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FOLDS, first-order logic with dependent sorts is intended to be a framework for a formalization, and the meta-theory of that formalization, of *structuralist mathematics*.

Ordinary first-order logic (FOL) provides the language for axiomatic set-theory, the accepted basis for pure mathematics nowadays. On the other hand, FOL has a model-theory where *isomorphism* of models is the operative identity concept: in the model theory of even extensions of first-order logic, infinitary or higher-order, two isomorphic structures cannot be distinguished by model-theoretic properties. The two sides of FOL, namely the foundational side and the model-theoretic side, come together in axiomatic set theory, where the interest is in models – sometimes in models in toposes other than the category of sets, e.g. Boolean valued models – of various combinations of set-theoretic axioms.

FOLDS is intended to serve in both roles: the foundational and the model-theoretical.

FOLDS is a specialized first-order logic: every FOLDS formula is equivalent to a multi-sorted first-order formula – but not conversely. FOLDS has a semantics that is simply inherited from first-order logic. As a result, the general model-theoretic facts of compactness and Loewenheim-Skolem theorems remain (automatically) true for FOLDS. The difference to ordinary first-order logic is that in FOLDS we require invariance of all defined predicates under the new notion of *FOLDS equivalence*. FOLDS equivalence is a parametrized family of so-called *L-equivalences*, one for each *FOLDS signature* *L*. In each case of a FOLDS signature *L*, it is meaningful to talk, also, about isomorphism of *L*-structures; but there are more *L*-equivalences than isomorphisms, and invariance under *L*-equivalence is, thus, a more stringent requirement than isomorphism invariance.

Bourbaki not only practiced structuralist mathematics (SM), but also gave a prescriptive *definition* for it. SM, first of all, is to be based on formal, axiomatic, set theory. The essential part of the prescription is that SM is based on a system of *species of structures*, with each species *S* given by an explicit set-theoretic definition. It is required that the definition of *S* be such that it is possible to *transfer a structure* *M* by a bijection *f* of the underlying set *A* of *M* to another set *B* (or a system of bijections *f1, f2* of underlying sets *A1, A2...* to *B1, B2* … respectively if there are more than one underlying set) producing a (uniquely defined) new structure *N* of species *S* with underlying set *B* which is *isomorphic to* *M*, by the *isomorphism* mapping *f* .

[Think of two examples:

1. Groups. Let **G** = *(G,m:GxG---->G)* be a group, *H* a set, and *f:G--->H* a bijection. Then there is a unique group **H** = (*H,n:HxH--->H)* such that the mapping *f:G--->H* becomes an isomorphism *f:***G–-->**H.

2. Topological spaces. Let **X** be a topological space, *X* its underlying set of points, *Y* another set, *f:X--->Y* a bijection. Then there is a unique topological space **Y** whose underlying set of points is *Y,* and such that *f:X---->Y* becomes an isomorphism = homeomorphism *f:***X---->Y**.]

Further, only those properties of structures of a given species are allowed in the theory which are *invariant under the isomorphism* notion of the species of S. As a result, one cannot (in polite company) talk about *identity*, sameness, of two "free-living" structures in a more stringent sense than the *isomophism* of them, by force of the requirement (Leibniz's rule) that tsructures M and N are identical *if* (and only if) they share all *meaningful* properties –- since now the meaningful ones are only the isomorphism-invariant ones. I put in the "free-living" proviso above, since, for instance, the *equality* in the usual sense of two subgroups of a given group remains, of course, an allowable predicate. A subgroup of a group is not just a group: it is equipped with the additional structure of its embedding in the container group.

Bourbaki's prescriptions had been instinctively followed by most of the practice of pure mathematics starting in early 20th century well before Bourbaki's definition of "structure" --- similarly to axiomatic set theory, before the latter's explicit statement. But since in (axiomatic) set theory (say ZFC), a randomly chosen property of a structure, say a group, (example: "the natural number 5 is an element of the group") is not isomorphism invariant, and constructions such as the union or intersection of two sets are not isomorphism-invariant, Bourbaki's prescription is far from empty.

FOLDS provides an "adequate" re-statement – in fact, at least two such re-statements – of axiomatic set-theory in which *all* properties of structures are automatically invariant under isomorphism. One such will be our second example: abstract set theory in the FOLDS language over the signature L-sub-absset. The other one is the "automatic" restatement of Bill Lawvere's "First-order theory of the category of sets", combined with Lawvere's topos theory, in the FOLDS language of categories, the latter being our first example.

With the advent of category theory, Bourbaki's structuralism became too narrow. To begin with, for categories, isomorphism turned out to be too restrictive. For most purposes, when categories *C* and *D* are *equivalent* – in the usual technical sense that is weaker than isomorphism --, we may regard them to be *identical* for most purposes. These purposes involve constructions *in C* and constructions involving *C* as *a whole*.

Example for the latter: the functor categories *(C,Set)* and *(D,Set)* are equivalent iff the idempotent completions of *C* and *D* are equivalent. This example involves two equivalence-invariant constructions, and their relation expressed through equivalence of categories. It would be silly to try to “improve” this result into something formulated in terms of isomorphism of categories.

If *C* and *D* are equivalent categories, then all the *isomorphism invariant* constructions in C, in the sense of “isomorphism” *in C* in the category-theoretical sense, can be *transferred to D*, and isomorphism-invariant properties of those constructions will be *preserved* for the transferred construction. An example of such a construction, one of a huge number of categorical constructions, is the binary product *(XxY, pi-0, pi-1)* of two objects *X* and *Y*.

As we will see, the capability of transferring content from one *L-structure C* to another *L*-structure *D* is at the heart of the general notion of “*L-equivalence*”. Here “*L*-structure” is a species of structures, taken as basic in FOLDS, depending on a so-called FOLDS-*signature* (similarity type) *L*. As to the example of categories, they will be regarded, or *re-coded,* as *L*-structures for a specific FOLDS-signature *L-sub-cat*, treated as our first example.

Further up from categories, we have 2-categories, bi-categories, and various kinds of *n*-categories, and even omega-categories. FOLDS is intended to, eventually, be relevant to many of these species. For instance, in the case of quasi-categories, since the “signature” of simplicial sets (the simplex category DELTA-opp) is not a FOLDS signature, a re-coding is needed to see them as *L*-structures for a FOLDS-signature *L*. In his case, the recoding is a map *X*|----> *M(X)* of a quasi-category *X* into an *L*-structure *M(X)*. On the other hand, this recoding can be usually abandoned because the FOLDS notion we want– for instance, *L*-equivalence for the re-coded *M(X)* – can be equivalently restated in the original language for the quasi-category *X*.

FOLDS was defined a little more than 25 years ago. There is only a few official publications on the subject. There are additional papers in manuscript form on my website. The present talk is only an elementary introduction to the basic concepts and the simplest examples. After the definition of *L*-equivalence in the pdf “Bohemian talk 3”, the part entitled “The Lindstroem context for FOLDS” is completely self-contained as far the definitions and statements of assertions are concerned; even the syntax of FOLDS is not needed.