

Intrinsic and internal equivalence in structures for a FOLDS signature

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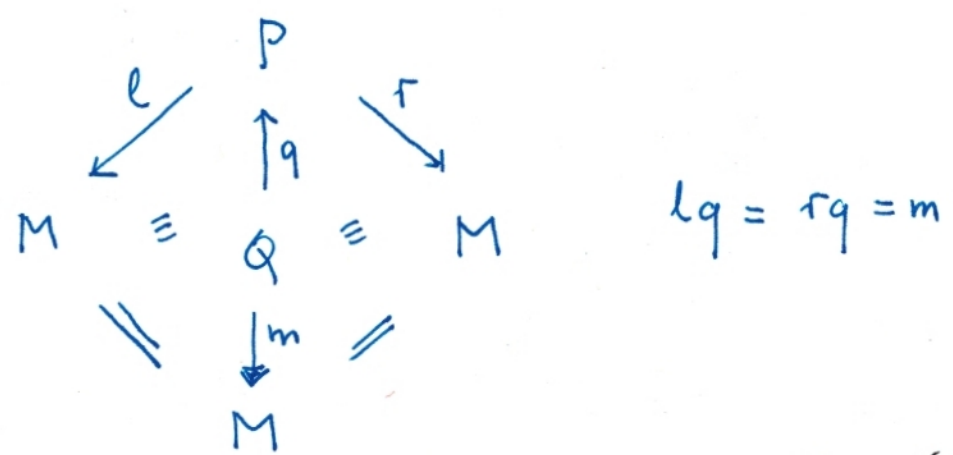
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Intrinsic and internal equivalence in structures for a FOLDS signature

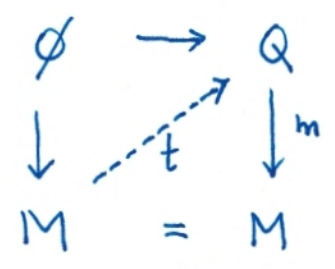
Introductory words

If you find, in the Introduction, or in the body of the write-up, notation or concepts with insufficient explanation, consult the slides for my Bohemian Cafe talk, especially the set "First-order logic with dependent sorts"; I'll refer to this as [B3]. There will be some slight changes with respect to [B3]. In [B3], I write \tilde{K} for the covariant representable functor $L(K, -): L \rightarrow \text{Set}$; here I've changed this to $\hat{K} = L(K, -): L \rightarrow \text{Set}$. Notation involving the symbol \cong has been simplified by writing \sim in place of \cong ; for instance, intrinsic equivalence is written as $\alpha \underset{\text{int}}{\sim} \beta$ in place of $\alpha \underset{\text{int}}{\cong} \beta$. In [B3], the equivalence span $\underline{P} = (P, \ell, r): M \underset{L}{\sim} M$ is said to extend the identity if there is $m: M \rightarrow P$ such that $\ell m = r m = 1_M$. Now I say for the same condition that \underline{P} is a reflexive self-equivalence, RSE for short, of M . Also, I make a change in the definition, the new one remaining equivalent to the old one, however. Now I say

that $\underline{P} = (P, \ell, r) : M \underset{L}{\sim} M$ is reflexive
 if there exist $Q \in \text{Set}^L$, $q : Q \rightarrow P$, $m : Q \rightarrow M$
 such that



and m is fiberwise surjective (FS). This is
 equivalent to the original definition: if the condition
 of the original definition holds, with $s : M \rightarrow P$
 (instead of $m : M \rightarrow P$ in [B3]), $\ell s = r s = 1_M$, then
 I can take $Q = M$, $m = 1_M$ and $q = s$ for the new
 definition. In the opposite direction, using that m is FS
 in the strong sense - having the right lifting property
 with respect to all monomorphisms, not just the sphere
 inclusions, we have $t : M \rightarrow Q$ such that $mt = 1_M$ (see



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we can take $s = qt: M \rightarrow P$, and have $ls = rs = 1_M$. Without the axiom of choice though, the new definition of reflexivity is weaker, and therefore it is better; it is easier to verify the new condition. (Note that for a finite monomorphism $U \xrightarrow{f} V$, the right lifting of \dots a map with respect to f is a consequence of same with respect to the sphere inclusions $i_K: \hat{K} \rightarrow \hat{K}$ ($K \in L$) in a constructive manner, without choice.)

The category theory used in the paper is confined to the Yoneda theory around the fixed category L . Initially, L is an arbitrary small category. Eventually, a condition is imposed on L , the existence of a sphere functor (see page 140), a condition that all FOLDS signatures satisfy, and so does the famed category $L = \Delta^{op}$ (with Set^L the category of simplicial sets). In the introduction proper that follows, some details will be relegated to the Appendix.

Introduction

For the introduction, unlike the body of the paper, L is a fixed FOLDS signature (see [B3]). To illustrate matters, I will take L to be L_{cat} - see the Bohemian Cafe set [B2] "The FOLDS signature for categories". For a while, I also fix an L -structure $M; M: L \rightarrow \text{Set}$. For illustration, we consider a small category \mathcal{C} , and the corresponding re-coded version of \mathcal{C} , $M = M(\mathcal{C}): L_{\text{cat}} \rightarrow \text{Set}$ (see [B2]).

We will be dealing with variable reflexive self-equivalences $\underline{P} = (P, \ell, r): M \underset{L}{\sim} M$ (the simplest and always available one is the 'identity' $\underline{\text{Id}}_M = (M, \mathbb{1}_M, \mathbb{1}_M)$), and with variable diagram types $U: L \rightarrow \text{Set}$.

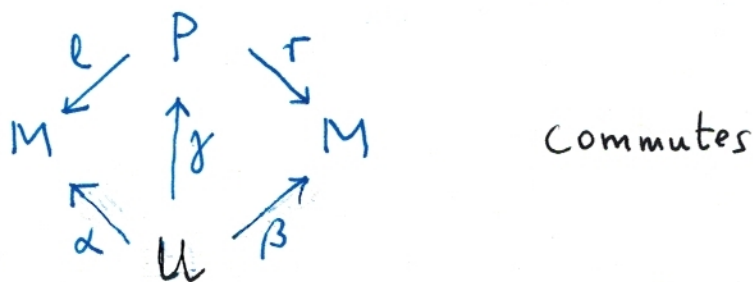
(For illustration, in case $L = L_{\text{cat}}$, think of finite diagrams in the category \mathcal{C} : an ordinary graph is, automatically, a diagram type $U: L_{\text{cat}} \rightarrow \text{Set}$ with all sets $U(K)$, $K \in L_{\text{cat}}$ except $K = 0$ and $K = A$ empty; and, a diagram in \mathcal{C} is the same as a diagram $\alpha: L \rightarrow M(\mathcal{C})$ in this case.)

For a specific \underline{P} and a specific U
 as above, we define the binary relation $Eg[M, \underline{P}, U]$
 abbreviated as $Eg[\underline{P}, U]$ with M fixed, on the set
 $Set^L(U, M)$ of all U -type diagrams in M , as follows:

for $\alpha, \beta: U \rightarrow M$

$(\alpha, \beta) \in Eg[\underline{P}, U] \Leftrightarrow$ there exists $\gamma: U \rightarrow P$

such that $\alpha = l\gamma, \beta = r\gamma$:



Most of the time, we write $\alpha \underset{P}{\sim} \beta$ for $(\alpha, \beta) \in Eg[\underline{P}, U]$

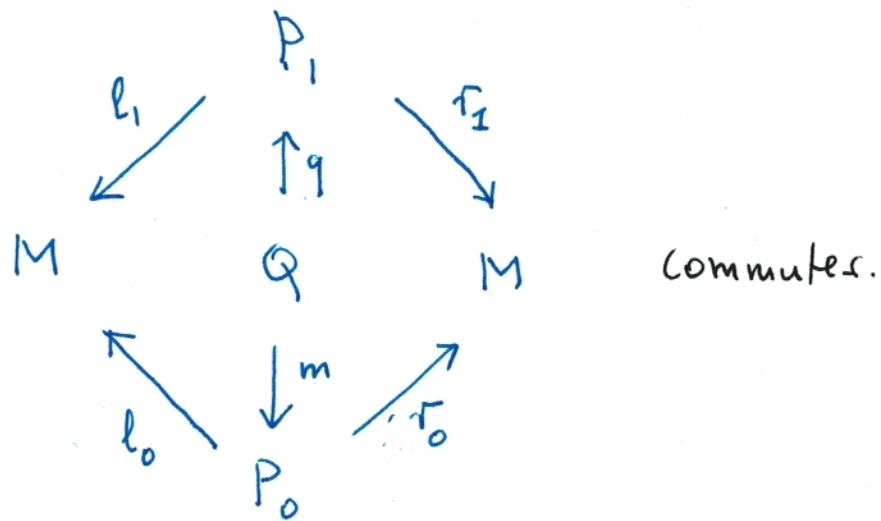
(even before we know that $Eg[\underline{P}, U]$ is an equivalence relation!).

For $\underline{P}_i = (P_i, l_i, r_i)$, $i=0$ and $i=1$, both RSE's on M

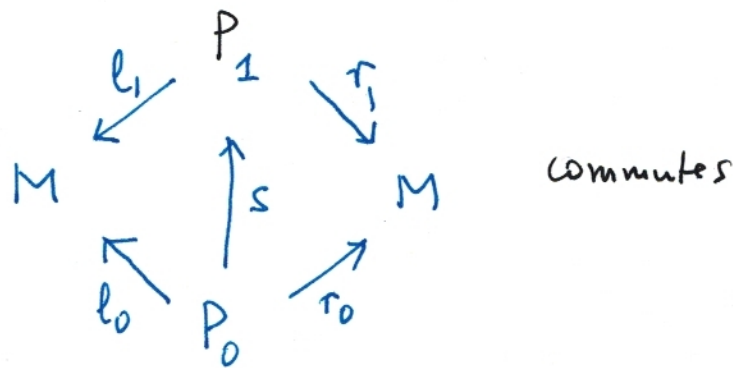
I say that \underline{P}_1 extends \underline{P}_0 if there are $Q \in Set^L$,

$q: Q \rightarrow P_1$, $m: Q \rightarrow P_0$ such that m is FS, and

$l_1 q = l_0 m$, $r_1 q = r_0 m$: the diagram



The simpler condition: there is $s: P_0 \rightarrow P_1$ such that $l_1 s = l_0$, $r_1 s = r_0$:



is equivalent non-constructively as before.

The very definition of RSE says that every RSE extends \underline{Id}_M .

It is immediate that if \underline{P}_1 extends \underline{P}_0 , then

$$Eq[\underline{P}_0, U] \subseteq Eq[\underline{P}_1, U] \quad (U \in \text{Set}^L)$$

(See Appendix)

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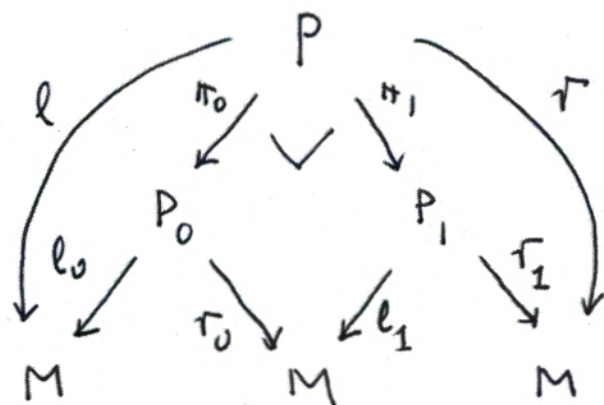
A simple but obviously basic fact is that for any two RSE's on M ,

$\underline{P}_i = (P_i, l_i, r_i)$, $i=0$ and $i=1$, there is a third one

$\underline{P} = (P, l, r)$ that extends both \underline{P}_0 and \underline{P}_1 .

In fact, \underline{P} can be taken to be the span-composite

$\underline{P} = \underline{P}_0 \circ \underline{P}_1$:



the same as the one that is used in showing that FOLDS equivalence in general is a transitive relation.

See the Appendix.

Let $U \in \text{Set}^L$. $\text{EqInt}[M, U] (= \text{EqInt}[U])$,
 intrinsic equivalence for U -type diagrams in M ,
 is the binary relation on $\text{Set}^L(U, M)$ defined as
 the union of all $\text{Eq}[P, U]$, with P ranging over

all RSE's on M : Writing $\alpha \underset{\text{int}}{\sim} \beta$
 for $(\alpha, \beta) \in \text{EqInt}[U]$, we have:

$\alpha \underset{\text{int}}{\sim} \beta \iff$ there is P , RSE on M
 such that $\alpha \underset{P}{\sim} \beta$.

I. 8.1

In a sequel to this write-up, I will show a (more) general calculus of composition of RSE's that can be used to prove the following:

for any structure $M \in \text{Set}_L$, we have an "almost canonically defined" single RSE on M , call it P_{\max} , that defines the total intrinsic equivalence:

$$\alpha \sim \beta \Leftrightarrow \alpha \sim \beta \text{ } P_{\max}$$

for all $U \in \text{Set}_L$ simultaneously. If I had an independent, or direct, definition of P_{\max} ,

I would have 'explained' intrinsic equivalence. Atlas,

P_{\max} is put together in a formal manner using all

RSE's on M . (In the spirit of Leibniz's philosophy,

one might try to 'explain' the notion of set by taking

"the set of all sets", call it MaxSet , and say that a set

is nothing else but an element of MaxSet...)

The definition of P_{max} is impredicative as we say in logic.
 (Note that the FOLDS signature L is kept fixed here. What happens when L is varied is the subject of further communications and investigations.)

The first property of intrinsic equivalence to be mentioned must be its invariance under L -equivalence: for $M, N \in \text{Set}^L$

$$\underline{P} = (P, l, r) : M \underset{L}{\sim} N$$

$$U \in \text{Set}^L, \quad \alpha, \beta : U \rightarrow M, \quad \bar{\alpha}, \bar{\beta} : U \rightarrow N$$

$$\text{such that } \underline{P} : \alpha \sim \bar{\alpha}, \quad \underline{P} : \beta \sim \bar{\beta}$$

(that is: there are $\varphi, \psi : U \rightarrow P$ such that

$$\alpha = l\varphi, \quad \bar{\alpha} = r\varphi, \quad \beta = l\psi, \quad \bar{\beta} = r\psi)$$

we have : $\alpha \underset{int}{\sim} \beta \iff \bar{\alpha} \underset{int}{\sim} \bar{\beta}$

i.e. $(\alpha, \beta) \in \text{EqInt}[M, U] \iff (\bar{\alpha}, \bar{\beta}) \in \text{EqInt}[N, U].$

For the verification, see the Appendix.

Each relation $\text{Eq}[M, \underline{P}, -]$, with varying $U \in \text{Set}^L$ for the blank, satisfies the transfer property: whenever $U \xrightarrow{f} V$ is a monomorphism, $\alpha, \beta: U \rightarrow M$; $\bar{\alpha}: V \rightarrow M$ such that $\alpha = \bar{\alpha}f$, then,

if $\underline{P}: \alpha \sim \beta$, then there exists $\bar{\beta}: V \rightarrow M$ such that $\underline{P}: \bar{\alpha} \sim \bar{\beta}$ and $\beta = \bar{\beta}f$.

(The short proof was given in [B3]. The significance of the transfer property was also indicated in [B3].) It immediately follows that intrinsic equivalence $\text{EqInt}[M, -]$ with varying $U: L \rightarrow \text{Set}$ for the blank also satisfies the transfer property.

For any $U \in \text{Set}^L$, $\text{EqInt}[M, U]$ is an equivalence relation on the set $\text{Set}^L(U, M)$: if $\underline{P}_0, \underline{P}_1$ are RSE's on M , then $\underline{P} \stackrel{\text{def}}{=} \underline{P}_0 \otimes \underline{P}_1$, the span-composite, is another RSE as was mentioned before; and if $\underline{P}_0: \alpha \sim \beta$, $\underline{P}_1: \beta \sim \gamma$ then $\underline{P}: \alpha \sim \gamma$ as easily seen - hence $\alpha \sim_{\text{int}} \gamma$

Showing transitivity of intrinsic equivalence.

Symmetry is even more obvious: if $\underline{P} = (P, l, r) : \alpha \rightsquigarrow \beta$,
then $\underline{P}^{\text{conv}} = (P, r, l) : \beta \rightsquigarrow \alpha$.

The phenomenon motivating the study of intrinsic equivalence is this: in several important and familiar examples, intrinsic equivalence coincides with a familiar and important notion of internal equivalence of diagrams. As a first application of the general results of the paper, I will work out the simple example of $L = L_{\text{cat}}$ and $M: L \rightarrow \text{Set}$ is a "category", that is, $M = M(\mathbb{C})$, the re-coded version of a small category \mathbb{C} (see the Bohemian notes [B2], "The FOLDS language for categories"), and show that intrinsic equivalence of diagrams in M is the same thing as isomorphisms of diagrams in \mathbb{C} . (Let me be precise: consider an ordinary graph G and diagrams $\alpha, \beta: G \rightarrow \mathbb{C}$ in \mathbb{C} in the usual sense. G can be regarded directly as a

diagram-type $U: L_{cat} \rightarrow Set,$

with $U(K) = \emptyset$ for all $K \in L_{cat}$ but $K=0$

and $K=A$, and the given α, β as U -type

diagrams $\alpha, \beta: U \rightarrow M(\mathbb{C})$. We obtain that

$\alpha \underset{int}{\sim} \beta$ if and only if α and β are

isomorphic as objects of the usual category \mathbb{C}^G of G -type diagrams in \mathbb{C} , with natural transformations as arrows.) For a general

L_{cat} -structure $M: L_{cat} \rightarrow Set$, or even for

$M: L \rightarrow Set$ where L is the full subcategory

of L_{cat} on the objects $0, A$ and T , I do not

have a description of intrinsic equivalence on M that is independent from the notion of L -equivalence.

I now give a description of the work in the paper.

Let us fix a FOLDS signature L .

I define the concept "internal equivalence

generator, IEG for short, for L ". An IEG H

for L consists of ingredients :

the functors $H, IH :$

$$H, IH : L^{op} \longrightarrow \text{Set}^L$$

and natural transformations

$$\ell, r : Y \longrightarrow H$$

$$q : H \longrightarrow IH$$

where Y is the Yoneda functor

$$Y : L^{op} \longrightarrow \text{Set}^L$$

$$K \longmapsto \hat{K} \stackrel{\text{def}}{=} L(K, -)$$

the covariant representable

It is required that we have the equalities

$$q\ell = qr : Y \rightarrow IH;$$

We denote the composite $m = q\ell = qr : Y \rightarrow IH$.

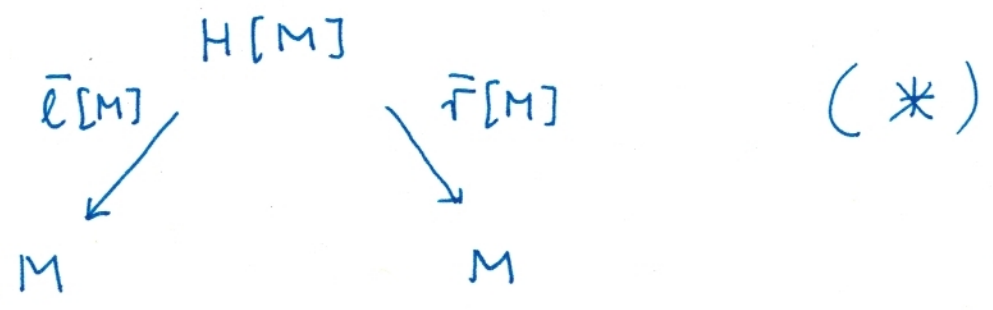
There is also a further condition I impose

on \underline{H} : \underline{H} is to be 'effective monomorphic'

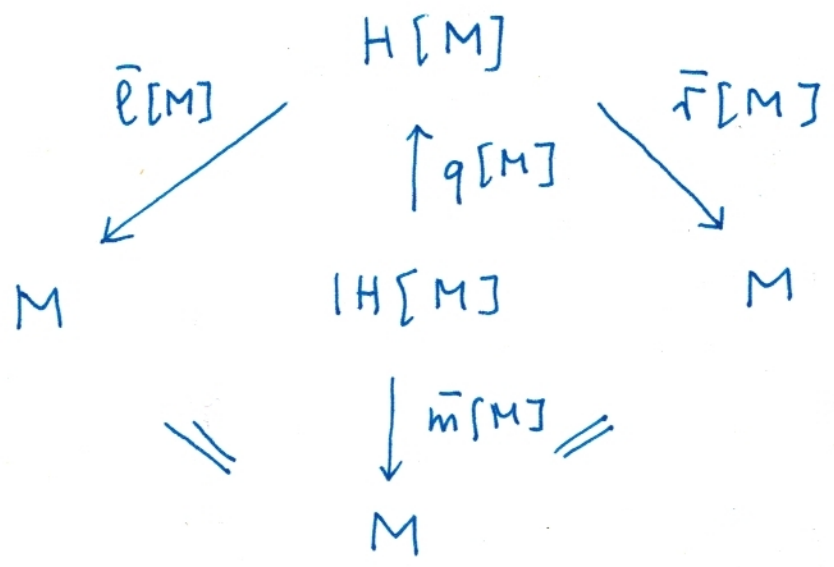
that I do not explain here. (see page 41).

Let \underline{H} be an IEG for L .

For any given L -structure $M: L \rightarrow \text{Set}$, \underline{H} induces, in a simple manner, the span



and the natural transformations $q[M]: H[M] \rightarrow IH[M]$, $\bar{m}[M]: IH[M] \rightarrow M$, resulting in the diagram



The Theorem says:

if M is such that

the morphisms $\bar{\ell}[M]$, $\bar{r}[M]$, $\bar{m}[M]$ are all fiberwise surjective, thereby making the span $(*)$ a reflexive self equivalence on M ,

$$\underline{H}[M] = (H[M], \bar{\ell}[M], \bar{r}[M]) : M \sim M$$

then, for any diagram $U : L \rightarrow \text{Set}$,

the relation $\text{Eq}[M, \underline{H}[M], U]$ on the set

$\text{Set}^L(U, M)$ equals the relation $\text{EqInt}[M, U]$ of intrinsic equivalence of U -type diagrams in M .

(The items in the theorem are defined in this

way: for $K \in L$,

$$H[M](K) \stackrel{\text{def}}{=} \text{Set}^L(H(K), M)$$

and similarly for $\bar{H}[M]$.

The component $\bar{\ell}[M]_K : \text{Set}^L(H(K), M) \rightarrow M(K)$

of the natural transformation

$$\bar{e}[M]: H[M] \rightarrow M$$

is:

$$\text{Set}^L(H(K), M) \rightarrow \text{Set}^L(Y(K), M) \cong M(K)$$

$$\sigma \longmapsto \sigma \circ l_K \longmapsto (\sigma \circ l_K)_K (1_K)$$

using a bit of "Yoneda theory". The

others, $\bar{r}[M]$, $q[M]$, $\bar{m}[M]$ are similar)

A property of the structure on M induced by an IEG \underline{H} , present even before we know the fiberwise surjective conditions in the theorem, is internalization: for any $\alpha, \beta: U \rightarrow M$ the condition of the existence of $\gamma: U \rightarrow H[M]$ such that

$$\begin{array}{ccccc}
 & & H[M] & & \\
 & \swarrow \bar{e}[M] & \uparrow \gamma & \searrow \bar{r}[M] & \\
 M & \xleftarrow{\alpha} & U & \xrightarrow{\beta} & M
 \end{array}$$

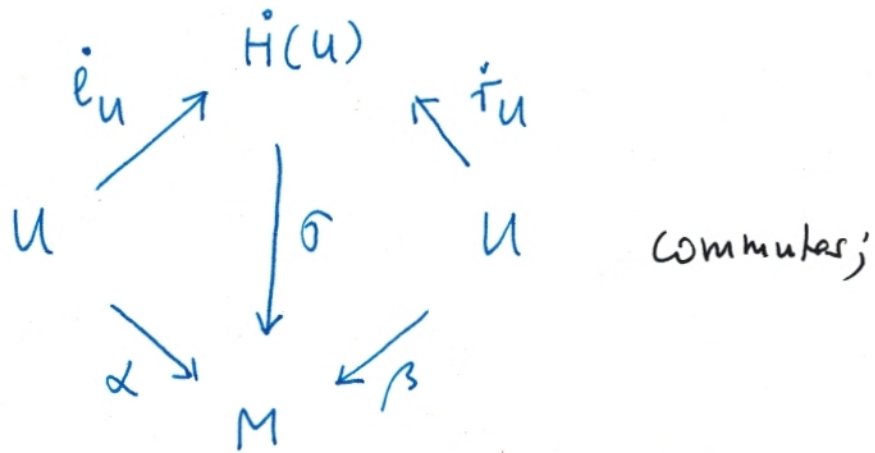
commutes

(which will be $\alpha \sim \beta$ when $\underline{H}[M]$ is a RSE)

is equivalent to the existence of

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$\sigma: \dot{H}(U) \rightarrow M$ such that



here $\dot{H}: \text{Set}^L \rightarrow \text{Set}^L$ is the left Kan extension of $H: L^{\text{op}} \rightarrow \text{Set}^L$ along $\gamma: L^{\text{op}} \rightarrow \text{Set}^L$; with $i, r: \gamma = \text{Id}_{\text{Set}^L} \rightarrow \dot{H}$ given by left Kan extension being a functor. The reason why we call this fact "internalization" is clear: whether or not $\underline{H[M]} = \alpha \sim \beta$ is the case depends now on the existence of a suitable diagram $\sigma: \dot{H}(U) \rightarrow M$ in \underline{M} , in distinction to the original condition referring to a diagram $\gamma: U \rightarrow H[M]$ in another structure $\underline{H[M]}$ (of 'homologies'?)

The fact that the IEG \underline{H} provides internalization is also key to the proof of the theorem, in which the main argument gives that if \underline{P} is any RSE on M and $\underline{P} : \alpha \sim \beta$, then $\underline{H}[M] : \alpha \sim \beta$.

A refinement - a strengthening - of the theorem introduces a constraint on the diagrams $\sigma : H(K) \rightarrow M$ in $H[M](K)$:

we have a system $\Phi = \langle \Phi_k \rangle_{k \in L}$

of "abstract FOLDS-formulas": each Φ_k is a class of $(L, H(K))$ -type

augmented structures $(N, \lambda : H(K) \rightarrow N)$;

Φ_k required to be invariant under

FOLDS equivalence (see the Bohemian set of slides "The Lindström context for FOLDS",

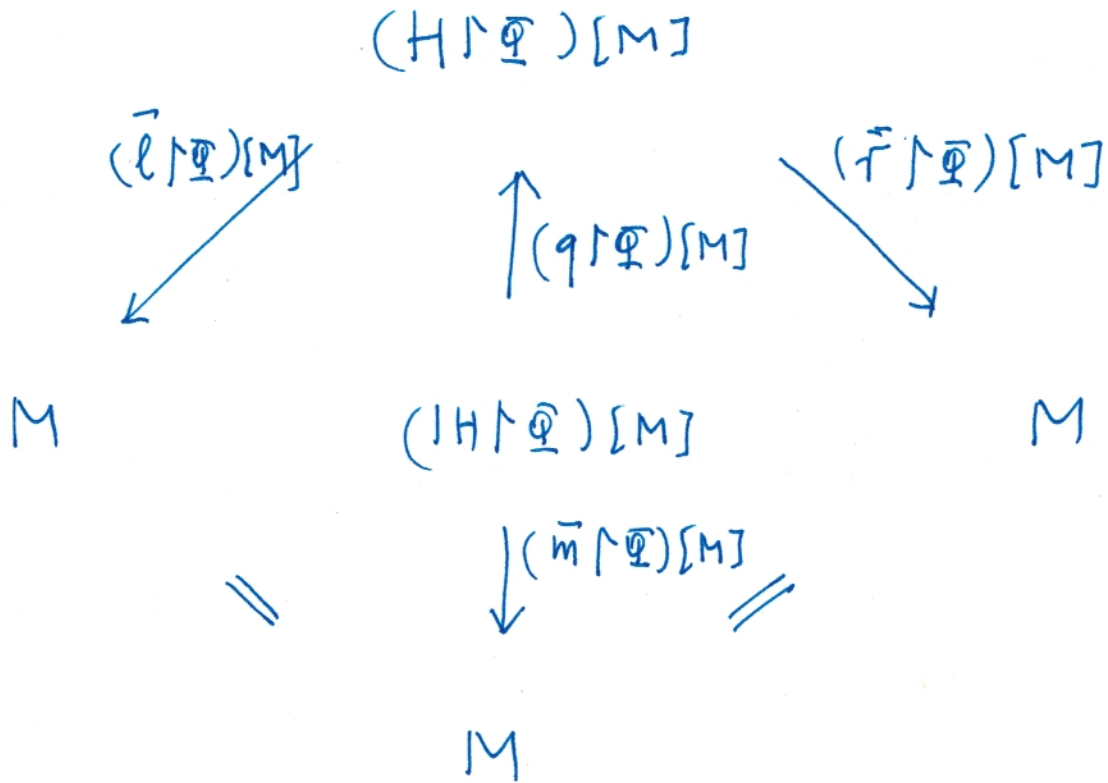
[B1]). We also impose an obvious functoriality

condition on the system $\langle \Phi_k \rangle_{k \in L}$.

We obtain, for each $K \in L$, a

subset $(H \upharpoonright \Phi)[M](K)$ of $H[M](K)$

and a subfunctor $(H \upharpoonright \Phi)[M]$ of $H[M]$; further, we obtain the items in the diagram



Of course:

$(H \upharpoonright \Phi)[M](K) \subseteq H[M](K)$ is defined thus:

$\sigma \in H[M](K)$ belongs to $(H \upharpoonright \Phi)[M](K)$ iff

$$(M, \sigma: H(K) \rightarrow M) \in \Phi_K$$

The generalized theorem says that

if $M: L \rightarrow \text{Set}$ is such that the morphisms

$$(\overline{\tau} \downarrow \Phi) [M], (\overline{m} \downarrow \Phi) [M], (\overline{m} \downarrow \Phi) [M]$$

and thus

$$(\overline{H} \downarrow \Phi) [M] = ((H \downarrow \Phi) [M], (\overline{\tau} \downarrow \Phi) [M], (\overline{m} \downarrow \Phi) [M])$$

is a RSE on M

then $\overline{\overline{H}} \downarrow \Phi$ induces the (total)

intrinsic equivalence

$$(\overline{H} \downarrow \Phi) [M] : \alpha \sim \sqrt{} \Leftrightarrow \alpha \sim \sqrt{}^{\text{int}}$$

(the left-to-right implication is automatic).

Given $M: L \rightarrow \text{Set}$, a RSE $P: M \sim M$ is

said to be schematic if it is isomorphic

to $(\overline{H} \downarrow \Phi) [M]$ for a suitable \overline{H} and Φ

as described above. A corollary, useful

in applications, is that any two schematic

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RSE's on any $M: L \rightarrow \text{Set}$, P_0 and P_1 ,
define the same ^{equivalence} relation $\text{Eq}[M, P_i, U]$
($i=0,1$) on U-type diagrams in M .

An efficient use of the constrained schematic-type of RSE's can be made to identify and internalize intrinsic equivalence in structures of L_{absset} that are models of a minimal theory of sets, hence in L_{absset} -structures that are models of any reasonably strong, even if quite weak, set theory.

For applications to L_{cat} , L_{absset} and other even more interesting examples, see a forthcoming sequel to this write-up.