MATH 255: Lecture 14

The Elementary Transcendental Functions: The Circular Functions

The name circular functions is derived from the fact that the points $(\cos x, \sin x)$ are precisely the points on the unit circle $x^2+y^2=1$. If we think of a point (a,b) in the plane as a complex number, we are led to introduce the complex-valued function $f(x) = \cos x + i\sin x$. This function satisfies $f'(x) = if(x)$, $f(0)=1$, where the derivative of a complex-valued function of one variable is defined by componentwise differentiation. We will show, without using any geometry, that there is a unique such function $f(x)$.

The Picard existence and uniqueness theorem for differential equations holds for complex-valued functions if the absolute value of a complex number $a+ib$ is defined to be $|a+ib| = \sqrt{a^2+b^2}$ and integration is done componentwise. We thus obtain that there is a unique complex-valued function $f(x)$ with $f'(x) = if(x)$, $f(0)=1$. This function is given by

$$f(x) = \exp(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

If we define $\cos x$ and $\sin x$ by $f(x) = \sin x + i\cos x$ and define $e^{ix} = \exp(ix)$, we obtain

$$e^{ix} = \cos x + i\sin x = (1 - x^2/2! + x^4/4! + \cdots) + i(x - x^3/3! + x^5/5! + \cdots)$$

In particular, we obtain that $\sin' = \cos$, $\cos' = -\sin$, $\sin 0 = 0$, $\cos 0 = 1$. One can also prove that $e^{i(x+y)} = e^{ix}e^{iy}$ exactly in the same way as in the real case. This gives the addition laws for sin and cos:

$$\sin(x+y) = \sin x\cos y + \cos x \sin y, \quad \cos(x+y) = \cos x\cos y - \sin x \sin y.$$

We also obtain $(\cos x + i\sin x)^n = \cos nx + i\sin x$, which is known as DeMoivre's Theorem.

Since $1 = e^{ix}e^{-ix} = \cos^2 x + \sin^2 x$, we obtain that e^{ix} is a point on the unit circle. We now show the we get all the points on the unit circle in this way. First note that $\cos 0 = 1$ and $\cos 2 < 1-2 = -1$. Let $\pi/2$ be the smallest zero of $\cos x$ which is > 0. Since $\cos x$ is the derivative of $\sin x$, we see that $\sin x$ is a strictly increasing function on the interval $[0,\pi/2]$ and that it attains a maximum of 1 at $x=\pi/2$. Since the derivative of $\cos x$ is $-\sin x$, we see that $\cos x$ strictly decreases from 1 to 0 on the interval $[0,\pi/2]$. Since $\sin(x+\pi/2) = \cos x$ and $\cos(x+\pi/2) = -\sin x$, $\sin x$ strictly decreases from 1 to 0 and $\cos x$ strictly decreases from 0 to -1 on $[\pi/2,\pi]$. In particular, we obtain $e^{i\pi} = -1$.

Since $\sin(x+\pi) = -\sin x$ and $\cos(x+\pi) = -\cos x$, we see that $\sin x$ decreases from 0 to -1 and on the interval $[\pi, 3\pi/2]$ and increases from -1 t0 0 on $[3\pi/2, 2\pi]$ while $\cos x$ increases from -1 to 1 on the interval $[\pi, 2\pi]$. Thus, as x increases from 0 to 2π the point e^{ix} goes through all the points of the unit circle exactly once with the exception that $e^{ix} = 1$ when $x=0$ and $x=2\pi$.

Since $e^{i(x+2\pi)} = e^{ix}$, we see that e^{ix} and hence $\sin x$, $\cos x$ are periodic with period 2π. We thus see that x is determined, up to an integral multiple of 2π by e^{ix}. For each non-zero complex number $z = a+bi$, there is a unique, up to the addition of an integral multiple of 2π, real number x such that $z = |z|e^{ix}$. This number is called the argument of z or the angle that the vector (a,b) makes with the positive x-axis.

One can avoid the use of the complex numbers to introduce sin and cos by noting that each of these functions satisfies the differential equation $y'' = -y$. If we set $y' = u$, then $u' = -y$. Conversely, if u, y are functions with $y' = u$, $u' = -y$, then $y'' = -y$. Thus solving $y'' = -y$ is equivalent to solving $y' = u$, $u' = -y$ for y and u.

Let $Y = \begin{bmatrix} y \\ u \end{bmatrix}$. Then the system $y' = u$, $u' = -y$ can be written in matrix form as $Y' = AY$, where

$$Y' = \begin{bmatrix} y' \\ u' \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This system of differential equations, together with the prescribing of an initial value for Y, is equivalent to solving the integral equation

$$Y(x) = Y(0) + \int_0^x AY(t)\,dt,$$

for a continuous Y, where the integration of a vector valued function is done componentwise and continuity is componentwise. The theorem of Picard applies in this case with the same proof, if the absolute value of a column vector is defined by

$$\left|\begin{bmatrix} a \\ b \end{bmatrix}\right| = \max(|a|, |b|).$$

In context of vector spaces, the term "norm" is used instead of absolute value. The term "absolute valued" is reserved for rings as the condition $|xy| = |x||y|$ is required. We will have more to say about this later. The Lipschitz condition is verified for $F(x, Y) = AY$ since $|AY - AZ| = |Y - Z|$. More generally, for any 2×2 matrix A, we have $|AY - AZ| \le |A||X - Y|$ where $|A|$ is the maximum of the sums of the absolute values of the rows of A.

Let us carry out the Picard iteration in the case $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The iterations $Y_n = \begin{bmatrix} y_n \\ u_n \end{bmatrix}$ are defined by

$$Y_{n+1}(x) = Y(0) + \int_0^x AY_n(t)\,dt.$$

This is equivalent to $y_{n+1}(x) = \int_0^x u_n(t)\,dt$, $u_{n+1}(x) = 1 - \int_0^x y_n(t)\,dt$. One shows inductively that, for $n \ge 1$,

$$\begin{aligned} y_{2n-1}(x) = y_{2n}(x) &= x - \frac{x^3}{3!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} \\ u_{2n}(x) = u_{2n+1}(x) &= 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}. \end{aligned}$$

It follows that the unique solution $Y = \begin{bmatrix} y \\ u \end{bmatrix}$ is given by

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

which define the sin and cosine functions respectively. To show the addition laws

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \quad \cos(x+y) = \cos x \cos yy - \sin x \sin y$$

we fix y and simply note that

$$U = \begin{bmatrix} \sin(x+y) \\ \cos(x+y) \end{bmatrix}, \quad V = \begin{bmatrix} \sin(x+y) = \sin x \cos y + \cos x \sin y \\ \cos(x+y) = \cos x \cos y - \sin x \sin y \end{bmatrix}$$

are solutions of the DE $Y' = AY$ and $U(0) = V(0)$. To show that $\sin^2 x + \cos^2 x = 1$, simply note that this holds if $x = 0$ and that the derivative of the LHS and RHS are both zero.

Exercise 1. Show that the solution space of the system $Y' = AY$ is a 2-dimensional subspace of $\mathbb{R}^2$-valued functions on $\mathbb{R}$. **Hint:** Show that the solutions U, V with $U(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $V(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are a basis.

Exercise 2. Show that the DE $y'' + ay' + by = 0$ is a 2-dimensional subspace of the vector space of $\mathbb{R}$-valued functions on $\mathbb{R}$.

Exercise 3. If $|A|$ is maximum of the sums of the absolute values of the rows of the matrix A, show that

$$|cA| = |c||A|, \quad |A + B| \leq |A| + |B|, \quad |AB| \leq |A||B|.$$

Exercise 4. Show that the unique solution of the $Y' = AY$, $Y(0) = B$ is $Y = e^{Ax}B$, where

$$e^{Ax} = \sum_{n=0}^{\infty} \frac{A^n x^n}{n!},$$

the convergence being componentwise. **Hint:** Show that the entries of the partial sums satisfy the uniform Cauchy criterion on $[-a, a]$ for any $a > 0$.