Solutions to Written Assignment 2

1. (a) Let $r = (t^3/3, 2t, 2/t)$. Then $\frac{dr}{dt} = (t^2, 2, -2/t^2)$ and $|\frac{dr}{dt}| = (t^4+2)/t^2$ so that $\mathbf{T} = (\frac{t^4}{t^4+2}, \frac{2t^2}{t^4+2}, \frac{-2}{t^4+2})$. Hence $= \frac{4t}{(t^4+2)^2}(2t^22 - t^4, 2t^2)$, $\kappa = |\frac{d\mathbf{T}}{dt}|/|\frac{dr}{dt}| = \frac{4t^3}{(t^4+2)^2}$ and $\mathbf{N} = \frac{d\mathbf{T}}{dt}/|\frac{d\mathbf{T}}{dt}| = (\frac{2t^2}{t^4+2}, \frac{2-t^4}{t^4+2}, \frac{-2t^2}{t^4+2})$.

   (b) Since $r(1) = (1/3, 2, 2), r'(1) = (1, 2, -2)$, the tangent line at $(1/3, 2, 2)$ is $x = t+1/3, y = 2t+2, z = 2-2t$.

2. We have $\frac{\partial u}{\partial x} = \frac{x}{r}\frac{du}{dr}, \frac{\partial u}{\partial y} = \frac{y}{r}\frac{du}{dr}, \frac{\partial u}{\partial z} = \frac{z}{r}\frac{du}{dr}$, so that

   (a) $(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial u}{\partial z})^2 = \frac{x^2+y^2+z^2}{r^r}(\frac{du}{dr})^2 = (\frac{du}{dr})^2$,

   (b) $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = \frac{1}{r}\frac{du}{dr}(x\vec{i} + y\vec{j} + z\vec{k})$.

3. (a) We have $\frac{\partial z}{\partial x} = 3e^y - 3x^2$, $\frac{\partial z}{\partial y} = 3xe^y - 3e^{3y}$ so that, at $(0,0)$, we have $\frac{\partial z}{\partial x} = 3$, $\frac{\partial z}{\partial y} = -3$. Hence the equations of the tangent plane and normal line at $(0, 0, -1)$ are respectively $z = -1 + 3x - 3y$ and $x = -3t, y = 3t, z = -1 + t$.

   (b) If $\mathbf{r} = (2s^2 + t^3, s^2t^3, s^2 - st^3)$ we have $\frac{\partial \mathbf{r}}{\partial s} = (4s, 2st^3, 2s - t^3)$, $\frac{\partial \mathbf{r}}{\partial t} = (3t^2, 3s^2t^2, -3st^2)$ so that, when $s = t = 1$, we have $\mathbf{r} = (3, 1, 0)$, $\frac{\partial \mathbf{r}}{\partial s} = (4, 2, 1)$, $\frac{\partial \mathbf{r}}{\partial t} = (3, 3, -3)$. A normal vector to the tangent plane at $(3, 1, 0)$ is $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial s} = (-9, 15, 6)$ so that the tangent plane at $(3, 1, 0)$ is $3x - 5y - 2z = 4$.

4. (a) We have $\nabla T = (3x^2y + z^3, 3y^2z + x^3, 3z^2x + y^3)$ and $\mathbf{u} = \overrightarrow{PQ} = (-1, 2, 2)$ so that, at $(2, -1, 0)$, we have $D_{\mathbf{u}T} = \nabla T \cdot \mathbf{u}/|\mathbf{u}| = (-12, 8, -1) \cdot (-1, 2, 2)/3 = 26/3$.

   (b) Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be the position of the mosquito at time t. Then $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of the mosquito at time t. We have $|\mathbf{v}| =$ speed of mosquito $=5$ and the direction of $\mathbf{v}$ is, up to sign, the gradient of f at $(2, -1, 0)$, namely $(8, -6, 0)/10 = (4, -3, 0)/5$ so that, at $(2, -1, 0)$, we have $\mathbf{v} = \pm((4, -3, 0)$. At time t, the temperature of the mosquito is $T(\mathbf{r}(t))$. The rate of change of the temperature of the mosquito per unit time is therefore

$$\frac{d}{dt}T(\mathbf{r}(t) = \nabla T(\mathbf{r}(t)) \cdot \mathbf{v}$$

   which, at the time the mosquito is at $(2, -1, 0)$, is $(-12, 8, -1) \cdot \pm(4, -3, 0) = \mp 72$. Since the mosquito is flying in the direction of increasing temperature, the rate must be positive so that $\mathbf{v} = (-4, 3, 0)$ and the rate is 72. (Things are getting hot for the mosquito!)