Part 1 Notation

$\mathbb{L}$ is a fixed small category throughout; $K, K', \dots$ objects of $\mathbb{L}$ ('K' for 'kind'...). The codomain of arrow p is denoted (sometimes) as K_p; $p: K \to K_p$.

$M, N, P, \dots$ denote objects of the functor category $Set^{\mathbb{L}}$, and so do $\mathcal{U}, \mathcal{V}, \dots$: $\mathcal{U}, \mathcal{V} \in Set^{\mathbb{L}}$.

$\mathcal{U}, \mathcal{V}$ play different roles than M, N, P. $\mathcal{U}, \mathcal{V}$ are 'diagram types'; M, N, P are 'structures'; an arrow $\alpha : \mathcal{U} \to M$ is a type-$\mathcal{U}$ diagram in M. However, no formal distinction is made between the $\mathcal{U}$'s and the M's. Unless explicitly said (and infrequently so), $\mathbb{L}$ is not assumed to be a FOLDS signature. Of the diagram types $\mathcal{U}, \mathcal{V}$ finiteness is not assumed — unless explicitly done so.

§ 1.1 Left Kan extension along Yoneda

Let $\mathcal{S}$ be any (small-) cocomplete category. Eventually $\mathcal{S}$ will be Set^L, but, for the first pieces of the notation, specifying $\mathcal{S}$ any further would only obscure the picture.

Let: $D : L^{op} \longrightarrow \mathcal{S}$.

$\mathring{D} : Set^L \longrightarrow \mathcal{S}$ denotes 'the' left Kan extension of D along the Yoneda functor

$Y : L^{op} \longrightarrow Set^L$.

$\mathring{D}$ is determined only up to isomorphism; the freedom of choosing a particular representative of the isomorphism class will be used.

For each $U \in Set^L$, the object $\mathring{D}(U)$ is determined as a colimit as follows. Let $el(U)$ be the category of elements of the functor $U : L \to Set$; the objects of $el(U)$ are pairs (K, u) where $K \in L$ and $u \in U(K)$; an arrow $(K, u) \xrightarrow{p} (K_p, v)$ is the same as $p : K \to K_p$ subject to the condition $U(p)(u) = v$. I write $[U]$ for the opposite category $(el(U))^{op}$. The objects of $[U]$

are the same pairs (K, u) as before. However when I write $p : (K, u) \to (K_p, v)$, I always mean the arrow in $el(\mathcal{U})$, not in $[\mathcal{U}]$. In $[\mathcal{U}]$, this would be $(K_p, v) \to (K, u)$, an unpleasant sight. This will result in some contravariance in the notation.

We have the forgetful functor

$$\Phi_{\mathcal{U}} : [\mathcal{U}] \longrightarrow L^{op}$$
$$(K, u) \longmapsto K$$

and the composite

$$D\Phi_{\mathcal{U}} : [\mathcal{U}] \longrightarrow \mathbb{S}$$
$$(K, u) \longmapsto D(K)$$

The promised (well-known) colimit expression of $\dot{D}(\mathcal{U})$ is:

$$\dot{D}(\mathcal{U}) = \operatorname{colim}(D\Phi_{\mathcal{U}})$$

or more easily

$$\dot{D}(\mathcal{U}) = \operatorname*{colim}_{(K,u) \in [\mathcal{U}]} D(K)$$

Let us write for the colimit coprojections:

$$\varphi^{D,\mathcal{U}}_{K,u} \;(= \varphi^{D}_{K,u} = \varphi_{K,u}) : D(K) \longrightarrow \dot{D}(\mathcal{U}).$$

The effect of the functor $\dot{D} : Set^{L} \longrightarrow \mathcal{S}$ on arrow $U \xrightarrow{f} V$ in Set^{L} is determined by the joint commutativity of the diagrams

$$\begin{tikzcd}
\dot{D}(U) \arrow[rr, "\dot{D}(f)"] & & \dot{D}(V) \\
& D(K) \arrow[ul, "\varphi^{D,U}_{K,u}"] \arrow[ur, "\varphi^{D,V}_{K,f_K(u)}"'] &
\end{tikzcd}$$

one for each $(K, u) \in [U]$.

It is useful to note that Set^{L} is the (small-) colimit completion of L^{op} via $Y : L^{op} \to$ and $\dot{D} : Set^{L} \longrightarrow \mathcal{S}$ preserves all small colimits in Set^{L}. Of course, the explicit formula given above for $\dot{D}(U)$ is an instance of this fact.

The left Kan extension $\dot{D} : Set^{L} \to \mathcal{S}$ comes with a canonical isomorphism (in this case):

$$\Theta : D \xrightarrow{\cong} \dot{D}Y \quad \text{in} \quad \mathcal{S}^{L^{op}}$$

whose component at $K \in L$

$$\Theta_K : D(K) \longrightarrow \dot{D}(\hat{K})$$

(where we write $\hat{K} = Y(K) = \mathrm{Set}^{L}(K, -)$

for the <u>covariant</u> representable, for now)

is, in terms of the notation already introduced

as follows:

$$\oplus_K \underset{\mathrm{def}}{=} \varphi^{D, \hat{K}}_{(K, 1_K \in \hat{K}(K))} : D(K) \longrightarrow \dot{D}(\hat{K})$$

Now we use our freedom of choosing the value

of $\dot{D}(\hat{K})$, together with the colimit cocone with

components

$$\varphi^{D, \hat{K}}_{(K_p,\, p : K \to K_p)} : D(K_p) \longrightarrow \dot{D}(\hat{K})$$

as follows: put $\dot{D}(\hat{K}) \underset{\mathrm{def}}{=} D(K)$

and

$$\varphi^{D, \hat{K}}_{(K_p,\, p : K \to K_p)} \underset{\mathrm{def}}{=} D(p)$$

In this way, we indeed obtain a colimit cocone

for the colimit $\dot{D}(\hat{K})$, since now, for $U = \hat{K}$,

$[\hat{K}]$ is a category with the terminal object $(K, 1_K)$, so for the diagram $[\hat{K}] \xrightarrow{D\Phi_{\hat{K}}} \mathcal{S}$ a correct choice of the colimit consists in

$$\operatorname{colim} D\Phi_{\hat{K}} \underset{def}{=} D\Phi_{\hat{K}}((K, 1_K)) = D(K)$$

with coprojections

$$D\Phi_{\hat{K}}((K_p, p : K \to K_p)) = D(p) : D(K_p) \longrightarrow D(K).$$

In particular, the canonical comparison isomorphism

$$\Theta_K = \varphi^{D, \hat{K}}_{(K, 1_K : K \to K)} : D(K) \longrightarrow \mathring{D}(\hat{K})$$

is $\Theta_K = D(1_K) = 1_{D(K)} : D(K) \to D(K)$

Thus, $\mathring{D}$ strictly extends D:

$$\begin{tikzcd} L^{op} \arrow[r, "D"] \arrow[d, "Y"'] \arrow[dr, phantom, "\equiv"] & \mathcal{S} \\ Set^L \arrow[ur, "\mathring{D}"'] & \end{tikzcd} \qquad \mathring{D}\, Y = D$$

Now, let, from now on, $\mathcal{S} = Set^L$. So, the Yoneda functor $Y : L^{op} \longrightarrow Set^L$ is one of the possible D's. For $U \in Set^L$, we define the colimit

$$\mathring{Y}(U) = \operatorname*{colim}_{(K,u) \in [U]} \hat{K}$$

by setting $\mathring{Y}(U) \underset{def}{=} U$

with coprojections $\varphi^{Y,U}_{K,u} : \underset{\underset{Y(K)}{\shortparallel}}{\hat{K}} \longrightarrow U$

to be $\varphi^{Y,U}_{K,u} \underset{def}{=} \hat{u}$,

where $\hat{u}$ is the Yoneda equivalent $\hat{u} : \hat{K} \to U$ of $u \in U(K)$, determined by the equality $\hat{u}_K(1_K) = u$.

This is the standard representation of $U \in Set^L$ as a colimit of representables.

When we look at how $\mathring{Y}(f) : \mathring{Y}(U) \longrightarrow \mathring{Y}(V)$ $(f : U \to V)$ is determined via coprojections (see above), we see that $\mathring{Y}(f) = f$. Thus

$$\mathring{Y} = Id_{Set^L}$$

We can simplify the notation by identifying the coprojections in this way:

$$\varphi^{D,\mathcal{U}}_{(K,u)} = \mathring{D}(\hat{u})$$

in the general case. (Both arrows are of the form $D(K) = \mathring{D}(\hat{K}) \longrightarrow \mathring{D}(\mathcal{U})$.) The reason: apply $\mathring{D} : Set^L \longrightarrow \mathcal{S}$ to the arrow $\hat{u} : \hat{K} \to \mathcal{U}$ in Set^L according to the recipe above; we get

$$\begin{tikzcd}
\mathring{D}(\hat{K}) \arrow[rr, "\mathring{D}(\hat{u})"] \arrow[dr, phantom, "\equiv"] & & \mathring{D}(\mathcal{U}) \\
& D(K) \arrow[ul, "{1_{D(K)} = D(1_K) = \varphi^{D,\hat{K}}_{K,1_K}}"] \arrow[ur, "{\varphi^{D,\mathcal{U}}_{K,\, \hat{u}_K(1_K) = u}}"'] &
\end{tikzcd}$$

.

Left Kan extension is a functor.

For $D, C : L^{op} \longrightarrow \mathcal{S}$, $\tau : D \to C$ $\left(\in (Set^L)^{L^{op}}(D,C)\right)$ we have $\mathring{\tau} : \mathring{D} \longrightarrow \mathring{C}$ $\left(\in (Set^L)^{(Set^L)}(\mathring{D}, \mathring{C})\right)$; for $\mathcal{U} \in Set^L$, $\mathring{\tau}_{\mathcal{U}} : \mathring{D}(\mathcal{U}) \longrightarrow \mathring{C}(\mathcal{U})$ is determined by: the commutativities

$$\begin{tikzcd}
D(K) \arrow[r, "\dot{D}(\hat{u})"] \arrow[d, "\tau_K"'] & \dot{D}(U) = \operatorname{colim}(D\Phi_U) \arrow[d, "\dot{\tau}_U"] \\
C(K) \arrow[r, "\dot{C}(\hat{u})"'] & \dot{C}(U)
\end{tikzcd}$$

one for each $(K,u) \in [U]$.

<u>Remark</u> For any functor $F : Set^L \to \mathcal{B}$ the arrows $F(\hat{u}) : F(\hat{K}) \to F(U)$ $((K,u) \in [U])$ form a cocone with vertex $F(U)$:

$$\begin{tikzcd}
(K,u) \arrow[d, "p"] & & F(\hat{K}) \arrow[dr, "F(\hat{u})"] \arrow[d, phantom, ""] & \\
(K_p, v) & & F(\hat{K}_p) \arrow[u, "F(Y(p))"] \arrow[r, phantom, "\equiv"] \arrow[r, "F(\hat{v})"'] & F(U)
\end{tikzcd}$$

$v = U(p)(u)$

since

$$\begin{tikzcd}
\hat{K} \arrow[dr, "\hat{u}"] & \\
\hat{K} \arrow[u, "Y(p)"] \arrow[r, "\hat{v}"'] \arrow[r, phantom, "\equiv"] & U
\end{tikzcd}$$

.

F is the left Kan extension of $D = F\circ Y$ along Y with transition $\Theta : D \to F\circ Y$ the identity iff the cocone $\langle F(\hat{u}) : F(\hat{K}) \to F(\mathcal{U})\rangle_{(K,u)\in[\mathcal{U}]}$ is a <u>colimit</u> cocone for all $\mathcal{U} \in Set^L$.

§ 1.2 Realizing the schemes $D : L^{op} \longrightarrow Set^L$ and $\mathring{D} : Set^L \longrightarrow Set^L$ in the structure $M \in Set^L$.

We have put $\mathbb{S} = Set^L$.

Given $D : L^{op} \longrightarrow Set^L$ and $M \in Set^L$,

$$D[M] : L \longrightarrow Set, \qquad D[M] \in Set^L$$

the "realization of D in M" is

$$D[M] \underset{def}{=} Set^L(D(-), M) : L \to Set;$$

$D[M]$ is the composite

$$L \xrightarrow{D^{op}} (Set^L)^{op} \xrightarrow[Set^L(-,M)]{} Set.$$

where $\mathrm{Set}^L(-, M)$ is the contravariant representable given by M. In particular, for $K \in L$:

$$D[M](K) = \mathrm{Set}^L(D(K), M) \, ;$$

and for $K \xrightarrow{p} K_p$ in L:

$$\begin{cases} D[M](K) \xrightarrow{D[M](p)} D[M](K_p) \\ D(K) \xrightarrow{\alpha} M \longmapsto D(K_p) \xrightarrow[D(p)]{} D(K) \xrightarrow{\alpha} M \\ D[M](p)(\alpha) = \alpha \circ D(p) \end{cases}$$

Similarly, "realizing" $\mathring{D}$ in M:

$$\mathring{D}[M] \underset{\mathrm{def}}{=} \mathrm{Set}^L(\mathring{D}(-), M) : (\mathrm{Set}^L)^{op} \longrightarrow \mathrm{Set}$$

$$\mathring{D}[M](U) = \mathrm{Set}^L(\mathring{D}(U), M)$$

We have:

$$\mathring{D}[M](\hat{K}) = D[M](K) \qquad (K \in L)$$

since $\dot{D} \circ Y = D$.

Let us see the special case for $D = Y$.

$$Y[M] = \mathrm{Set}^{L}(Y(-), M)$$

$$Y[M](K) = \mathrm{Set}^{L}(\hat{K}, M)$$

We have the Yoneda isomorphism, for each $K \in L$:

$$y[M]_K : \mathrm{Set}^{L}(\hat{K}, M) \xrightarrow{\cong} M(K)$$

such that for $a \in M(K)$

$$\hat{K} \xrightarrow{\hat{a}} M \overset{y[M]_K}{\longmapsto} \hat{a}_K(1_K) = a$$

$$(\hat{K}(K) \xrightarrow[\hat{a}_K]{} M(K);\ 1_K \mapsto \hat{a}_K(1_K)).$$

$y[M]_K$ is natural in K; we have

$$y[M] : Y[M] \xrightarrow{\cong} M$$

Also: $\quad \mathring{Y}[M](U) = \mathrm{Set}^{L}(\dot{Y}(U), M) = \mathrm{Set}^{L}(U, M)$;

$$\mathring{Y}[M] = \mathrm{Set}^{L}(-, M) : (\mathrm{Set}^{L})^{op} \longrightarrow \mathrm{Set}$$

the contravariant representable given by M.

Given $\tau : D \to C$ in $(\mathbf{Set}^L)^{L^{op}}$,

and the items induced as in the diagram

$$\begin{tikzcd}
L^{op} \arrow[r, shift left=2, "D"] \arrow[r, shift right=2, "C"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[dr, "Y"'] & \mathbf{Set}^L \\
& \mathbf{Set}^L \arrow[u, shift right=2, "\dot{D}"] \arrow[u, shift left=2, "\dot{C}"'] \arrow[u, phantom, "\Downarrow\dot{\tau}"]
\end{tikzcd}
\qquad (\dot{\tau} Y = \tau)$$

and also $M \in \mathbf{Set}^L$, we <u>define</u>

$$\tau[M] : C[M] \longrightarrow D[M] \quad \text{(contravariance!)}$$

and

$$\dot{\tau}[M] : \dot{C}[M] \longrightarrow \dot{D}[M]$$

giving rise to

$$\begin{tikzcd}
L \arrow[r, shift left=2, "{C[M]}"] \arrow[r, shift right=2, "{D[M]}"'] \arrow[r, phantom, "{\Downarrow\tau[M]}"] \arrow[dr, "Y^{op}"'] & \mathbf{Set} \\
& (\mathbf{Set}^L)^{op} \arrow[u, shift right=2, "{\dot{C}[M]}"] \arrow[u, shift left=2, "{\dot{D}[M]}"'] \arrow[u, phantom, "{\Downarrow\dot{\tau}[M]}"]
\end{tikzcd}
\qquad (\dot{\tau}[M]\, Y^{op} = \tau[M])$$

by taking opposites in the previous diagram, and whiskering by $\mathrm{Set}(-,M) : (\mathrm{Set}^L)^{op} \to \mathrm{Set}$:

$$\begin{tikzcd}
L \arrow[rr, bend left, "C^{op}"] \arrow[rr, bend right, "D^{op}"'] \arrow[rr, phantom, "\Downarrow \tau^{op}"] \arrow[dr] & & (\mathrm{Set}^L)^{op} \arrow[r, "\mathrm{Set}(-,M)"] & \mathrm{Set} \\
& (\mathrm{Set}^L)^{op} \arrow[ur, bend left, "\dot{C}^{op}"] \arrow[ur, bend right, "\mathring{D}^{op}"'] \arrow[ur, phantom, "\Downarrow \mathring{\tau}^{op}"] & &
\end{tikzcd}$$

In particular, for $K \in L$ and $\gamma : C(K) \to M$

$$\tau[M]_K = (-)\circ \tau_K : C[M](K) \longrightarrow D[M](K)$$

$$\tau[M]_K(\gamma) = \gamma\, \tau_K .$$

For $U \in \mathrm{Set}^L$ and $\xi : \dot{C}(U) \to M$:

$$\dot{\tau}[M]_U = (-)\circ \mathring{\tau}_U : \dot{C}[M](U) \to \mathring{D}[M](U)$$

$$\dot{\tau}[M]_U(\xi) = \xi\, \mathring{\tau}_U .$$

Proposition

Part (i). Given: $D : L^{op} \longrightarrow Set^L$

$$U \in Set^L$$

$$M \in Set^L$$

we have the bijection

$$\Sigma[D,M](U) : Set^L(U, D[M]) \xrightarrow{\cong} Set^L(\dot{D}(U,M))$$

pictured as

$$\frac{U \xrightarrow{\delta} D[M]}{\dot{D}(U) \xrightarrow[\delta^{\#}]{} M}$$

or even

$$\Sigma[D] : \delta \longleftrightarrow \delta^{\#}$$

where the connection between δ and $\delta^{\#}$ is:

$$\odot \quad \begin{tikzcd} D(K) \arrow[rr, "\mathring{D}(\hat{u})"] \arrow[dr, "\delta_K(u)"'] & \equiv & \mathring{D}(U) \arrow[dl, "\delta^{\#}"] \\ & M & \end{tikzcd} \quad \text{for all } (K,u) \in [U]$$

Note that $\delta_K : U(K) \longrightarrow D[M](K)$

$$= Set^L(D(K), M)$$

hence, for $u \in U(K)$, $\delta_K(u) : D(K) \to M$.

Part (ii) Given $D, C : L^{op} \longrightarrow Set^L$,

$$\tau : D \to C$$

and $\Sigma[D] : \delta \longleftrightarrow \delta^{\#}$,

$$\Sigma[C] : \gamma \longleftrightarrow \gamma^{\#}$$

we have that ① commutes iff ② commutes, where ① and ② are the diagrams

$$\begin{tikzcd} U \arrow[rr, "\gamma"] \arrow[dr, "\delta"'] \arrow[rr, phantom, shift right=4ex, "1"] & & C[M] \arrow[dl, "\tau[M]"] \\ & D[M] & \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} \dot{D}(U) \arrow[rr, "\dot{\tau}_U"] \arrow[dr, "\delta^{\#}"'] \arrow[rr, phantom, shift right=4ex, "2"] & & \dot{C}(U) \arrow[dl, "\gamma^{\#}"] \\ & M & \end{tikzcd}$$

[For proof: see Appendix]

We explore the contents of the Proposition for the special case when $D = Y$, the Yoneda functor $Y : L^{op} \to Set^L$. In what follows, $\mathcal{U}$ and M are fixed objects of Set^L.

The first observation is that the bijection

$$\Sigma[Y] : Set^L(\mathcal{U}, Y[M]) \xrightarrow{\cong} Set^L(\mathring{Y}(\mathcal{U}), M)$$

reduces to the equality $\delta^\# = y[M] \circ \delta$, with the Yoneda isomorphism $y[M] : Y[M] \xrightarrow{\cong} M$ ensuring that "$\delta^\#$ is almost the same as δ":

In picture:

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\delta"] \arrow[dr, "\delta^\#"'] & \equiv & Y[M] \arrow[dl, "y[M]", "\cong"'] \\ & M & \end{tikzcd} \qquad (*)$$

[For proof: see Appendix]

Now, consider and fix a functor $C : L^{op} \to Set$, and a natural transformation $\tau : Y \to C$.

We have the induced $\tau[M] : C[M] \longrightarrow Y[M]$.

I write

$$\bar{\varepsilon}[M] \underset{def}{=} y[M] \circ \varepsilon[M]$$

$$\bar{\varepsilon}[M] : C[M] \to M$$

$$\begin{tikzcd}
C[M] \arrow[rr, "{\varepsilon[M]}"] \arrow[dr, "{\bar{\varepsilon}[M]}"'] \arrow[rr, phantom, "\equiv", yshift=-6ex] & & Y[M] \arrow[dl, "{y[M]}", "\cong"'] \\
& M &
\end{tikzcd}$$

$\bar{\varepsilon}[M]$ is almost the same as $\varepsilon[M]$ – but only almost.

The commutativity of 1, p $\boxed{16}$ (if true) is the same as the commutativity of

$$\begin{tikzcd}
U \arrow[rr, "\gamma"] \arrow[dr, "\delta^{\#}"'] \arrow[rr, phantom, "\bar{1}", yshift=-6ex] & & C[M] \arrow[dl, "{\bar{\varepsilon}[M]}"] \\
& M &
\end{tikzcd}$$

because :

$$\begin{tikzcd}
U \arrow[rr, "\gamma"] \arrow[dr, "\delta"] \arrow[ddr, bend right, "\delta^{\#}"'] \arrow[rr, phantom, "1", yshift=-6ex] & & C[M] \arrow[dl, "{\varepsilon[M]}"'] \arrow[ddl, bend left, "{\bar{\varepsilon}[M]}"] \\
& Y[M] \arrow[d, "{y[M]}", "\cong"'] \arrow[l, phantom, "(*)"] \arrow[r, phantom, "\text{def'n}"] & \\
& M &
\end{tikzcd}$$

② p. $\boxed{16}$ be

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\mathring{\iota}_{\mathcal{U}}"] \arrow[dr, "\delta^{\#}"'] & & \mathring{C}(\mathcal{U}) \arrow[dl, "\gamma^{\#}"] \\ & M & \end{tikzcd}$$

<u>Conclusion</u>: Given the data

$\iota : Y \longrightarrow C$

$\mathcal{U}, M \in Set^{L}$

$\gamma : \mathcal{U} \longrightarrow C[M]$

$\alpha : \mathcal{U} \to M$

we have:

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\gamma"] \arrow[dr, "\alpha"'] \arrow[r, phantom, shift right=4ex, "\equiv"] & & C[M] \arrow[dl, "\bar{\iota}[M]"] \\ & M & \end{tikzcd} \quad \text{if and only if} \quad \begin{tikzcd} \mathcal{U} \arrow[rr, "\mathring{\iota}_{\mathcal{U}}"] \arrow[dr, "\alpha"'] \arrow[r, phantom, shift right=4ex, "\equiv"] & & \mathring{C}(\mathcal{U}) \arrow[dl, "\gamma^{\#}"] \\ & M & \end{tikzcd}$$

where $\gamma^{\#} \underset{\text{def}}{=} \Sigma[C, M](\mathcal{U})(\gamma)$

<u>Remark</u> A fact about the bijection

$$\Sigma[C, M](U) : \quad \gamma \longmapsto \gamma^{\#}$$

in general: the square

$$\begin{tikzcd}
U \arrow[r, "\gamma"] \arrow[d, "\dot{\varepsilon}_U"'] \arrow[dr, phantom, "="] & C[M] \arrow[d, "\bar{\varepsilon}[M]"] \\
\dot{C}(U) \arrow[r, "\gamma^{\#}"'] & M
\end{tikzcd}$$

commutes (see the Appendix)

Anticipating the use of the foregoing material:

let: $\quad H : L^{op} \longrightarrow Set^L \qquad (\text{"}H = C\text{"})$

$$Y \underset{r}{\overset{l}{\rightrightarrows}} H$$

$$U, M \in Set^L$$

$$\alpha, \beta : U \to M$$

$$\gamma : U \to H[M]$$

Then:

$$\begin{tikzcd}
& H[M] \arrow[dl, "{\bar{\ell}[M]}"'] \arrow[dr, "{\bar{r}[M]}"] & \\
M & & M \\
& U \arrow[ul, "\alpha"] \arrow[uu, "\gamma"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

commutes if and only if

$$\begin{tikzcd}
& \mathring{H}(U) \arrow[dd, "\gamma^{\#}"] & \\
U \arrow[ur, "\mathring{\ell}_U"] \arrow[dr, "\alpha"'] & & U \arrow[ul, "\mathring{r}_U"'] \arrow[dl, "\beta"] \\
& M &
\end{tikzcd}$$

commutes

here, again, $\gamma^{\#} = \Sigma[H,M](U)(\gamma)$.

Since $\Sigma[H,M](U)$ is a bijection,

we have:

$\boxed{\text{there exists}}$ $\gamma : U \longrightarrow H[M]$

Such that

$$\begin{tikzcd}
& H[M] \arrow[dl, "{\bar{\ell}[M]}"'] \arrow[dr, "{\bar{r}[M]}"] & \\
M & & M \\
& U \arrow[ul, "\alpha"] \arrow[uu, "\gamma"'] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

commutes

if and only if

$\boxed{\text{there exists}}$ $\sigma : \mathring{H}(U) \longrightarrow M$

Such that

$$\begin{tikzcd}
& \mathring{H}(U) \arrow[dd, "\sigma"] & \\
U \arrow[ur, "\mathring{\ell}_U"] \arrow[dr, "\alpha"'] & & U \arrow[ul, "\mathring{r}_U"'] \arrow[dl, "\beta"] \\
& M &
\end{tikzcd}$$

Commutes