

First-order logic with dependent sorts

FOLDS (I)

Terminology: L a category, K, K_p, \dots objects of L
(K for 'kind')

$\dim(K)$ = largest $n \in \mathbb{N}$ such that

there exists:

$$K = K_n \xrightarrow[\neq 1]{f_n} K_{n-1} \xrightarrow[\neq]{f_{n-1}} \dots \xrightarrow[\neq 1]{f_1} K_0$$

of proper (= non-identity) arrows f_i .

If no such n , $\dim(K) = \infty$.

L is a (FOLDS-) signature if for all $K \in \text{Ob}(L)$:

1) $\dim(K) < \infty$

2) $\tilde{K} = L(K, -) : L \rightarrow \text{Set}$ (covariant representable)

is a finite functor ($\text{el}(\tilde{K})$, the category of elements of \tilde{K}) has finitely many objects

Consequences: L is 1-way: $\text{End}(K) = \{1_K\}$

and more generally

$$K \xrightarrow[\neq 1]{p} K_p \text{ proper} \Rightarrow \dim(K) > \dim(K_p)$$

L -structure : $M: L \rightarrow \text{Set}$ functor

(or: $M: L \rightarrow \text{SET}$)

We write $\text{Str}(L)$ for the functor category Set^L

Three examples to be discussed:

L_{cat}

'cat' for 'category'

L_{absset}

'absset' for 'abstract set'

L_{sanafun}

'sanafun' for 'sana-functor'
(saturated anafunctor)

Re-coding turns a classical structure into an L -structure:

\mathbb{C} category $\longmapsto M(\mathbb{C}) \in \text{Str}(L_{\text{cat}})$

\underline{A} concrete category (discrete iso fibration $\begin{matrix} A \\ \downarrow a \\ \text{Set} \end{matrix}$) $\longmapsto M(\underline{A}) \in \text{Str}(L_{\text{absset}})$

free-living functor $X \xrightarrow{F} A$

$\longmapsto M(F) \in \text{Str}(L_{\text{sanafun}})$

L - equivalence

Equ II

(L, K, K', ... as before)

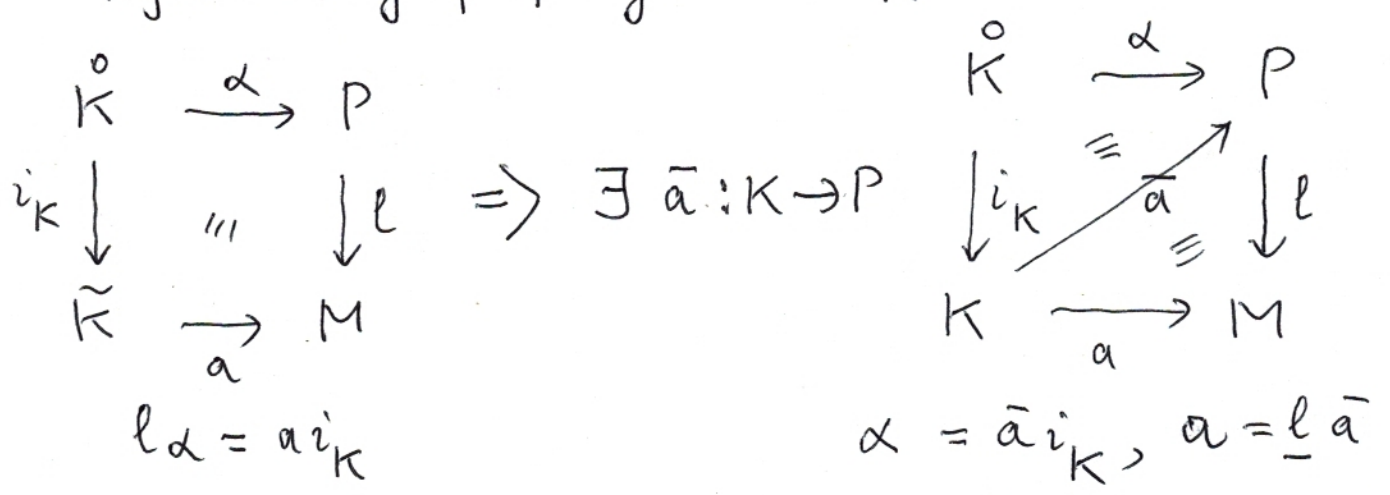
$K^{\circ} : L \rightarrow \text{Set}$ is the functor, subfunctor of $\tilde{K} = L(K, -)$, for which $\text{ob}(el(K^{\circ})) = \text{ob}(el(\tilde{K})) = \{(K, 1_K)\}$

$K^{\circ} \xrightarrow{i_K} \tilde{K}$: "sphere into ball" inclusion

Let $P, M \in \text{str}(L)$, $l : P \rightarrow M$ (nat. transf.)

Definition l is fiberwise surjective (FS)

if, for all $K \in \text{ob}(L)$, l has the right lifting property wrt i_K



I'd say "trivial fibration" if there were "fibrations",
 (fibrations may come later)

EQU 1.1

As a consequence, if l is FS
 it has the RLP wrt to all monomorphisms
 in $\text{Str}(L)$:

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & P \\
 \downarrow i \text{ mono} & \llcorner & \downarrow l \\
 Y & \xrightarrow{a} & M
 \end{array}
 \Rightarrow \exists \bar{a} : \begin{array}{ccc}
 X & \xrightarrow{\alpha} & P \\
 \downarrow i & \swarrow \bar{a} & \downarrow l \\
 Y & \xrightarrow{a} & M
 \end{array}$$

This is because the class of monomorphisms is
 the Gabriel-Zisman saturation of the sphere-inclusions.

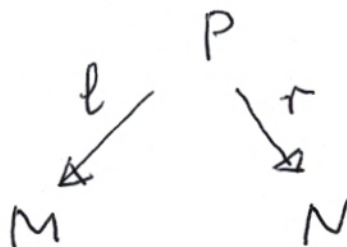
Let $M, N \in \text{Str}(L)$:

Equ [2]

An L -equivalence \underline{P} of M and N

$$\underline{P} : M \simeq_L N$$

$\underline{P} = (P, \ell, \tau)$



with both ℓ and τ being FS.

$$M \simeq_L N \stackrel{\text{def}}{\Leftrightarrow} \exists \underline{P} : \underline{P} : M \simeq_L N$$

More generally: let $X \in \text{Set}^L$, and

$$M \xleftarrow{\alpha} X \xrightarrow{\beta} N$$

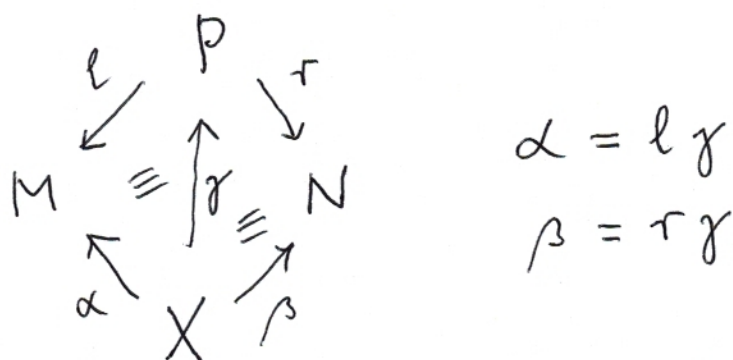
Usually, X will be finite. X is a context/system of typed variables, α is an evaluation of X in M , similarly for β . We also say: α is an X -element of M .

Write: $\underline{P} : (M, \alpha) \simeq_L (N, \beta)$ if

$$\underline{P} : M \simeq_L N$$

and (next page)

... and there exists $\gamma: X \rightarrow P$ s.t.



$(M, \alpha), (N, \beta)$ = "augmented structures"

Fact: \cong_L is an equivalence relation on augmented structures (plain structures included: $X = \emptyset$)

Equivalence transfer:

Suppose $M \xleftarrow{\alpha} X \xrightarrow{\beta} N$

$$(M, \alpha) \cong_L (N, \beta)$$

$X \xrightarrow{i} Y$ monomorphism

$Y \xrightarrow{\bar{\alpha}} M$ extending α : $\alpha = \bar{\alpha}i$

Then: there is $Y \xrightarrow{\bar{\beta}} N$ extending β : $\beta = \bar{\beta}i$

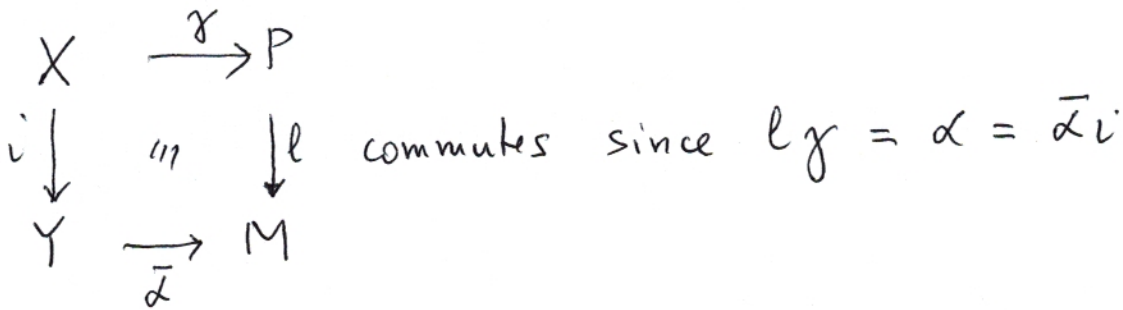
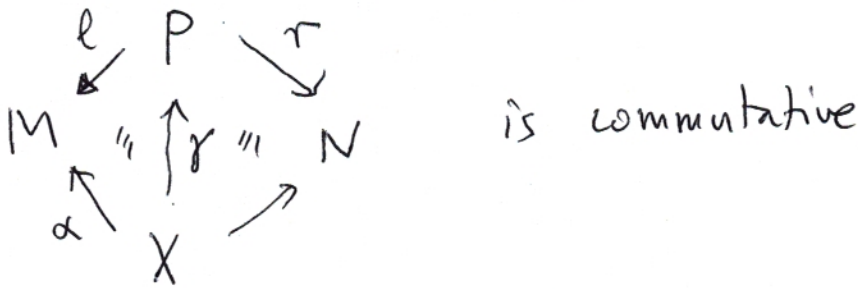
such that $(M, \bar{\alpha}) \cong_L (N, \bar{\beta})$.

because: we have

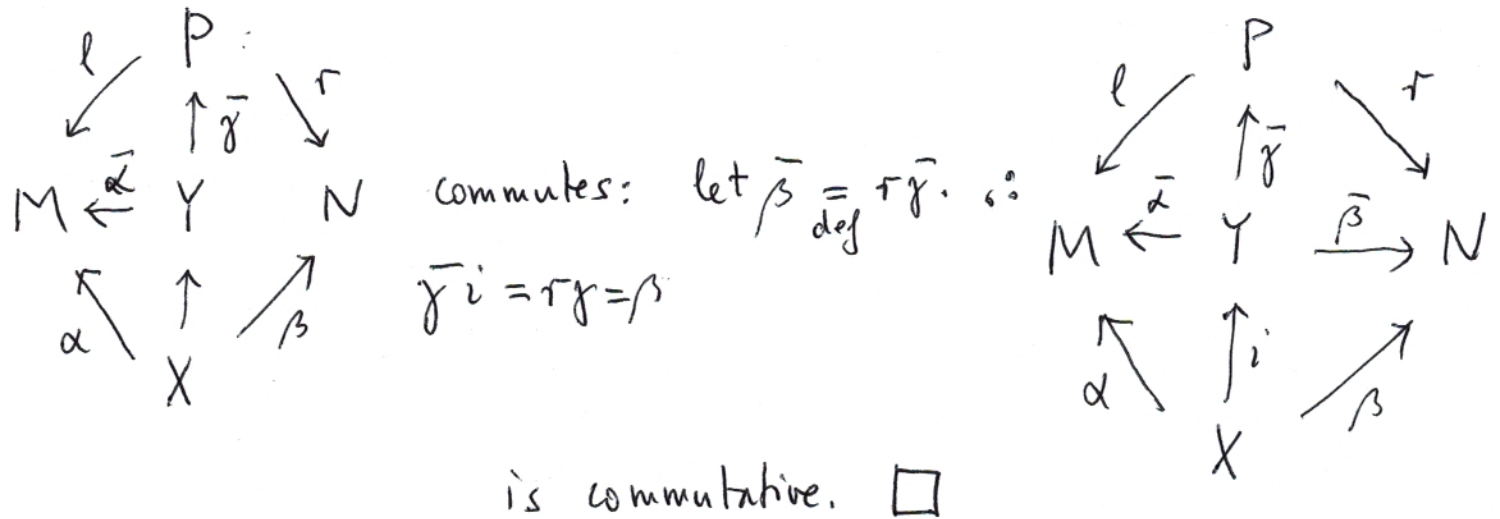
$$(P, l, r) : (M, \alpha) \cong_L (N, \beta)$$

and

$$X \xrightarrow{\gamma} P \text{ such that}$$



\therefore since i is a mono, there is a diagonal $\bar{\gamma}: Y \rightarrow P$ such that $\gamma = \bar{\gamma}i$ & $\bar{\alpha} = l\bar{\gamma}$.



Equivalence transfer is used

to show the soundness of L-equivalence

wrt FOLDS properties of (augmented) structures: for a FOLDS formula $\varphi(X)$,

$$(M, \alpha) \cong_L (N, \beta) \ \& \ M \models \varphi[\alpha/X]$$

$$\Rightarrow N \models \varphi[\beta/X]$$

(with $i: X \rightarrow Y$, think of the quantifiers

\exists_i, \forall_i ; see later!)

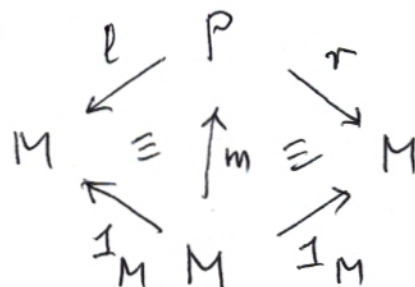
Intrinsic L-equivalence

L : FOLDS signature, M : L -structure

The self-equivalence $\underline{P} = (P, \ell, r): M \cong_L M$ of M

extends the identity if there is $m: M \rightarrow P$

such that $\ell m = r m = 1_M$:



The X -elements α, β of M :

EQU [6]

$$X \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} M$$

are intrinsically (L-)equivalent if

there exists $\underline{P} : (M, \alpha) \cong_L (M, \beta)$ extending the identity.

Notation: $\underline{P} : \alpha \underset{\text{int}}{\cong} \beta$, $\alpha \underset{\text{int}}{\cong} \beta$.

In the 'usual' cases, intrinsic equivalence is the "expected" relation.

For C a category, x, y objects of C

$$x \underset{\text{int}}{\cong} y \text{ in } M(C) \in \text{Str}(L_{\text{cat}})$$

iff $x \cong y$ (isomorphism) in C .

More generally, if X is a graph, or even a 'category sketch' (with some triangles in the graph marked commutative, some arrows as identities)

then $X \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} C$ are intrinsically equivalent in $M(C)$

(a structure over L_{cat}) iff they are isomorphic

as objects of the diagram category \mathcal{C}^X .

For any signature L , $X \xrightarrow{\alpha} M$
 $\xrightarrow{\beta}$

and $\varphi(X)$ a FOLDS(L)-formula

Then: $\alpha \underset{\text{int}}{\cong} \beta$ & $M \models \varphi[\alpha/X] \Rightarrow M \models \varphi[\beta/X]$.

as a special case of the soundness of L -equivalence.

In the case of $L = L_{\text{absset}}$ and $M \in \text{Str}(L)$

being a model of the

"minimal theory of abstract sets", Σ_{min}

(essentially, $M \underset{L}{\cong} M(\underline{A})$ for a concrete

category $\underline{A} = (A, a: A \rightarrow \text{Set})$)

$\alpha \underset{\text{int}}{\cong} \beta$ is the same as isomorphism:

$$\alpha \cong \beta$$

Thus: $\alpha \cong \beta$ & $M \models \varphi[\alpha/X] \Rightarrow M \models \varphi[\beta/X]$

"Abstract set theory is Bourbakian"