

a divergent p-series. Since a power series represents a continuous function on its interval of convergence, the function

$$g(x) = \sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{5^n \sqrt{n}}$$

is a continuous function on the interval $[-2, 8)$. Since you can integrate term by term in the interval of convergence we have

$$\int_3^x g(t)\, dt = f(x)$$

for $-2 \le x < 8$. Differentiating with respect to x, we get $f'(x) = g(x)$ for $-2 \le x < 8$.

3. (a) We have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Differentiating twice with respect to x, we get

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

for $|x| < 1$ so that $\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}$ for $|x| < 1$. Hence, for $|x| < 1$,

$$\begin{aligned}
\frac{x + x^2}{(1-x)^3} &= \frac{x}{(1-x)^3} + \frac{x^2}{(1-x)^3} \\
&= \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-1} + \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^n \\
&= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n + \sum_{n=1}^{\infty} \frac{n(n-1)}{2} x^n \\
&= \sum_{n=1}^{\infty} (\frac{n(n+1)}{2} + \frac{n(n-1)}{2}) x^n = \sum_{n=1}^{\infty} n^2 x^n.
\end{aligned}$$

(b) Since $\frac{x+x^2}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n$ for $|x| < 1$ it holds for $x = 1/2$ which gives

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6.$$

4. (a) We have $\vec{r}'(t) = (1, \sqrt{2t}, t)$ so that the tangent vector to the curve at $(2, 2\sqrt{2}/3 + 362, 1/2)$ is $(1, \sqrt{2}, 1)$. The equation of the tangent line at $t = 1$ is therefore

$$\vec{r} = (2, 2\sqrt{2}/3 + 362, 1/2) + t(1, \sqrt{2}, 1).$$

(b) The arc length $= \int_1^2 |\vec{r}'(t)|\, dt = \int_1^2 (t+1)\, dt = 5/2$.