

Part 1 Notation

L is a fixed small category throughout;
 K, K', \dots objects of L (' K ' for 'kind'...).
 The codomain of arrow p is denoted (sometimes) as K_p ; $p: K \rightarrow K_p$.
 M, N, P, \dots denote objects of the functor category Set^L , and so do U, V, \dots ; $U, V \in Set^L$.
 U, V play different roles than M, N, P .
 U, V are 'diagram types'; M, N, P are 'structures';
 an arrow $\alpha: U \rightarrow M$ is a type- U diagram in M .
 However, no formal distinction is made between the U 's and the M 's. Unless explicitly said (and infrequently so), L is not assumed to be a FOLDS signature. Of the diagram types U, V finiteness is not assumed - unless explicitly done so.

§ 1.1 Left Kan extension along Yoneda 2

Let \mathcal{S} be any (small-) cocomplete category. Eventually \mathcal{S} will be Set^L , but, for the first pieces of the notation, specifying \mathcal{S} any further would only obscure the picture.

$$\text{Let: } D: L^{\text{op}} \longrightarrow \mathcal{S}.$$

$\hat{D}: \text{Set}^L \longrightarrow \mathcal{S}$ denotes 'the' left Kan extension of D along the Yoneda functor

$$Y: L^{\text{op}} \longrightarrow \text{Set}^L.$$

\hat{D} is determined only up to isomorphism; the freedom of choosing a particular representative of the isomorphism class will be used.

For each $U \in \text{Set}^L$, the object $\hat{D}(U)$ is determined as a colimit as follows. Let $\text{el}(U)$ be the category of elements of the functor $U: L \rightarrow \text{Set}$; the objects of $\text{el}(U)$ are pairs (K, u) where $K \in L$ and $u \in U(K)$; an arrow $(K, u) \xrightarrow{p} (K_p, v)$ is the same as $p: K \rightarrow K_p$ subject to the condition $U(p)(u) = v$. I write $[U]$ for the opposite category $(\text{el}(U))^{\text{op}}$. The objects of $[U]$

are the same pairs (K, u) as before.

However when I write $p: (K, u) \rightarrow (K_p, v)$, I always mean the arrow in $el(U)$, not in $[U]$. In $[U]$, this would be $(K_p, v) \rightarrow (K, u)$, an unpleasant sight. This will result in some contravariance in the notation.

We have the forgetful functor

$$\begin{aligned} \Phi_U : [U] &\longrightarrow L^{op} \\ (K, u) &\longmapsto K \end{aligned}$$

and the composite

$$\begin{aligned} D\Phi_U : [U] &\longrightarrow \mathcal{S} \\ (K, u) &\longmapsto D(K) \end{aligned}$$

The promised (well-known) colimit expression of $\mathring{D}(U)$ is:

$$\mathring{D}(U) = \text{colim} (D\Phi_U)$$

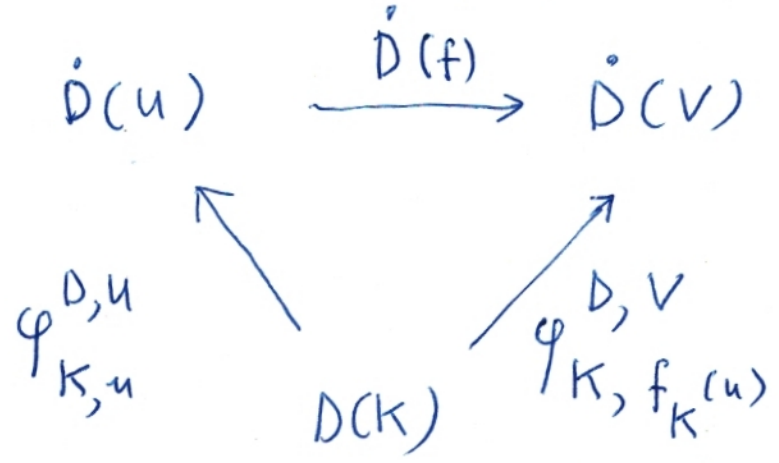
or more easily

$$\mathring{D}(U) = \text{colim}_{(K, u) \in [U]} D(K)$$

Let us write for the colimit coprojections:

$$\varphi_{K, u}^{D, U} (= \varphi_{K, u}^D = \varphi_{K, u}) : D(K) \longrightarrow \mathring{D}(U).$$

The effect of the functor $\dot{D}: \text{Set}^L \rightarrow \mathcal{S}$ on arrow $U \xrightarrow{f} V$ in Set^L is determined by the joint commutativity of the diagrams



one for each $(K, u) \in [U]$.

It is useful to note that Set^L is the (small-) colimit completion of L^{op} via $Y: L^{op} \rightarrow \text{Set}^L$ and $\dot{D}: \text{Set}^L \rightarrow \mathcal{S}$ preserves all small colimits in Set^L . Of course, the explicit formula given above for $\dot{D}(U)$ is an instance of this fact.

The left Kan extension $\dot{D}: \text{Set}^L \rightarrow \mathcal{S}$ comes with a canonical isomorphism (in this case):

$$\Theta: D \xrightarrow{\cong} \dot{D}Y \quad \text{in } \mathcal{S}^{L^{op}}$$

whose component at $K \in L$

$$\Theta_K: D(K) \longrightarrow \dot{D}(K)$$

(where we write $\hat{K} = Y(K) = \text{Set}^L(K, -)$

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for the covariant representable, for now)

is, in terms of the notation already introduced as follows:

$$\oplus_K \stackrel{\text{def}}{=} \varphi_{D, \hat{K}}^{(K, \perp_K \in \hat{K}(K))} : D(K) \longrightarrow \dot{D}(\hat{K})$$

Now we use our freedom of choosing the value of $\dot{D}(\hat{K})$, together with the colimit cocone with components

$$\varphi_{D, \hat{K}}^{(K_p, p: K \rightarrow K_p)} : D(K_p) \longrightarrow \dot{D}(\hat{K})$$

as follows: put $\dot{D}(\hat{K}) \stackrel{\text{def}}{=} D(K)$

and

$$\varphi_{D, \hat{K}}^{(K_p, p: K \rightarrow K_p)} \stackrel{\text{def}}{=} D(p)$$

In this way, we indeed obtain a colimit cocone for the colimit $\dot{D}(\hat{K})$, since now, for $U = \hat{K}$,

$[\hat{K}]$ is a category with the terminal object $(K, 1_K)$, so for the diagram $[\hat{K}] \xrightarrow{D\Phi_{\hat{K}}} \mathcal{S}$ a correct choice of the colimit consists in

$$\operatorname{colim} D\Phi_{\hat{K}} \stackrel{\text{def}}{=} D\Phi_{\hat{K}}((K, 1_K)) = D(K)$$

with coprojections

$$D\Phi_{\hat{K}}((K_p, p: K \rightarrow K_p)) = D(\beta) = D(K_p) \rightarrow D(K).$$

In particular, the canonical comparison isomorphism

$$\Theta_K = \varphi_{(K, 1_K: K \rightarrow K)}^{D, \hat{K}} : D(K) \longrightarrow \overset{\circ}{D}(\hat{K})$$

$$\text{is } \Theta_K = D(1_K) = 1_{D(K)} : D(K) \rightarrow D(K)$$

Thus, $\overset{\circ}{D}$ strictly extends D :

$$\begin{array}{ccc} L^{\text{op}} & \xrightarrow{D} & \mathcal{S} \\ \gamma \searrow & \cong & \nearrow \overset{\circ}{D} \\ & \text{Set}^L & \end{array} \quad \overset{\circ}{D}\gamma = D$$

Now, let, from now on, $\mathcal{S} = \text{Set}^L$. So, the Yoneda functor $\gamma: L^{\text{op}} \rightarrow \text{Set}^L$ is one of

The possible D 's. For $U \in \text{Set}^L$,
we define the colimit

$$\dot{Y}(U) = \text{colim}_{(K, u) \in [U]} \hat{K}$$

by setting $\dot{Y}(U) \stackrel{\text{def}}{=} U$

with coprojections $\varphi_{K, u}^{Y, U} : \hat{K} \rightarrow U$
" $Y(K)$

to be $\varphi_{K, u}^{Y, U} \stackrel{\text{def}}{=} \hat{u}$,

where \hat{u} is the Yoneda equivalent $\hat{u} : \hat{K} \rightarrow U$
of $u \in U(K)$, determined by the equality $\hat{u}_K(1_K) = u$.

This is the standard representation of $U \in \text{Set}^L$
as a colimit of representables.

When we look at how $\dot{Y}(f) : \dot{Y}(U) \rightarrow \dot{Y}(V)$
($f : U \rightarrow V$) is determined via coprojections (see above),
we see that $\dot{Y}(f) = f$. Thus

$$\dot{Y} = \text{Id}_{\text{Set}^L}$$

We can simplify the notation by identifying the coprojections in this way:

$$\varphi_{(K, u)}^{D, U} = \dot{D}(\hat{u})$$

in the general case. (Both arrows are of the form $D(K) = \dot{D}(\hat{K}) \rightarrow \dot{D}(U)$.) The reasons:

apply $\dot{D} : \text{Set}^L \rightarrow \mathcal{S}$ to the arrow $\hat{u} : \hat{K} \rightarrow U$ in Set^L according to the recipe above; we get

$$\begin{array}{ccc}
 \dot{D}(\hat{K}) & \xrightarrow{\dot{D}(\hat{u})} & \dot{D}(U) \\
 \swarrow & \cong & \nearrow \\
 \varphi_{K, 1_K}^{D, \hat{K}} & & \varphi_{K, \hat{u}_K(1_K)}^{D, U} = u
 \end{array}$$

$1_{D(K)} = D(1_{\hat{K}}) =$

Left Kan extension is a functor.

For $D, C : L^{op} \rightarrow \mathcal{S}$, $\tau : D \rightarrow C$ ($\in (\text{Set}^L)^{L^{op}}(D, C)$)

we have $\dot{\tau} : \dot{D} \rightarrow \dot{C}$ ($\in (\text{Set}^L)^{(\text{Set}^L)}(\dot{D}, \dot{C})$);

for $U \in \text{Set}^L$, $\dot{\tau}_U : \dot{D}(U) \rightarrow \dot{C}(U)$ is determined

by: the commutativities

$$\begin{array}{ccc}
 D(K) & \xrightarrow{\dot{D}(\hat{u})} & \dot{D}(U) = \text{colim}(D\Phi_u) \\
 \downarrow \tilde{r}_K & & \downarrow \tilde{r}_U \\
 C(K) & \xrightarrow{\dot{C}(\hat{u})} & \dot{C}(U)
 \end{array}$$

one for each $(K, u) \in [U]$.

Remark For any functor $F: \text{Set}^L \rightarrow \mathcal{G}$
 the arrows $F(\hat{u}) : F(\hat{K}) \rightarrow F(U) \ ((K, u) \in [U])$
 form a cocone with vertex $F(U)$:

$$\begin{array}{ccc}
 (K, u) & & F(\hat{K}) \\
 \downarrow p & & \searrow F(\hat{u}) \\
 (K_p, v) & & F(U) \\
 & & \nearrow F(\hat{v}) \\
 & & F(\hat{K}_p) \\
 & & \uparrow F(Y(p)) \\
 & & F(\hat{K})
 \end{array}$$

$v = U(p)(u)$

since

$$\begin{array}{ccc}
 \hat{K} & & \hat{u} \\
 \uparrow Y(p) & \rightsquigarrow & \searrow \\
 \hat{K} & & U \\
 & \nearrow \hat{v} & \\
 & &
 \end{array}$$

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F is the left Kan extension of $D = F \circ Y$ along Y with transition $\Theta: D \rightarrow F \circ Y$ the identity iff the cocone $\langle F(\hat{u}): F(\hat{K}) \rightarrow F(U) \rangle_{(K,u) \in [U]}$ is a colimit cocone for all $U \in \text{Set}^L$.

§ 1.2 Realizing the schemes $D: L^{\text{op}} \rightarrow \text{Set}^L$ and $\hat{D}: \text{Set}^L \rightarrow \text{Set}^L$ in the structure $M \in \text{Set}^L$.

We have put $\mathcal{S} = \text{Set}^L$.

Given $D: L^{\text{op}} \rightarrow \text{Set}^L$ and $M \in \text{Set}^L$,

$D[M]: L \rightarrow \text{Set}$, $D[M] \in \text{Set}^L$

the "realization of D in M " is

$D[M] \stackrel{\text{def}}{=} \text{Set}^L(D(-), M): L \rightarrow \text{Set};$

$D[M]$ is the composite

$$L \xrightarrow{D^{\text{op}}} (\text{Set}^L)^{\text{op}} \xrightarrow{\text{Set}^L(-, M)} \text{Set}.$$

where $\text{Set}^L(-, M)$ is the contravariant representable given by M . In particular, for $K \in L$:

$$D[M](K) = \text{Set}^L(D(K), M) ;$$

and for $K \xrightarrow{p} K_p$ in L :

$$\left\{ \begin{array}{l} D[M](K) \xrightarrow{D[M](p)} D[M](K_p) \\ D(K) \xrightarrow{\alpha} M \quad \xrightarrow{D(p)} \quad D(K_p) \rightarrow D(K) \xrightarrow{\alpha} M \\ D[M](p)(\alpha) = \alpha \circ D(p) \end{array} \right.$$

Similarly, "realizing" \dot{D} in M :

$$\dot{D}[M] \stackrel{\text{def}}{=} \text{set}^L(\dot{D}(-), M) : (\text{Set}^L)^{\text{op}} \longrightarrow \text{Set}$$

$$\dot{D}[M](U) = \text{Set}^L(\dot{D}(U), M)$$

We have:

$$\dot{D}[M](\hat{K}) = D[M](K) \quad (K \in L)$$

since $\dot{D} \circ \gamma = D$.

Let us see the special case for $D = Y$.

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$$Y[M] = \text{Set}^L(Y(-), M)$$

$$Y[M](K) = \text{Set}^L(\hat{K}, M)$$

We have the Yoneda isomorphism, for each $K \in L$:

$$y[M]_K : \text{Set}^L(\hat{K}, M) \xrightarrow{\cong} M(K)$$

such that for $a \in M(K)$

$$\hat{K} \xrightarrow{\hat{a}} M \xrightarrow{y[M]_K} \hat{a}_K(1_K) = a$$

$$(\hat{K}(K) \xrightarrow{\hat{a}_K} M(K); 1_K \mapsto \hat{a}_K(1_K)).$$

$y[M]_K$ is natural in K ; we have

$$y[M] : Y[M] \xrightarrow{\cong} M$$

Also: $\overset{\circ}{Y}[M](U) = \text{Set}^L(\overset{\circ}{Y}(U), M) = \text{Set}^L(U, M)$;

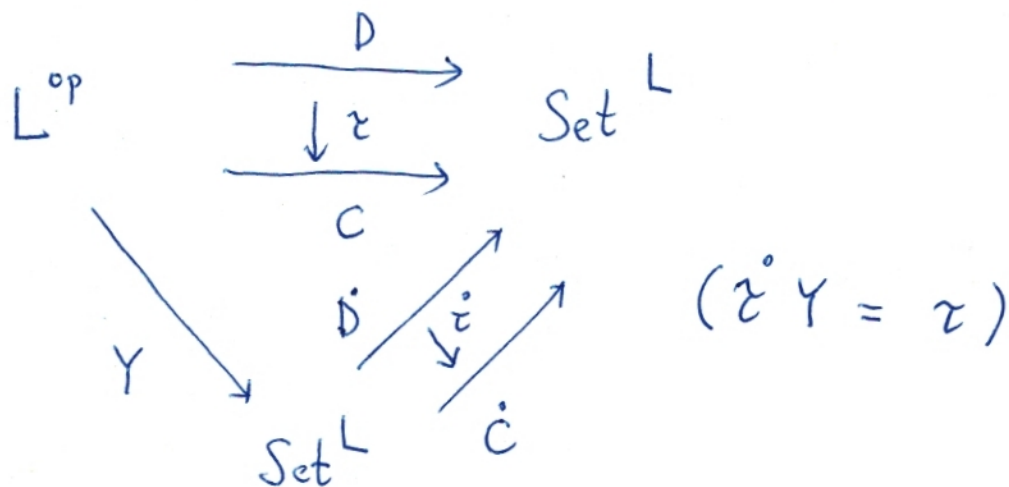
$$\overset{\circ}{Y}[M] = \text{Set}^L(-, M) : (\text{Set}^L)^{\text{op}} \rightarrow \text{Set}$$

the contravariant representable given by M .

Given $\tau: D \rightarrow C$ in $(\text{Set}^L)^{L^{\text{op}}}$,

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and the items induced as in the diagram



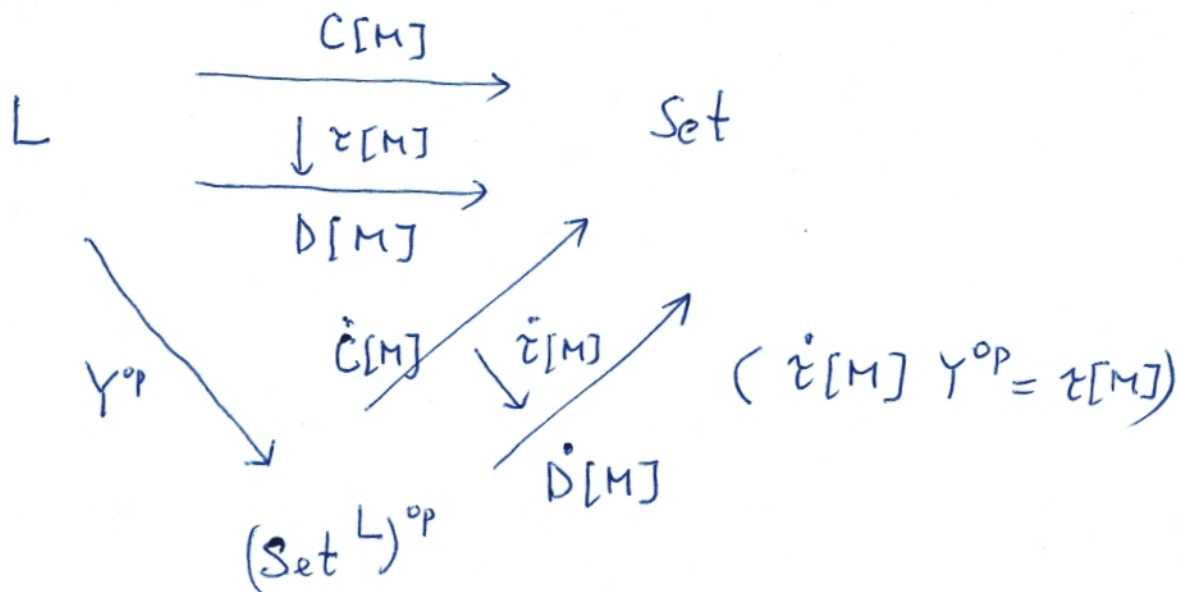
and also $M \in \text{Set}^L$, we define

$$\tau[M]: C[M] \longrightarrow D[M] \quad (\text{contravariance!})$$

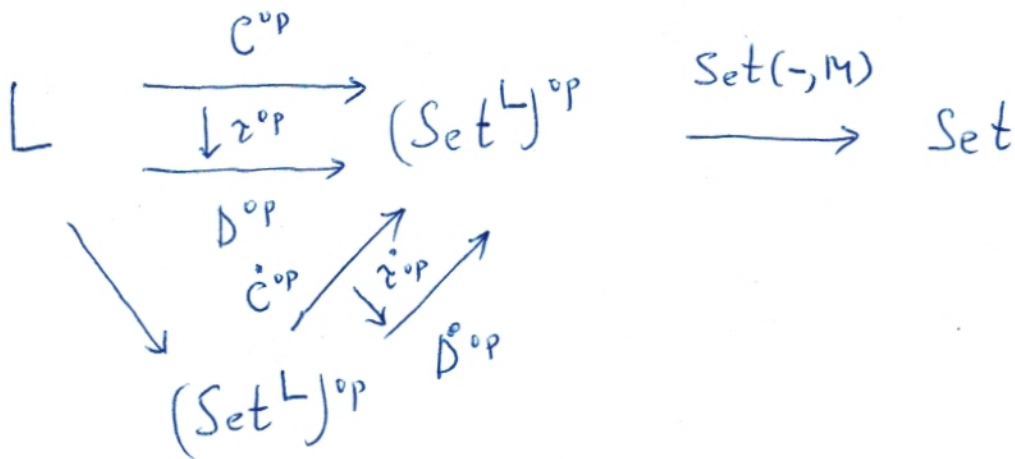
and

$$\dot{\tau}[M]: \dot{C}[M] \longrightarrow \dot{D}[M]$$

giving rise to



by taking opposites in the previous diagram, and whiskering by $\text{Set}(-, M) : (\text{Set}^L)^{\text{op}} \rightarrow \text{Set}$:



In particular, for $K \in L$ and $\gamma : C(K) \rightarrow M$

$$\tau[M]_K = (-) \circ \tau_K : C[M](K) \rightarrow D[M](K)$$

$$\tau[M]_K(\gamma) = \gamma \tau_K$$

For $U \in \text{Set}^L$ and $\xi : \hat{C}(U) \rightarrow M$:

$$\dot{\tau}[M]_U = (-) \circ \dot{\tau}_U : \dot{C}[M](U) \rightarrow \dot{D}[M](U)$$

$$\dot{\tau}[M]_U(\xi) = \xi \dot{\tau}_U$$

Proposition

Part (i). Given: $D: L^{op} \rightarrow Set^L$

$$U \in Set^L$$

$$M \in Set^L$$

We have the bijection

$$\Sigma[D, M](U) : Set^L(U, D[M]) \xrightarrow{\cong} Set^L(\dot{D}(U, M))$$

pictured as

$$\frac{U \xrightarrow{\delta} D[M]}{\dot{D}(U) \xrightarrow{\delta^\#} M}$$

or even

$$\Sigma[D] : \delta \longleftrightarrow \delta^\#$$

where the connection between δ and $\delta^\#$ is:

$$D(K) \xrightarrow{\dot{D}(u)} \dot{D}(U)$$



$$\begin{array}{ccc} & \cong & \\ \delta_K(u) \searrow & & \swarrow \delta^\# \\ & M & \end{array} \quad \text{for all } (K, u) \in [U]$$

Note that $\delta_K : U(K) \rightarrow D[M](K)$

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$$= \text{Set}^L(D(K), M)$$

hence, for $u \in U(K)$, $\delta_K(u) : D(K) \rightarrow M$.

Part (ii) Given $D, C : L^{\text{op}} \rightarrow \text{Set}^L$,

$$\gamma : D \rightarrow C$$

and $\Sigma[D] : \delta \leftrightarrow \delta^\#$,

$\Sigma[C] : \gamma \leftrightarrow \gamma^\#$

we have that ① commutes iff ② commutes,

where ① and ② are the diagrams

$$U \xrightarrow{\gamma} C[M]$$

$$D(u) \xrightarrow{\gamma_u} C(u)$$

$$\begin{array}{ccc} \delta \searrow & \textcircled{1} & \swarrow \gamma[M] \\ & & \\ & D[M] & \end{array} \quad (\text{iff})$$

$$\begin{array}{ccc} \delta^\# \searrow & \textcircled{2} & \swarrow \gamma^\# \\ & & \\ & M & \end{array}$$

[For proof: see Appendix]

We explore the contents of the Proposition for the special case when $D = Y$, the Yoneda functor $Y: L^{op} \rightarrow Set^L$. In what follows, U and M are fixed objects of Set^L .

The first observation is that the bijection

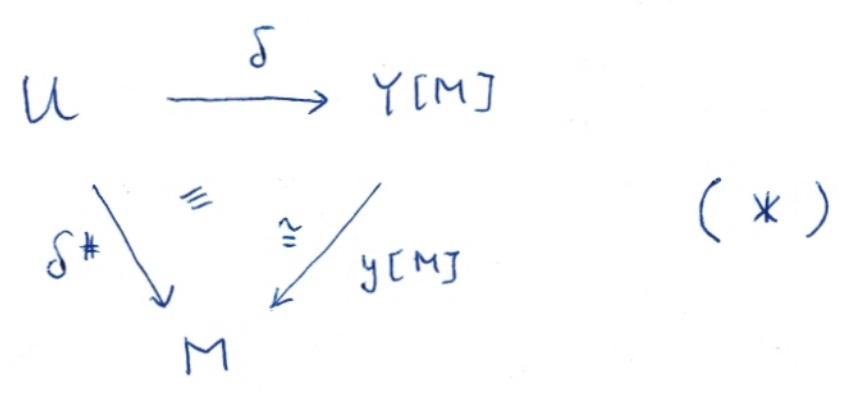
$$\Sigma[Y]: Set^L(U, Y[M]) \xrightarrow{\cong} Set^L(Y(U), M)$$

reduces to the equality $\delta^\# = y[M] \circ \delta$, with

the Yoneda isomorphism $y[M]: Y[M] \xrightarrow{\cong} M$

ensuring that " $\delta^\#$ is almost the same as δ ," :

In picture:



[For proof: see Appendix]

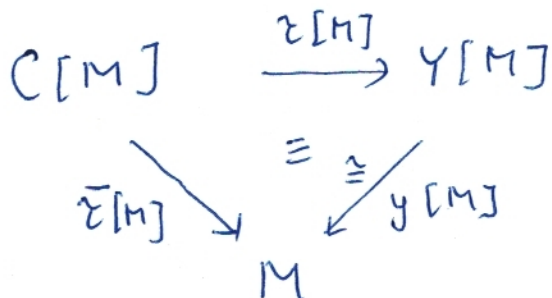
Now, consider and fix a functor $C: L^{op} \rightarrow Set$, and a natural transformation $\tau: Y \rightarrow C$.

We have the induced $\tau[M]: C[M] \rightarrow Y[M]$.

I write

$$\bar{\varepsilon}[M] \stackrel{\text{def}}{=} \gamma[M] \circ \varepsilon[M]$$

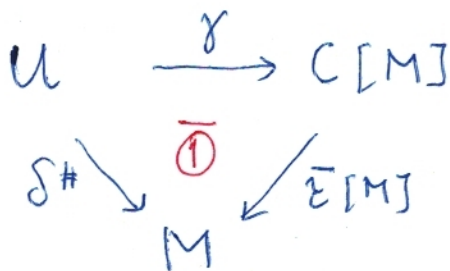
$$\bar{\varepsilon}[M]: C[M] \rightarrow M$$



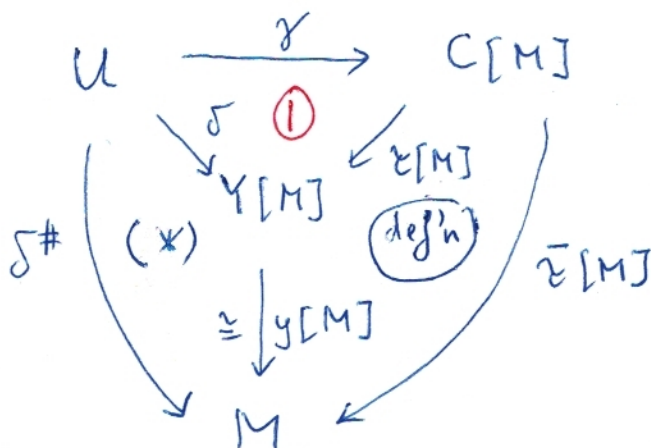
$\bar{\varepsilon}[M]$ is almost the same as $\varepsilon[M]$ - but only almost.

The commutativity of $\textcircled{1}$, p 16 (if true) is

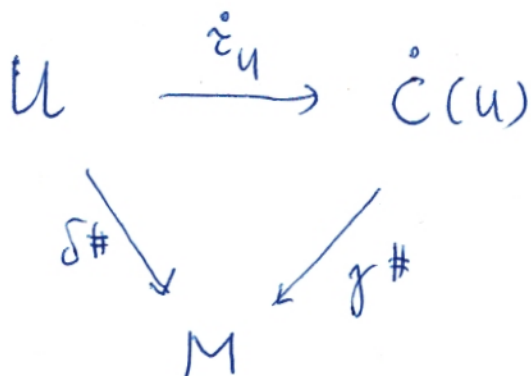
the same as the commutativity of



because:



② p. 16 be.



Conclusion: Given the data

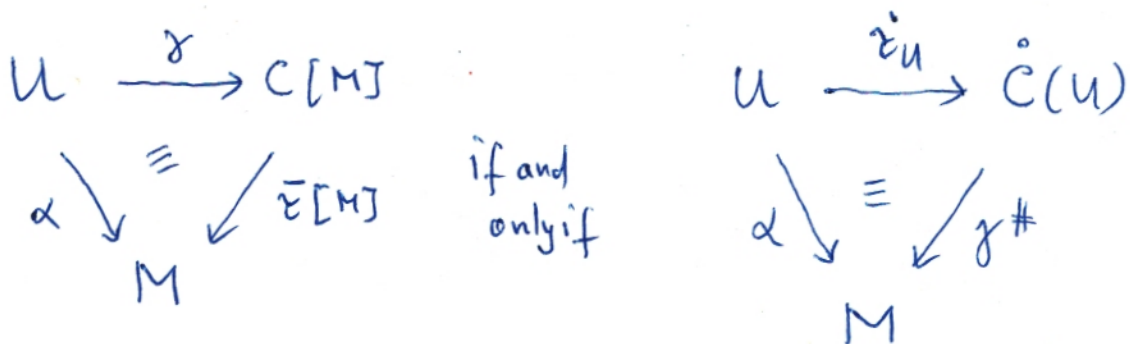
$$z: Y \rightarrow C$$

$$U, M \in \text{Set}^L$$

$$\gamma: U \rightarrow C[M]$$

$$\alpha: U \rightarrow M$$

We have:



where $\gamma\# \stackrel{\text{def}}{=} \Sigma[C, M](U)(\gamma)$

Remark A fact about the bijection

$$\Sigma [C, M](U) : \gamma \longmapsto \gamma^\#$$

in general: the square

$$\begin{array}{ccc} U & \xrightarrow{\gamma} & C[M] \\ \downarrow \tilde{\gamma}_U & \cong & \downarrow \tilde{\gamma}[M] \\ \dot{C}(U) & \xrightarrow{\gamma^\#} & M \end{array}$$

Commutates (see the Appendix)

Anticipating the use of the foregoing material:

let: $H : L^{op} \longrightarrow \text{Set}^L$ ("H = C")

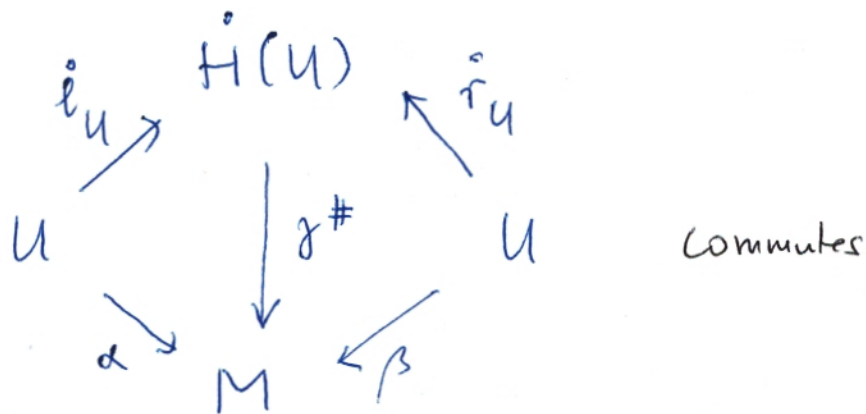
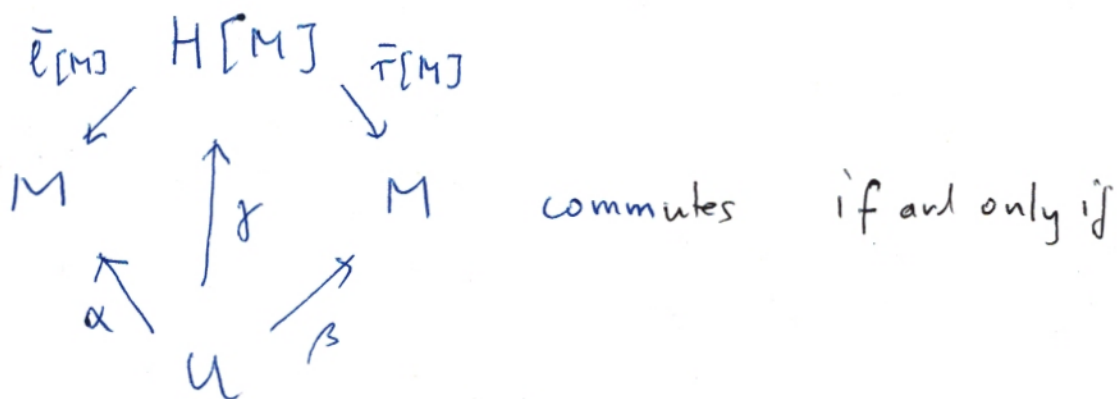
$$\begin{array}{ccc} Y & \xrightarrow{\ell} & H \\ & \xrightarrow{\tau} & \end{array}$$

$$U, M \in \text{Set}^L$$

$$\alpha, \beta : U \rightarrow M$$

$$\gamma : U \rightarrow H[M]$$

Then:



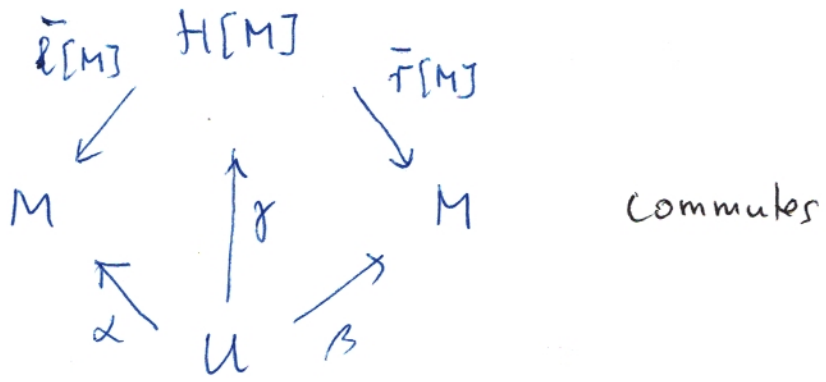
here, again, $\hat{\gamma} = \Sigma[H, M](U)(\gamma)$.

Since $\Sigma[H, M](U)$ is a bijection,

we have:

there exists $\gamma: U \rightarrow H[M]$

Such that



if and only if

there exists $\sigma: H(U) \rightarrow M$

Such that

