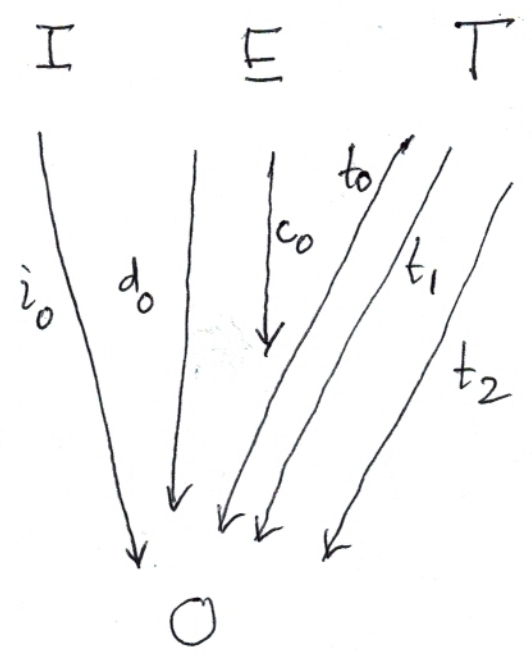
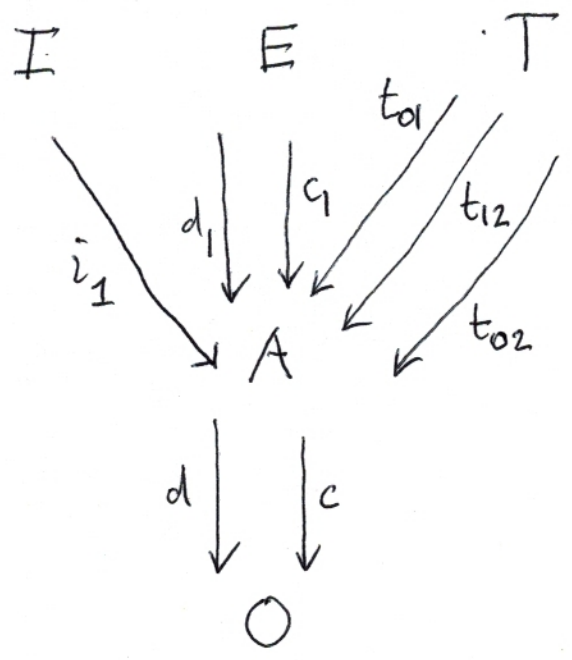


The FOLDS language for categories

$L_{cat}$   $\square$

Signature  $L_{cat}$ :



$$i_0 = di_1 = ci_1$$

$$d_0 = dd_1 = dc_1$$

$$c_0 = cd_1 = cc_1$$

$$t_0 = dt_{01} = dt_{02}$$

$$t_1 = ct_{01} = dt_2$$

$$t_2 = ct_{02} = ct_{12}$$

$C$  : category
 $\rightsquigarrow$   
 re-coding

$$M(C) : L_{\text{cat}} \rightarrow \text{SET}$$

 $M = M(C) :$ 

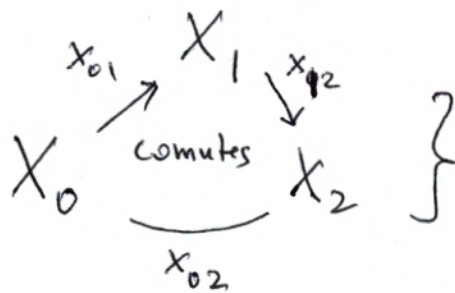
$$M(O) = \text{ob}(C)$$

$$M(A) = \text{Arr}(C) = \{(X, Y, f) \mid f: X \rightarrow Y \text{ in } C\}$$

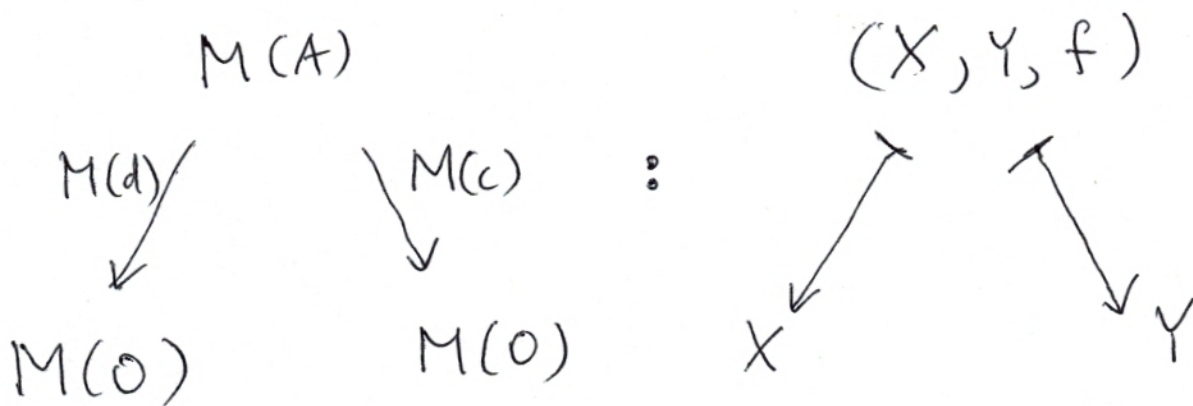
$$M(I) = \{1_X \mid X \in \text{ob}(C)\} = \{(X, f) \mid f = 1_X\}$$

$$M(E) = \{(X, Y, f_1, f_2) : X \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} Y \text{ and } \underbrace{f_1 = f_2}\}$$

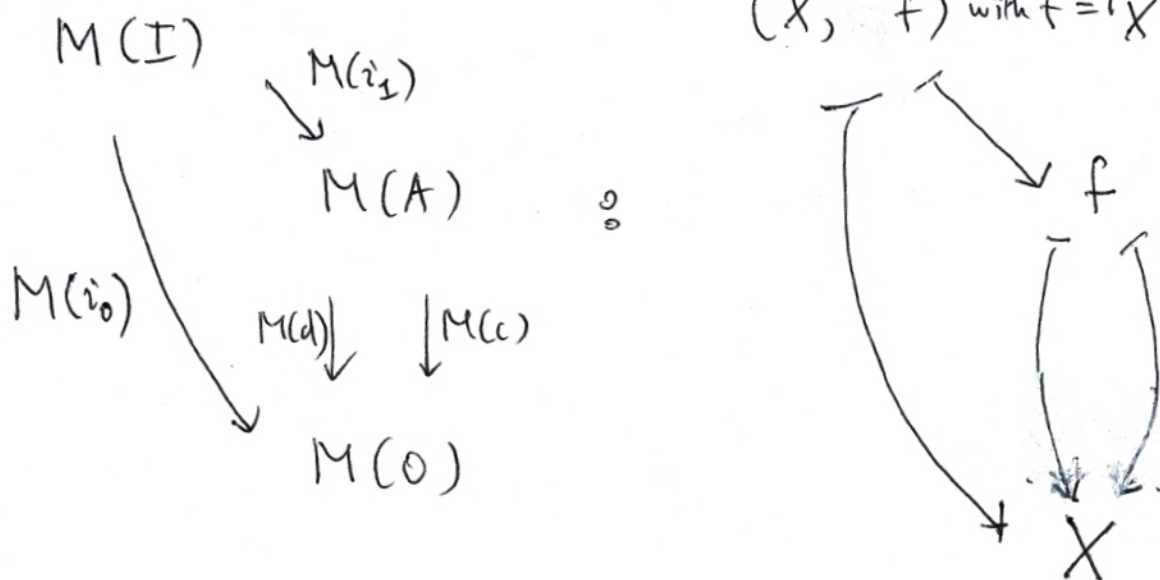
$$M(T) = \{(X_0, X_1, X_2, x_{01}, x_{12}, x_{02}) \mid$$



$M(d), M(c) :$



$M(i_0), M(i_1) :$



Example for  $\varphi \in FOLDS(L_{cat})$ :

"existence of composite of composable arrows"

$\varphi :=$

$$\forall X_0: O. \forall X_1: O. \forall X_2: O$$

$$\forall x_{01}: A(X_0, X_1). \forall x_{12}: A(X_1, X_2)$$

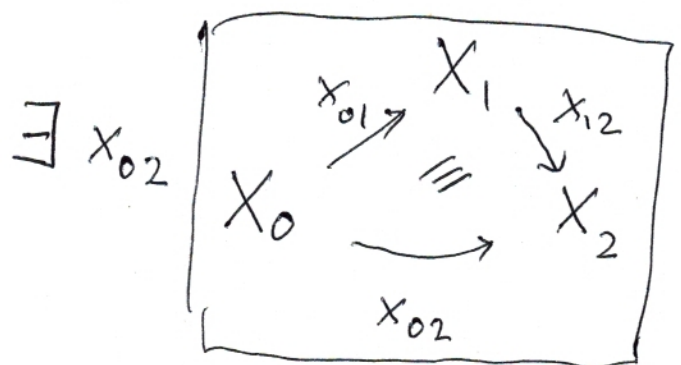
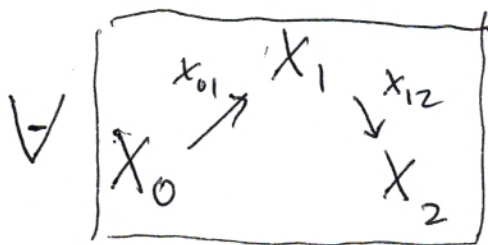
$\boxed{\begin{array}{cc} d & c \end{array}}$

$$\exists x_{02}: A(X_0, X_2). \exists x_{012}: T(X_0, X_1, X_2, x_{01}, x_{12}, x_{02}). \text{TRUE}$$

$\uparrow$   
 witness  
 for commutativity

$\boxed{(t_0 \ t_1 \ t_2 \ t_{01} \ t_{12} \ t_{02})}$

$M(C) \models \varphi \iff$  in terms of  $C$ :



$\varphi = \varphi_{comp}$  for later reference

Separating typings of variables  
from quantifications:

Context of variables:

$$C := \left[ \begin{array}{l} X_0 : 0, X_1 : 0, X_2 : 0 \\ x_{01} : A(X_0, X_1), x_{12} : A(X_1, X_2) \\ x_{02} : A(X_0, X_2), x_{012} : T(X_0, X_1, X_2, x_{01}, x_{12}, x_{02}) \end{array} \right.$$

formula:

$$\varphi := \forall X_0 \forall X_1 \forall X_2 \forall x_{01} \forall x_{12} \exists x_{02} \exists x_{012} \cdot \text{TRUE}$$

Full form:  $C :: \varphi$  "in context  $C$ ,  $\varphi$  holds"

$C$  "is" the discrete opfibration

$$\text{el}(\tilde{T})$$



where  $\tilde{T} = L(T, -)$   
:  $L \rightarrow \text{Set}$

Briefly:  $C = |\tilde{T}|$

Let  $C_0$  be the context

obtained by omitting the last two items:  $x_{02}$  and  $x_{012}$

Let  $U: L \rightarrow \text{Set}$  be the functor

for which

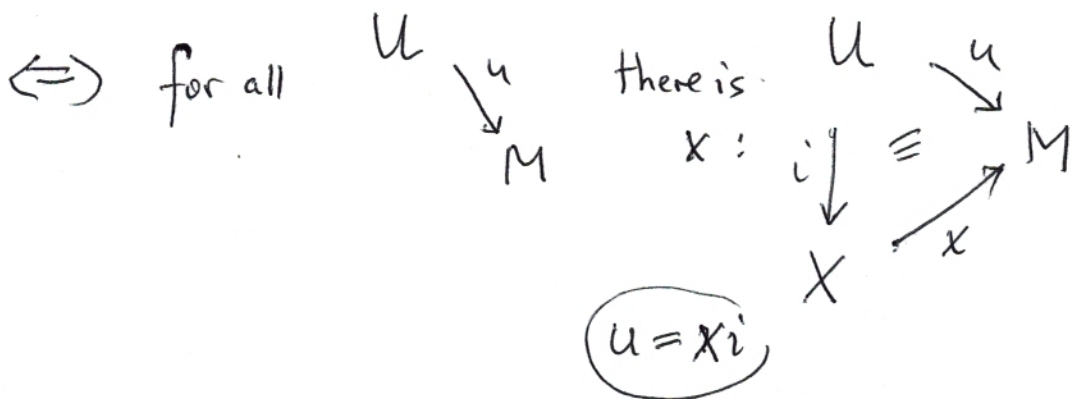
$$C_0 = |U|$$

We have the inclusion

$$U \xrightarrow{i} \tilde{T}$$

and for  $M \in \text{Str}(L)$ ,

$M \models \varphi \iff M$  is injective wrt  $i$



$L_{\text{cat}} [7]$ 

Exercise: Write down a

$\text{FOLDS}(L_{\text{cat}})$  sentence  $\varphi$  such

that

$M(C) \models \varphi \iff C$  has

binary products.

This  $\varphi$  will have a multiple alternation of quantifiers — will not be "equivalent" to an injectivity condition.



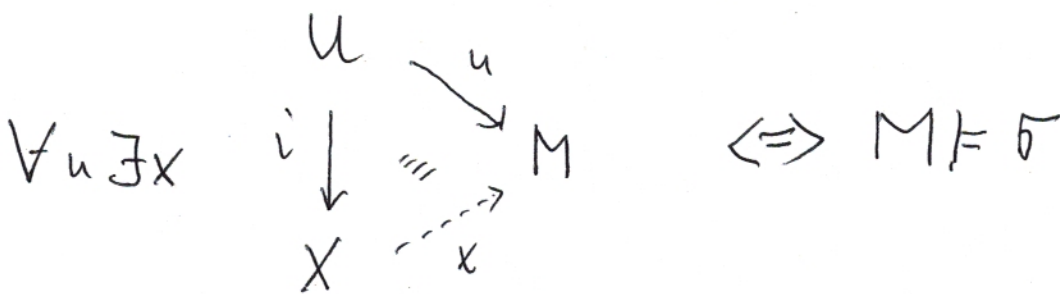
$L_{cat}$  [8]

(to be read after  $\approx_{L_{cat}}$  has been defined)

Proposition There is a finite set  $\Sigma_{cat}$  of FOLDS( $L_{cat}$ ) sentences such that

$$\underbrace{M \in \text{Mod}(\Sigma_{cat})}_{\substack{\text{class of models} \\ \text{of } \Sigma_{cat}}} \Leftrightarrow \exists C \text{ category} \\ M \approx_{L_{cat}} M(C).$$

$\varphi_{comp}$  above is an element of  $\Sigma_{cat}$ ; the particular  $\Sigma_{cat}$  has 12 elements; each  $\sigma \in \Sigma_{cat}$  is an injectivity condition on  $M$ :





The Proposition more abstractly:

Let  $L = L_{cat}$ .

For a class  $(\mathcal{H})$  of  $L$ -structures:

$$(\mathcal{H}) \stackrel{\approx_L}{=} \stackrel{\text{def}}{=} \approx_L\text{-closure of } (\mathcal{H}) :$$

$$N \in (\mathcal{H}) \stackrel{\approx_L}{=} \Leftrightarrow \exists M \in (\mathcal{H}), N \approx_L M.$$

$$Th_L(\mathcal{H}) \stackrel{\text{def}}{=} L\text{-theory of } (\mathcal{H}) :$$

for  $\sigma \in \text{FOLDS}(L)$

$$\sigma \in Th_L(\mathcal{H}) \Leftrightarrow \text{for all } M \in (\mathcal{H}), M \models \sigma.$$

$$\text{Let : } (\mathcal{H}) \stackrel{\text{def}}{=} M[\text{cat}] = \{M(C) : C \in \text{Cat}\}.$$

Then

Proposition\*

$$M[\text{cat}] \stackrel{\approx_L}{=} \text{Mod } Th_L(M[\text{cat}])$$

$\subseteq$

$\uparrow$  is automatic

Since FOLDS is invariant under  $\approx_L$

As a matter of fact, as we know,

$Th_L$  can be replaced by  $Th_L^{inj}$

which involves only sentences that are equivalent to (finite) injectivity conditions.

The usual 'elementary' conditions on a category ("having binary products", or "being an elementary topos")

can all be axiomatized in  $L_{cat}$ . For

instance, for the class  $ElTop$  of elementary toposes, a subclass of  $Cat$ , we (also) have

$$M[ElTop] \stackrel{\sim}{=}_{L_{cat}} Mod Th_{L_{cat}}(M[ElTop])$$

This requires the obvious translation of the 'Elementary' (first-order) conditions for elementary toposes into  $FOLDS(L_{cat})$ .

Let  $\text{Mod}(\Sigma_{\text{cat}})$  denote (also)  $\text{L}_{\text{cat}} \square$

the full subcategory of  $\text{Set}^{\text{L}}$  whose objects are the models of  $\Sigma_{\text{cat}}$ . Temporarily,

we write  $C^*$  for  $M(C)$ . We have

the functor  $(-)^* : \text{Cat} \rightarrow \text{Mod}(\Sigma_{\text{cat}})$

as it is easily seen

Proposition There is a functor

$$\begin{aligned} (-)^{\#} : \text{Mod}(\Sigma_{\text{cat}}) &\longrightarrow \text{Cat} \\ M &\longmapsto M^{\#} \end{aligned}$$

such that :

$$\begin{array}{ccc} (-)^{\#*} \circlearrowleft \left( \begin{array}{c} \cong \\ \downarrow \\ \text{Cat} \end{array} \right) & \begin{array}{c} \xleftarrow{(-)^{\#}} \\ \xrightarrow{(-)^*} \end{array} & \text{Mod}(\Sigma_{\text{cat}}) \begin{array}{c} \xrightarrow{\cong} \\ \downarrow \\ \text{L} \end{array} \\ & & \begin{array}{c} \xleftarrow{(-)^{\#*}} \end{array} \end{array}$$

$$\boxed{C^{\#*} \cong C}$$

$$\boxed{M \cong_{\text{L}_{\text{cat}}} M^{\#*}}$$

In particular :

$$\boxed{C \cong D \Leftrightarrow C^* \cong_{\text{L}} D^* \Leftrightarrow M(C) \cong_{\text{L}_{\text{cat}}} M(D)}$$