Solutions to Selected WebWork Problems

Set 10 Problem 4

Compute

$$\iint_R (x^2+y^2)\,dx\,dy,$$

where R is the region bounded by the ellipse $ax^2+2bxy+ay^2=c$.

Since we are given the hint that the axes of the ellipse are $x=y, x=-y$ or $x-y=0, x+y=0$ we make the substitution $u=x-y, v=x+y$ so that the equations of the axes are $u=0, v=0$ (recall that the axes of an ellipse in standard position are the x and y axes whose equations are $y=0, x=0$ respectively). Solving for x, y, we get

$$x=\frac{1}{2}(u+v), y=\frac{1}{2}(v-u).$$

The Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}=\frac{1}{2}.$$

The integrand becomes

$$x^2+y^2=[\frac{1}{2}(u+v)]^2+[\frac{1}{2}(v-u)]^2=\frac{1}{2}(u^2+v^2).$$

The region S in the uv-plane that corresponds to R is that bounded by the ellipse

$$\begin{aligned} ax^2+2bxy+ay^2&=c\\ a[\frac{1}{2}(u+v)]^2+2b[\frac{1}{2}(u+v)][\frac{1}{2}(v-u)]+a[\frac{1}{2}(v-u)]^2&=c\\ \frac{1}{2}[(a-b)u^2+(a+b)v^2]&=c\\ \left(\frac{u}{\sqrt{\frac{2c}{a-b}}}\right)^2+\left(\frac{v}{\sqrt{\frac{2c}{a+b}}}\right)^2&=1\\ \left(\frac{u}{\alpha}\right)^2+\left(\frac{v}{\beta}\right)^2&=1 \end{aligned}$$

where $\alpha=\sqrt{\frac{2c}{a-b}}, \beta=\sqrt{\frac{2c}{a+b}}$. Thus, using the change of variables formula, we have

$$\iint_R (x^2+y^2)\,dx\,dy=\iint_S \frac{1}{2}(u^2+v^2)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\frac{1}{4}\iint_S (u^2+v^2)\,du\,dv.$$

Now, to transform the ellipse into the unit circle, we make the substitution

$$u=\alpha s, v=\beta t.$$

Thus the equation becomes $s^2+t^2=1$. Note that the integrand becomes

$$u^2+v^2=\alpha^2s^2+\beta^2t^2.$$

The Jacobian of this transformation is

$$\frac{\partial(u,v)}{\partial(s,t)}=\begin{vmatrix} \alpha & 0 \\ 0 & \beta \end{vmatrix}=\alpha\beta.$$

Hence the integral becomes

$$\begin{aligned}\iint_R (x^2+y^2)\,dx\,dy &= \frac{1}{4}\iint_S (u^2+v^2)\,du\,dv = \frac{1}{4}\iint_T (\alpha^2 s^2+\beta^2 t^2)\left|\frac{\partial(u,v)}{\partial(s,t)}\right| ds\,dt \\ &= \frac{\alpha\beta}{4}\iint_T (\alpha^2 s^2+\beta^2 t^2)\,ds\,dt\end{aligned}$$

where T is the unit disc in the st-plane.

We can now use polar coordinates $s = r\cos\theta, t = r\sin\theta$ to get

$$\begin{aligned}\iint_R (x^2+y^2)\,dx\,dy = \frac{\alpha\beta}{4}\iint_T (\alpha^2 s^2+\beta^2 t^2)\,ds\,dt &= \frac{\alpha\beta}{4}\int_0^{2\pi}\int_0^1 (\alpha^2 r^2\cos^2\theta+\beta^2 r^2\sin^2\theta) r\,dr\,d\theta \\ &= \frac{\alpha\beta}{4}\int_0^{2\pi}\int_0^1 (\alpha^2\cos^2\theta+\beta^2\sin^2\theta) r^3\,dr\,d\theta = \frac{\alpha\beta}{4}\int_0^{2\pi}(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\frac{r^4}{4}\bigg|_{r=0}^{r=1} d\theta \\ &= \frac{\alpha\beta}{16}\int_0^{2\pi}(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta = \frac{\alpha^3\beta}{16}\int_0^{2\pi}\cos^2\theta\,d\theta + \frac{\alpha\beta^3}{16}\int_0^{2\pi}\sin^2\theta\,d\theta \\ &= \frac{\pi\alpha^3\beta+\pi\alpha\beta^3}{16} = \frac{\pi\alpha\beta(\alpha^2+\beta^2)}{16}.\end{aligned}$$

Note that

$$\alpha\beta = \sqrt{\frac{2c}{a-b}}\sqrt{\frac{2c}{a+b}} = \frac{2c}{\sqrt{a^2-b^2}}, \alpha^2+\beta^2 = \frac{2c}{a-b}+\frac{2c}{a+b} = \frac{4ac}{a^2-b^2},$$

so

$$\iint_R (x^2+y^2)\,dx\,dy = \frac{\pi\alpha\beta(\alpha^2+\beta^2)}{16} = \frac{\pi ac^2}{2(a^2-b^2)^{\frac{3}{2}}}.$$

Set 10 Problem 7

Find the surface area of the part of the sphere $x^2+y^2+z^2=a^2$ that lies outside of the cylinder $x^2+y^2=ax$.

We can find the surface area of the part of the sphere outside the cylinder by finding the surface area inside the cylinder and subtracting it from the total surface area of the cylinder. Note that there are two parts of the sphere within the cylinder, one above the xy-plane and one below, each of the same area. We shall find the area of the part above.

The cylinder passes through the sphere, so the region R in the xy-plane we will be integrating over is simply the circle $x^2+y^2=ax$. The surface area will be given by

$$A_{\text{inside}} = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

It is easiest to find the partial derivatives of z by differentiating implicitly. Differentiating the equation of the sphere with respect to x, we get

$$2x+2z\frac{\partial z}{\partial x}=0,$$

which gives $\partial z/\partial x = -x/z$. Similarly, $\partial z/\partial y = -y/z$. Thus

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \sqrt{\frac{z^2+x^2+y^2}{z^2}} = \sqrt{\frac{a^2}{z^2}} = \frac{a}{z},$$

recalling that $z^2+x^2+y^2=a^2$. Note that on the upper hemisphere, $z=\sqrt{a^2-x^2-y^2}$. Therefore, the area is given by

$$\iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy.$$

Now, both the integrand and the region of integration indicate that it would be easier to use polar coordinates. The equation of the circle $x^2+y^2=ax$ in polar coordinates is $r=a\cos\theta$; recall that to trace out this circle,

θ ranges from $-\pi/2$ to $\pi/2$. Thus the surface area of the part of the sphere within the cylinder and above the xy-plane is given by

$$\begin{aligned}
A_{\text{inside}} &= \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy = \iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy \\
&= a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\sqrt{a^2-r^2}\right]_{r=0}^{r=a\cos\theta} d\theta \\
&= a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a-a|\sin\theta|\,d\theta = a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1-|\sin\theta|\,d\theta
\end{aligned}$$

noting that $\sqrt{a^2-a^2\cos^2\theta}=a\sqrt{\sin^2\theta}=a|\sin\theta|$. Since $\sin\theta\le 0$ when $\theta\in[-\pi/2,0]$, we integrate the above as

$$a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1-|\sin\theta|\,d\theta = a^2\left[\int_{-\frac{\pi}{2}}^{0} 1+\sin\theta\,d\theta+\int_0^{\frac{\pi}{2}} 1-\sin\theta\,d\theta\right] = a^2(\frac{\pi}{2}-1+\frac{\pi}{2}-1).$$

Thus we get $A_{\text{inside}}=a^2(\pi-2)$. Alternatively, we could have used symmetry in the x-axis and simply integrated over the region above the x-axis, to get

$$A_{\text{inside}} = 2a\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = a^2(\pi-2).$$

In either case we find that the area of the sphere inside the cylinder is twice this, namely $2a^2(\pi-2)$. Since the surface area of a sphere of radius a is $4\pi a^2$, the area of the sphere outside the cylinder must be $4\pi a^2-2a^2(\pi-2)=2\pi a^2+4a^2=2a^2(\pi+2)$.

Set 11 Problem 4

Find the centroid $(\overline{x},\overline{y},\overline{z})$ of the region bounded by the surfaces $(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2=1, x,y,z\ge 0$.

We first make the transformation $x=au, y=bv, z=cw$ so that the ellipsoid becomes the unit sphere $u^2+v^2+w^2=1$ in uvw-space. The Jacobian of this transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

Using the change of variables formula for triple integrals to compute the volume gives

$$V = \iiint_R dx\,dy\,dz = \iint_S \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = abc\iiint_S du\,dv\,dw,$$

where S is the part of the solid unit sphere in the first octant $u,v,w\ge 0$ in uvw-space. We now switch to spherical coordinates ($u=\rho\sin\phi\cos\theta, v=\rho\sin\phi\sin\theta, w=\rho\cos\phi$), to get

$$V = abc\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{abc}{3}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi\,d\theta = \frac{abc}{3}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi abc}{6}.$$

Now, remark that $x=au=a\rho\sin\phi\cos\theta, y=bv=b\rho\sin\phi\sin\theta, z=cw=c\rho\cos\phi$. Thus

$$\begin{aligned}
\overline{x} &= \frac{1}{V}\iiint_R x\,dV = \frac{6}{\pi abc}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (a\rho\sin\phi\cos\theta)abc\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{6a}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta = \frac{3a}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^2\phi\cos\theta\,d\phi\,d\theta \\
&= \frac{3a}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^2\phi\cos\theta\,d\theta\,d\phi = \frac{3a}{2\pi}\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi \\
&= \frac{3a}{4\pi}\int_0^{\frac{\pi}{2}} 1-\cos 2\phi\,d\phi = \frac{3a}{4\pi}\left[\phi-\frac{\sin 2\phi}{2}\right]_{\phi=0}^{\phi=\frac{\pi}{2}} = \frac{3a}{8}.
\end{aligned}$$

Similarly, we find $\overline{y} = 3b/8, \overline{z} = 3c/8$.

Set 11 Problem 6

Find the mass of the solid in the shape of the region

$$a^2 \leq x^2 + y^2 + z^2 \leq b^2, \sqrt{3(x^2 + y^2)} \leq z$$

if the density at (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$.

The region is the space within the cone $z = \sqrt{(3(x^2 + y^2)}$ and between the spheres $x^2 + y^2 + z^2 = a^2, x^2 + y^2 + z^2 = b^2$. We use spherical coordinates here. The equations of the spheres are $\rho = a, \rho = b$. Note also that the equation of the cone is $z = \sqrt{3r^2} = \sqrt{3}r$, so $r/z = 1\sqrt{3}$ and hence the cone can be described

$$\phi = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

The density function is $\delta = \sqrt{x^2 + y^2 + z^2} = \rho$. Therefore the mass is given by

$$\begin{aligned}
m &= \iiint_R \delta \, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_a^b \rho\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_a^b \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \frac{b^4 - a^4}{4} \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \sin\phi \, d\phi \, d\theta = \frac{b^4 - a^4}{4} \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \sin\phi \, d\theta \, d\phi = \frac{\pi(b^4 - a^4)}{2} \int_0^{\frac{\pi}{6}} \sin\phi \, d\phi \\
&= \frac{\pi(b^4 - a^4)}{2} [-\cos\phi]_{\phi=0}^{\phi=\frac{\pi}{6}} = \frac{\pi(b^4 - a^4)(2 - \sqrt{3})}{4}
\end{aligned}$$