Intrinsic and internal equivalence in structures for a FOLDS signature

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Introductory words

If you find, in the Introduction, or in the body of the write-up, notation or concepts with insufficient explanation, consult the slides for my Bohemian Cafe talk, especially the set "First-order logic with dependent sorts"; I'll refer to this as [B3]. There will be some slight changes with respect to [B3]. In [B3], I write $\tilde{K}$ for the covariant representable functor $L(K,-):L\to Set$; here I've changed this to $\hat{K}=L(K,-):L\to Set$. Notation involving the symbol $\simeq$ has been simplified by writing $\sim$ in place of $\simeq$; for instance, intrinsic equivalence is written as $\alpha\underset{int}{\sim}\beta$ in place of $\alpha\underset{int}{\simeq}\beta$. In [B3], the equivalence span $\underline{P}=(P,\ell,r):M\underset{L}{\sim}M$ is said to extend the identity if there is $m:M\to P$ such that $\ell m=rm=1_M$. Now I say for the same condition that $\underline{P}$ is a reflexive self-equivalence, RSE for short, of M. Also, I make a change in the definition, the new one remaining equivalent to the old one, however. Now I say that $\underline{P} = (P, \ell, r) : M \underset{L}{\sim} M$ is reflexive if there exist $Q \in \mathrm{Set}^L$, $q : Q \to P$, $m : Q \to M$ such that

$$\begin{tikzcd} & P \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ M \arrow[r, phantom, "\equiv"] \arrow[dr, equal] & Q \arrow[u, "q"] \arrow[d, "m"] \arrow[r, phantom, "\equiv"] & M \\ & M \arrow[ur, equal] & \end{tikzcd}$$

$$\ell q = r q = m$$

and m is fiberwise surjective (FS). This is equivalent to the original definition: if the condition of the original definition holds, with $s : M \to P$ (instead of $m : M \to P$ in [B3]), $\ell s = r s = 1_M$, then I can take $Q = M$, $m = 1_M$ and $q = s$ for the new definition. In the opposite direction, using that m is FS in the strong sense – having the right lifting property with respect to all monomorphisms, not just the sphere inclusions, we have $t : M \to Q$ such that $mt = 1_M$ (see

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Q \arrow[d, "m"] \\ M \arrow[r, equal] \arrow[ur, dashed, "t"] & M \end{tikzcd}$$

we can take $s = qt : M \to P$, and have $\ell s = r s = 1_M$. Without the axiom of choice though, the new definition of reflexivity is weaker, and therefore it is better; it is easier to verify the new condition. (Note that for a finite monomorphism $U \overset{f}{\rightarrowtail} V$, the right lifting of a map with respect to f is a consequence of same with respect to the sphere inclusions $i_K : \mathring{K} \to \hat{K}$ $(K \in L)$ in a constructive manner, without choice.)

The category theory used in the paper is confined to the Yoneda theory around the fixed category L. Initially, L is an arbitrary small category. Eventually, a condition is imposed on L, the existence of a sphere functor (see page $\boxed{40}$), a condition that all FOLDS signatures satisfy, and so does the famed category $L = \Delta^{op}$ (with Set^L the category of simplicial sets). In the introduction proper that follows, some details will be relegated to the Appendix.

Introduction

For the introduction, unlike the body of the paper, L is a fixed FOLDS signature (see [B3]). To illustrate matters, I will take L to be L_{cat} - see the Bohemian Cafe set [B2] "The FOLDS signature for categories". For a while, I also fix an L-structure M; $M: L \to Set$. For illustration, we consider a small category $\mathbb{C}$, and the corresponding re-coded version of $\mathbb{C}$, $M = M(\mathbb{C}) : L_{cat} \to Set$ (see [B2]. We will be dealing with variable reflexive self-equivalences $\underline{P} = (P, \ell, r) : M \underset{L}{\sim} M$ (the simplest and always available one is the 'identity' $\underline{Id}_M = (M, 1_M, 1_M)$), and with variable diagram types $\mathcal{U} : L \to Set$. (For illustration, in case $L = L_{cat}$, think of finite diagrams in the category $\mathbb{C}$: an ordinary graph is, automatically, a diagram type $\mathcal{U} : L_{cat} \to Set$ with all sets $\mathcal{U}(K)$, $K \in L_{cat}$ except $K = 0$ and $K = A$ empty; and, a diagram in $\mathbb{C}$ is the same as a diagram $\alpha : \mathcal{L} \to M(\mathbb{C})$ in this case.)

For a specific $\underline{P}$ and a specific $\mathcal{U}$ as above, we define the binary relation $\mathrm{Eq}[M,\underline{P},\mathcal{U}]$ abbreviated as $\mathrm{Eq}[\underline{P},\mathcal{U}]$ with M fixed, on the set $\mathbf{Set}^{L}(\mathcal{U},M)$ of all $\mathcal{U}$-type diagrams in M, as follows: for $\alpha,\beta:\mathcal{U}\to M$

$$(\alpha,\beta)\in\mathrm{Eq}[\underline{P},\mathcal{U}] \iff \text{there exists } \gamma:\mathcal{U}\to P$$

such that $\alpha=\ell\gamma$, $\beta=r\gamma$:

$$\begin{tikzcd} & P \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ M & & M \\ & \mathcal{U} \arrow[ul, "\alpha"] \arrow[uu, "\gamma"] \arrow[ur, "\beta"'] & \end{tikzcd}$$ commutes

Most of the time, we write $\alpha \underset{P}{\sim} \beta$ for $(\alpha,\beta)\in\mathrm{Eq}[\underline{P},\mathcal{U}]$ (even before we know that $\mathrm{Eq}[\underline{P},\mathcal{U}]$ is an equivalence relation!).

For $\underline{P}_i=(P_i,\ell_i,r_i)$, $i=0$ and $i=1$, both RSE's on M I say that $\underline{P}_1$ extends $\underline{P}_0$ if there are $Q\in\mathbf{Set}^L$, $q:Q\to P_1$, $m:Q\to P_0$ such that m is FS, and $\ell_1q=\ell_0m$, $r_1q=r_0m$: the diagram

$$\begin{tikzcd} & P_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & \\ M & Q \arrow[u, "q"] \arrow[d, "m"] & M \\ & P_0 \arrow[ul, "l_0"] \arrow[ur, "r_0"'] & \end{tikzcd}$$ commutes.

The simpler condition: there is $s : P_0 \to P_1$ such that $l_1 s = l_0$, $r_1 s = r_0$:

$$\begin{tikzcd} & P_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & \\ M & & M \\ & P_0 \arrow[uu, "s"] \arrow[ul, "l_0"] \arrow[ur, "r_0"'] & \end{tikzcd}$$ commutes

is equivalent non-constructively as before.

The very definition of RSE says that every RSE extends $\underline{\mathrm{Id}}_M$.

It is immediate that if $\underline{P}_1$ extends $\underline{P}_0$, then

$$\mathrm{Eq}[\underline{P}_0, u] \subseteq \mathrm{Eq}[\underline{P}_1, u] \quad (u \in \mathrm{Set}^L)$$

(See Appendix)

A simple but obviously basic fact is that for any two RSE's on M, $\underline{P}_i = (P_i, \ell_i, r_i)$, $i=0$ and $i=1$, there is a third one $\underline{P} = (P, \ell, r)$ that extends both $\underline{P}_0$ and $\underline{P}_1$. In fact, $\underline{P}$ can be taken to be the span-composite $\underline{P} = \underline{P}_0 \circledast \underline{P}_1$:

$$\begin{tikzcd}
 & & P \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[ddll, bend right, "\ell"'] \arrow[ddrr, bend left, "r"] \arrow[d, phantom, "\vee"] & & \\
 & P_0 \arrow[dl, "\ell_0"'] \arrow[dr, "r_0"] & {} & P_1 \arrow[dl, "\ell_1"'] \arrow[dr, "r_1"] & \\
M & & M & & M
\end{tikzcd}$$

the same as the one that is used in showing that FOLDS equivalence in general is a transitive relation. See the Appendix.

Let $\mathcal{U} \in \mathbf{Set}^L$. $\mathrm{EqInt}[M, \mathcal{U}]$ $(= \mathrm{EqInt}[\mathcal{U}])$, intrinsic equivalence for $\mathcal{U}$-type diagrams in M, is the binary relation on $\mathbf{Set}^L(\mathcal{U}, M)$ defined as the union of all $\mathrm{Eq}[\underline{P}, \mathcal{U}]$, with $\underline{P}$ ranging over all RSE's on M: Writing $\alpha \underset{\mathrm{int}}{\sim} \beta$ for $(\alpha, \beta) \in \mathrm{EqInt}[\mathcal{U}]$, we have:

$$\alpha \underset{\mathrm{int}}{\sim} \beta \iff \text{there is } \underline{P}\text{, RSE on } M \text{ such that } \alpha \underset{\underline{P}}{\sim} \beta.$$

In a sequel to this write-up, I will show a (more) general calculus of composition of RSE's that can be used to prove the following:

for any structure $M \in \mathbf{Set}^{L}$, we have an "almost canonically defined" single RSE on M, call it $\underline{P}_{max}$, that defines the total intrinsic equivalence:

$$\alpha \underset{int}{\sim} \beta \iff \alpha \underset{\underline{P}_{max}}{\sim} \beta$$

for all $\mathcal{U} \in \mathbf{Set}^{L}$ simultaneously. If I had an independent, or direct, definition of $\underline{P}_{max}$, I would have 'explained' intrinsic equivalence. Alas, $\underline{P}_{max}$ is put together in a formal manner using <u>all</u> RSE's on M. (In the spirit of Leibniz's philosophy, one might try to 'explain' the notion of set by taking "the set of all sets", call it MaxSet, and say that a set

is nothing else but an element of MaxSet ...)
The definition of $\underline{P}_{max}$ is <u>impredicative</u> as we say in logic.
(Note that the FOLDS signature L is kept fixed here. What happens when L is varied is the subject of further communications and investigations.)

The first property of intrinsic equivalence to be mentioned <u>must</u> be its invariance under L-equivalence: <u>for</u> $M, N \in \mathrm{Set}^L$

$$\underline{P} = (P, \ell, r) : M \underset{L}{\sim} N$$

$U \in \mathrm{Set}^L$, $\alpha, \beta : U \to M$, $\bar{\alpha}, \bar{\beta} : U \to N$

such that $\underline{P} : \alpha \sim \bar{\alpha}$, $\underline{P} : \beta \sim \bar{\beta}$

(that is: there are $\varphi, \psi : U \to P$ such that
$\alpha = \ell\varphi$, $\bar{\alpha} = r\varphi$, $\beta = \ell\psi$, $\bar{\beta} = r\psi$)

<u>we have</u>: $\alpha \underset{int}{\sim} \beta \iff \bar{\alpha} \underset{int}{\sim} \bar{\beta}$

i.e. $(\alpha, \beta) \in \mathrm{EqInt}[M, U] \iff (\bar{\alpha}, \bar{\beta}) \in \mathrm{EqInt}[N, U]$.

For the verification, see the Appendix.

Each relation $Eq[M, \underline{P}, -]$, with varying $U \in Set^L$ for the blank, satisfies the transfer property: whenever $U \overset{f}{\rightarrowtail} V$ is a monomorphism, $\alpha, \beta : U \to M$; $\bar{\alpha} : V \to M$ such that $\alpha = \bar{\alpha} f$, then,

if $\underline{P} : \alpha \sim \beta$, then there exists $\bar{\beta} : V \to M$ such that $\underline{P} : \bar{\alpha} \sim \bar{\beta}$ and $\beta = \bar{\beta} f$.

(The short proof was given in [B3]. The significance of the transfer property was also indicated in [B3].) It immediately follows that intrinsic equivalence $Eq\,Int[M, -]$ with varying $U : L \to Set$ for the blank also satisfies the transfer property.

For any $U \in Set^L$, $Eq\,Int[M, U]$ is an equivalence relation on the set $Set^L(U, M)$: if $\underline{P}_0$, $\underline{P}_1$ are RSE's on M, then $\underline{P} \underset{def}{=} \underline{P}_0 \otimes \underline{P}_1$, the span-composite, is another RSE as was mentioned before; and if $\underline{P}_0 : \alpha \sim \beta$, $\underline{P}_1 : \beta \sim \gamma$ then $\underline{P} : \alpha \sim \gamma$ as easily seen — hence $\alpha \underset{int}{\sim} \gamma$

showing transitivity of intrinsic equivalence.

Symmetry is even more obvious: if $\underline{P}=(P,\ell,r):\alpha\sim\beta$, then $\underline{P}^{conv}=(P,r,\ell):\beta\sim\alpha$.

The phenomenon motivating the study of intrinsic equivalence is this: in several important and familiar examples, intrinsic equivalence coincides with a familiar and important notion of internal equivalence of diagrams. As a first application of the general results of the paper, I will work out the simple example of $L=L_{cat}$ and $M:L\to Set$ is a "category", that is, $M=M(\mathbb{C})$, the re-coded version of a small category $\mathbb{C}$ (see the Bohemian notes [B2], "The FOLDS language for categories"), and show that intrinsic equivalence of diagrams in M is the same thing as isomorphisms of diagrams in $\mathbb{C}$. (Let me be precise: consider an ordinary graph G and diagrams $\alpha,\beta: G\to\mathbb{C}$ in $\mathbb{C}$ in the usual sense. G can be regarded directly as a

diagram-type $\mathcal{U} : L_{cat} \to Set$,
with $\mathcal{U}(K) = \emptyset$ for all $K \in L_{cat}$ but $K = O$ and $K = A$, and the given α, β as $\mathcal{U}$-type diagrams $\alpha, \beta : \mathcal{U} \to M(\mathbb{C})$. We obtain that $\alpha \underset{int}{\sim} \beta$ if and only if α and β are isomorphic as objects of the usual category $\mathbb{C}^G$ of G-type diagrams in $\mathbb{C}$, with natural transformations as arrows.) For a general L_{cat}-structure $M : L_{cat} \to Set$, or even for $M : L \to Set$ where L is the full subcategory of L_{cat} on the objects O, A and T, I do not have a description of intrinsic equivalence on M that is independent from the notion of L-equivalence.

I now give a description of the work in the paper.
Let us fix a FOLDS signature L.
I define the concept "internal equivalence generator, IEG for short, for L". An IEG $\underline{H}$ for L consists of ingredients :

the functors H, $\mathbb{H}$:

$$H, \mathbb{H} : L^{op} \longrightarrow Set^L$$

and natural transformations

$$\ell, r : Y \longrightarrow H$$
$$q : H \longrightarrow \mathbb{H}$$

where Y is the Yoneda functor

$$Y : L^{op} \longrightarrow Set^L$$
$$K \longmapsto \hat{K} \underset{def}{=} L(K, -)$$

the <u>covariant</u> representable

It is required that we have the equalities

$$q\ell = qr : Y \to \mathbb{H};$$

we denote the composite $m = q\ell = qr : Y \to \mathbb{H}$.

There is also a further condition I impose on $\underline{H}$: $\underline{H}$ is to be 'effective monomorphic' that I do not explain here. (see page $\boxed{41}$).

Let $\underline{H}$ be an IEG for L.
For any given L- structure $M : L \to Set$, $\underline{H}$ induces, in a simple manner, the span

$$\begin{tikzcd} & H[M] \arrow[dl, "\bar{\ell}[M]"'] \arrow[dr, "\bar{r}[M]"] & \\ M & & M \end{tikzcd} \qquad (*)$$

and the natural transformations $q[M] : H[M] \to IH[M]$, $\bar{m}[M] : IH[M] \to M$, resulting in the diagram

$$\begin{tikzcd} & H[M] \arrow[dl, "\bar{\ell}[M]"'] \arrow[dr, "\bar{r}[M]"] & \\ M \arrow[dr, equal] & IH[M] \arrow[u, "q[M]"'] \arrow[d, "\bar{m}[M]"] & M \arrow[dl, equal] \\ & M & \end{tikzcd}$$

The <u>Theorem</u> says:

<u>if</u> M is such that

the morphisms $\bar{\ell}[M]$, $\bar{r}[M]$, $\bar{m}[M]$ are all fiberwise surjective, thereby making the span ($*$) a reflexive self equivalence on M,

$$\underline{H}[M] = (H[M], \bar{\ell}[M], \bar{r}[M]) : M \sim M$$

<u>then</u>, for any diagram $\mathcal{U}: L \to \mathrm{Set}$, the relation $\mathrm{Eq}[M, \underline{H}[M], \mathcal{U}]$ on the set $\mathrm{Set}^L(\mathcal{U}, M)$ <u>equals</u> the relation $\mathrm{EqInt}[M, \mathcal{U}]$ of intrinsic equivalence of $\mathcal{U}$-type diagrams in M.

(The items in the theorem are defined in this way: for $K \in L$,

$$H[M](K) \underset{\text{def}}{=} \mathrm{Set}^L(H(K), M)$$

and similarly for $IH[M]$.

The component $\bar{\ell}[M]_K : \mathrm{Set}^L(H(K), M) \longrightarrow M(K)$

of the natural transformation

$$\bar{\ell}[M] : H[M] \to M$$

is:

$$\begin{aligned} Set^L(H(K), M) &\longrightarrow Set^L(Y(K), M) \cong M(K) \\ \sigma &\longmapsto \sigma \circ \ell_K \mapsto (\sigma \circ \ell_K)_K (1_K) \end{aligned}$$

using a bit of "Yoneda theory". The others, $\bar{r}[M]$, $q[M]$, $\bar{m}[M]$ are similar)

A property of the structure on M induced by an IEG $\underline{H}$, present even before we know the fiberwise surjective conditions in the theorem, is <u>internalization</u>: for any $\alpha, \beta : U \to M$ the condition of the <u>existence</u> of $\gamma : U \to H[M]$ such that

$$\begin{tikzcd} & H[M] \arrow[dl, "{\bar{\ell}[M]}"'] \arrow[dr, "{\bar{r}[M]}"] & \\ M & U \arrow[l, "\alpha"] \arrow[r, "\beta"'] \arrow[u, "\gamma"'] & M \end{tikzcd}$$

commutes

(which will be $\alpha \sim \beta$ when $\underline{H}[M]$ is a RSE)

is <u>equivalent</u> to the <u>existence</u> of
$\sigma : \mathring{H}(U) \to M$ such that

$$
\begin{tikzcd}
 & \mathring{H}(U) \arrow[dd, "\sigma"] & \\
U \arrow[ur, "\dot{e}_U"] \arrow[dr, "\alpha"'] & & U \arrow[ul, "\dot{r}_U"'] \arrow[dl, "\beta"] \\
 & M &
\end{tikzcd}
$$

commutes;

here $\mathring{H} : Set^L \longrightarrow Set^L$ is the left Kan extension of $H : L^{op} \longrightarrow Set^L$ along $Y : L^{op} \to Set^L$; with $\dot{e}, \dot{r} : \dot{Y} = Id_{Set^L} \to \mathring{H}$ given by left Kan extension being a functor.
The reason why we call this fact "internalization" is clear: whether or not $\underline{H}[M] : \alpha \sim \beta$ is the case depends now on the existence of a suitable diagram $\sigma : \mathring{H}(U) \to M$ *in* M, in distinction to the original condition referring to a diagram $\gamma : U \to H[M]$ *in* another structure $H[M]$ (of 'homotopies'?)

The fact the the IEG $\underline{H}$ provides internalization is also key to the proof of the theorem, in which the main argument gives that if $\underline{P}$ is ~any~ RSE on M and $\underline{P} : \alpha \sim \beta$, then $\underline{H}[M] : \alpha \sim \beta$.

A refinement – a strengthening – of the theorem introduces a constraint on the diagrams $\sigma : H(K) \to M$ in $H[M](K)$: we have a system $\Phi = \langle \Phi_K \rangle_{K \in L}$ of "abstract FOLDS-formulas": each Φ_K is a class of $(L, H(K))$-type augmented structures $(N, \lambda : H(K) \to N)$; Φ_K required to be invariant under FOLDS equivalence (see the Bohemian set of slides "The Lindström context for FOLDS", [B17]). We also impose an obvious functoriality condition on the system $\langle \Phi_K \rangle_{K \in L}$.

We obtain, for each $K \in L$, a subset $(H\restriction\Phi)[M](K)$ of $H[M](K)$ and a subfunctor $(H\restriction\Phi)[M]$ of $H[M]$; further, we obtain the items in the diagram

$$\begin{tikzcd}
 & (H\restriction\Phi)[M] \arrow[dl, "(\bar{l}\restriction\Phi){[M]}"'] \arrow[dr, "(\bar{r}\restriction\Phi){[M]}"] & \\
M \arrow[dr, equal] & (|H\restriction\Phi)[M] \arrow[u, "(q\restriction\Phi){[M]}"'] \arrow[d, "(\bar{m}\restriction\Phi){[M]}"] & M \arrow[dl, equal] \\
 & M &
\end{tikzcd}$$

Of course:

$(H\restriction\Phi)[M](K) \subseteq H[M](K)$ is defined thus:

$\sigma \in H[M](K)$ belongs to $(H\restriction\Phi)[M](K)$ iff

$$(M, \sigma : H(K) \to M) \in \Phi_K$$

The generalized theorem says that

<u>if</u> $M : L \to Set$ is such that the morphisms

$$(\bar{\ell} \upharpoonright \Phi)[M],\ (\bar{r} \upharpoonright \Phi)[M],\ (\bar{m} \upharpoonright \Phi)[M] \text{ are all FS}$$

and thus

$$(\underline{H} \upharpoonright \Phi)[M] = \big((H \upharpoonright \Phi)[M],\ (\bar{\ell} \upharpoonright \Phi)[M],\ (\bar{r} \upharpoonright \Phi)[M]\big)$$

is a $RS\underline{E}$ on M

<u>then</u> $(\underline{H} \upharpoonright \Phi)[M]$ induces the (total) intrinsic equivalence

$$(\underline{H} \upharpoonright \Phi)[M] : \alpha \sim \beta \iff \alpha \underset{int}{\sim} \beta$$

(the left-to-right implication is automatic).

Given $M : L \to Set$, a $RS\underline{E}$ $\underline{P} : M \sim M$ is said to be <u>schematic</u> if it is isomorphic to $(\underline{H} \upharpoonright \Phi)[M]$ for a suitable $\underline{H}$ and Φ as described above. A corollary, useful in applications, is that any two schematic

RSE's on any $M : L \to Set$, $\underline{P}_0$ and $\underline{P}_1$, define the same $\overset{\text{equivalence}}{\text{relation}}$ $Eq[M, \underline{P}_i, U]$ $(i = 0, 1)$ on U-type diagrams in M.

An efficient use of the constrained schematic-type of RSE's can be made to identify and internalize intrinsic equivalence in structures of L_{absset} that are models of a minimal theory of sets, hence in L_{absset}-structures that are models of any reasonably strong, even if quite weak, set theory.
For applications to L_{cat}, L_{absset} and other even more interesting examples, see a forthcoming sequel to this write-up.