The FOLDS language for categories

Signature $\boxed{L_{cat}}$:

$$
\begin{tikzcd}
I \arrow[dr, "i_1"'] & E \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "c_1"] & T \arrow[dl, shift right=2, "t_{01}"'] \arrow[dl, "t_{12}"] \arrow[dl, shift left=2, "t_{02}"] \\
& A \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & \\
& O &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[dr, "i_0"'] & E \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "c_0"] & T \arrow[dl, shift right=2, "t_0"'] \arrow[dl, "t_1"] \arrow[dl, shift left=2, "t_2"] \\
& O &
\end{tikzcd}
$$

$$i_0 = d i_1 = c i_1$$

$$t_0 = d t_{01} = d t_{02}$$

$$t_1 = c t_{01} = d t_{12}$$

$$d_0 = d d_1 = d c_1$$

$$t_2 = c t_{02} = c t_{12}$$

$$c_0 = c d_1 = c c_1$$

C : category

$$\xrightarrow[\text{re-coding}]{} \boxed{M(C) : L_{cat} \longrightarrow SET}$$

$M = M(C)$:

$$M(O) = ob(C)$$

$$M(A) = Arr(C) = \{(X, Y, f) \mid f : X \to Y \text{ in } C\}$$

$$M(I) = \{1_X \mid X \in ob(C)\} = \{(X, f) \mid f = 1_X\}$$

$$M(E) = \{(X, Y, f_1, f_2) : X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \text{ and } f_1 = f_2\}$$

$$M(T) = \Big\{(X_0, X_1, X_2, x_{01}, x_{12}, x_{12}) \Big|$$

$$\begin{tikzcd} & X_1 \arrow[dr, "x_{12}"] & \\ X_0 \arrow[ur, "x_{01}"] \arrow[rr, bend right, "x_{02}"'] & \text{comutes} & X_2 \end{tikzcd} \Big\}$$

$M(d), M(c)$:

$$\begin{tikzcd} & M(A) \arrow[dl, "M(d)"'] \arrow[dr, "M(c)"] & \\ M(0) & & M(0) \end{tikzcd} \quad : \quad \begin{tikzcd} & (X, Y, f) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ X & & Y \end{tikzcd}$$

$M(i_0), M(i_1)$:

$$\begin{tikzcd} M(I) \arrow[dr, "M(i_1)"] \arrow[ddr, bend right, "M(i_0)"'] & \\ & M(A) \arrow[d, shift right, "M(d)"'] \arrow[d, shift left, "M(c)"] \\ & M(0) \end{tikzcd} \quad : \quad \begin{tikzcd} (X, f) \text{ with } f = 1_X \arrow[dr, mapsto] \arrow[ddr, mapsto, bend right] & \\ & f \arrow[d, mapsto, shift right] \arrow[d, mapsto, shift left] \\ & X \end{tikzcd}$$

Example for $\varphi \in \mathrm{FOLDS}(L_{cat})$:

"existence of composite of composable arrows"

$\varphi :=$

$$\forall X_0 : O.\ \forall X_1 : O.\ \forall X_2 : O$$

$$\forall x_{01} : A(\underset{d}{X_0}, \underset{c}{X_1}).\ \forall x_{12} : A(X_1, X_2)$$

$$\exists x_{02} : A(X_0, X_2).\ \underset{\substack{\uparrow \\ \text{witness} \\ \text{for commutativity}}}{\exists x_{012}} : T(\underset{t_0}{X_0}, \underset{t_1}{X_1}, \underset{t_2}{X_2}, \underset{t_{01}}{x_{01}}, \underset{t_{12}}{x_{12}}, \underset{t_{02}}{x_{02}}).\ \mathrm{TRUE}$$

$M(C) \models \varphi \iff$ in terms of C:

$$\forall \boxed{\begin{tikzcd} & X_1 \arrow[dr, "x_{12}"] & \\ X_0 \arrow[ur, "x_{01}"] & & X_2 \end{tikzcd}} \quad \exists x_{02} \boxed{\begin{tikzcd} & X_1 \arrow[dr, "x_{12}"] \arrow[d, phantom, "\equiv"] & \\ X_0 \arrow[ur, "x_{01}"] \arrow[rr, bend right, "x_{02}"'] & {} & X_2 \end{tikzcd}}$$

$\varphi = \varphi_{comp}$ for later reference

Separating typings of variables from quantifications:

Context of variables:

$$C :\equiv \left[\begin{array}{l} X_0 : O .\ X_1 : O .\ X_2 : O \\ \quad x_{01} : A(X_0, X_1) .\ x_{12} : A(X_1, X_2) \\ \\ x_{02} : A(X_0, X_2) .\ x_{012} : T(X_0, X_1, X_2, x_{01}, x_{12}, x_{02}) \end{array} \right.$$

formula:

$$\varphi :\equiv \forall X_0\, \forall X_1\, \forall X_2\, \forall x_{01}\, \forall x_{12}\, \exists x_{02}\, \exists x_{012} .\ \mathrm{TRUE}$$

Full form: $C :: \varphi$ "in context C, φ holds"

C "is" the discrete opfibration

$$\begin{array}{c} el(\tilde{T}) \\ p \big\downarrow \\ L \end{array} \qquad \text{where } \tilde{T} = L(T, -) : L \to Set$$

Briefly: $\boxed{C = |\tilde{T}|}$

Let C_0 be the context

obtained by omitting the last two

items: x_{02} and x_{012}

Let $\mathcal{U} : L \to \mathrm{Set}$ be the functor

for which

$$C_0 = |\mathcal{U}|$$

We have the inclusion

$$\mathcal{U} \xrightarrow[i]{} \tilde{T}$$

and for $M \in \mathrm{Str}(L)$,

$M \models \varphi \iff M$ is injective wrt i

$\iff$ for all $\mathcal{U} \xrightarrow{u} M$ there is x:

$$\begin{tikzcd} \mathcal{U} \arrow[d, "i"'] \arrow[dr, "u"] \arrow[dr, phantom, "\equiv" near start, shift right=3] & \\ X \arrow[r, "x"'] & M \end{tikzcd}$$

$u = xi$

<u>Exercise</u>: Write down a FOLDS(L_{cat}) sentence φ such that

$$M(\mathcal{C}) \models \varphi \iff \mathcal{C} \text{ has binary products.}$$

This φ will have a multiple alternation of quantifiers — will not be "equivalent" to an injectivity condition.

(to be read after $\simeq_{L_{cat}}$ has been defined)

<u>Proposition</u> There is a finite set Σ_{cat} of FOLDS(L_{cat}) sentences such that

$$\boxed{M \in \underbrace{Mod(\Sigma_{cat})}_{\text{class of models of } \Sigma_{cat}} \iff \exists C \text{ category } M \simeq_{L_{cat}} M(C).}$$

φ_{comp} above is an element of Σ_{cat}; the particular Σ_{cat} has 12 elements; each $\sigma \in \Sigma_{cat}$ is an injectivity condition on M:

$$\forall u \exists x \quad \begin{tikzcd} U \arrow[d, "i"'] \arrow[dr, "u"] & \\ X \arrow[r, dashed, "x"'] \arrow[ur, phantom, "="] & M \end{tikzcd} \iff M \models \sigma$$

The Proposition more abstractly:

Let $L = L_{cat}$.

For a class Θ of L-structures:

$$\Theta^{\simeq_L} =_{def} \simeq_L\text{-closure of } \Theta :$$

$$N \in \Theta^{\simeq_L} \iff \exists M \in \Theta.\ N \simeq_L M.$$

$$Th_L(\Theta) =_{def} L\text{-theory of } \Theta :$$

for $\sigma \in FOLDS(L)$

$$\sigma \in Th_L(\Theta) \iff \text{for all } M \in \Theta,\ M \models \sigma.$$

Let: $\Theta =_{def} M[Cat] = \{M(C) : C \in Cat\}$.

Then

Proposition*

$$\boxed{M[Cat]^{\simeq_L} = Mod\ Th_L(M[Cat])}$$

$\subseteq$

$\uparrow$ is automatic

since FOLDS is invariant under $\simeq_L$

As a matter of fact, as we know,

Th_L can be replaced by Th_L^{inj}

which involves only sentences that
are equivalent to (finite) injectivity conditions.

The usual 'elementary' conditions on a
category ("having binary products", ...
"being an elementary topos")
can all be axiomatized in L_{cat}. For
instance, for the class ElTop of elementary
toposes, a subclass of Cat, we (also) have

$$\boxed{M[\mathrm{ElTop}]^{\sim L_{cat}} = \mathrm{Mod}\ Th_{L_{cat}}(M[\mathrm{ElTop}])}$$

This requires the obvious translation of
the 'elementary' (first-order) conditions
for elementary toposes into FOLDS(L_{cat}).

Let $\mathrm{Mod}(\Sigma_{cat})$ denote (also) the full subcategory of Set^L whose objects are the models of Σ_{cat}. Temporarily, we write C^* for $M(C)$. We have the functor $(-)^* : \mathrm{Cat} \longrightarrow \mathrm{Mod}(\Sigma_{cat})$ as it is easily seen

Proposition There is a functor

$$\begin{aligned}(-)^\# : \mathrm{Mod}(\Sigma_{cat}) &\longrightarrow \mathrm{Cat}\\ M &\longmapsto M^\#\end{aligned}$$

such that:

$$(-)^{*\#} \simeq 1_{\mathrm{Cat}} \circlearrowright \mathrm{Cat} \underset{(-)^*}{\overset{(-)^\#}{\leftrightarrows}} \mathrm{Mod}(\Sigma_{cat}) \circlearrowleft \simeq_L (-)^{\#*}$$

$$\boxed{C^{*\#} \simeq C} \qquad\qquad \boxed{M \simeq_{L_{cat}} M^{\#*}}$$

In particular: $\boxed{C \simeq D \iff C^* \simeq_L D^* \iff M(C) \simeq_{L_{cat}} M(D)}$