

# § 1.3 Miscellany

Let  $\mathcal{S}$  be a cocomplete category in which pushouts of mono(morphisms) are mono's:

$$\begin{array}{ccc}
 A & \longrightarrow & C \\
 i \downarrow & \lrcorner_{\text{po}} \downarrow j & \\
 B & \longrightarrow & D
 \end{array}
 \quad \& \quad i \text{ mono} \quad \Rightarrow \quad j \text{ mono}$$

An effective monomorphic (EM) square (with left vertical  $i$ ) in  $\mathcal{S}$  is a commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{f} & C \\
 i \downarrow & & \downarrow j \\
 B & \xrightarrow{g} & D
 \end{array}
 \quad \text{such that } 1) \text{ the left vertical } i$$

is a mono, and 2) for the pushout  $P = B \sqcup_A C$ :

the canonical morphism  $P \xrightarrow{m} D$  is a mono:

$$\begin{array}{ccccc}
 & & f & & \\
 & & \longrightarrow & & \\
 A & & & & C \\
 & \lrcorner_{\text{po}} & & \swarrow i' & \\
 & P & & & \downarrow j \\
 i \downarrow & \nearrow f' & & \searrow m & \\
 B & & \xrightarrow{g} & & D
 \end{array}
 \quad mi' = j, \quad mf' = g$$

In particular, as a consequence, the right vertical  $j$  is also a mono.

Assume:  $F, G: \mathcal{U} \rightarrow \mathcal{S}$  functors

$$\varphi: F \rightarrow G \in \mathcal{S}^{\mathcal{U}}(F, G)$$

Let  $U \xrightarrow{f} V$  be an arrow in  $\mathcal{U}$ . We say

that  $\varphi$  is EM at  $f$  if the naturality square of  $\varphi$  at  $f$  is EM in  $\mathcal{S}$ :

$$\begin{array}{ccc} FU & \xrightarrow{Ff} & FV \\ \varphi_U \downarrow & & \downarrow \varphi_V \\ GU & \xrightarrow{Gf} & GV \end{array} \text{ is EM.}$$

$EM[\varphi]$  denotes the class of all  $f \in \text{Arr}(\mathcal{U})$  such that  $\varphi$  is EM at  $f$ .

Assume, from now on, that both  $\mathcal{U}$  and  $\mathcal{S}$  are cocomplete categories, and both  $F$  and  $G$  are cocontinuous. For a class  $I$  of arrows in  $\mathcal{U}$ ,

let  $I^\#$  denote the (generalized) Gabriel/Zisman saturation of  $I$ , i.e. the least class of arrows in  $\mathcal{U}$  that contains  $I$ , and is closed under pushouts, transfinite compositions and retracts.

Proposition. Assume:  $\mathcal{U}, \mathcal{S}$  cocomplete categories,  $F, G: \mathcal{U} \rightarrow \mathcal{S}$  cocontinuous functors;  $\varphi: F \rightarrow G$ .

In addition, also assume that  $\mathcal{S}$  is locally finitely presentable (in the sense of Gabriel/Ulmer) and that pushouts of mono's in  $\mathcal{S}$  are mono's.

(Note that  $\mathcal{S} = \text{Set}^L$  meets all these conditions.)

Suppose  $I \subseteq \text{Arr}(\mathcal{U})$  such that  $I \subseteq \text{EM}(\varphi)$ .

Then  $I^\# \subseteq \text{EM}(\varphi)$  as well.

In particular, with  $\varnothing_{\mathcal{U}}$  the initial object of  $\mathcal{U}$ , if  $\varnothing_{\mathcal{U}} \xrightarrow{!} \mathcal{U}$  belongs to  $I^\#$ , then

$$\varphi_{\mathcal{U}}: F(\mathcal{U}) \rightarrow G(\mathcal{U})$$

is a monomorphism.

(For proof: see Appendix)

Let  $L$  be a FOLDS signature. Then, for any  $K \in L$ , we have the "sphere"  $\overset{\circ}{K} \in \text{Set}^L$ , the subfunctor of  $\hat{K} = L(K, -)$ , with inclusion

$$i_K: \overset{\circ}{K} \rightarrow \hat{K}$$

$\overset{\circ}{K}$  misses just one element of  $\text{el}(\hat{K})$ , the identity  $(K, 1_K)$ .

$$\overset{\circ}{K}(K') = \begin{cases} \hat{K}(K') & \text{if } K' \neq K \\ \emptyset & \text{if } K' = K \end{cases}$$

(The fact that  $L$  is a FOLDS signature ensures that  $\overset{\circ}{K}$  is well-defined.)

Proposition When  $L$  is a FOLDS signature, the class  $\{i_K: \overset{\circ}{K} \rightarrow \hat{K} \mid K \in \text{Ob}(L)\}^\#$ ,

the saturation of the class of sphere inclusions, equals the class of all monomorphisms in  $\text{Set}^L$

[For proof: see Appendix]



Part 2. Internal equivalences

§ 2.1 Internal equivalence generators (IEG's)

$L$  is an arbitrary small category as in

Part 1.

An IEG for  $L$  is a six-tuple

$$\underline{H} = (H, IH, \ell, r, q, m)$$

where  $H, IH$  are functors  $L^{op} \rightarrow Set^L$ , and with  $Y$  the Yoneda functor  $L^{op} \rightarrow Set^L$ ,  $\ell, r, q, m$  are natural transformations

$$\ell, r: Y \rightarrow H$$

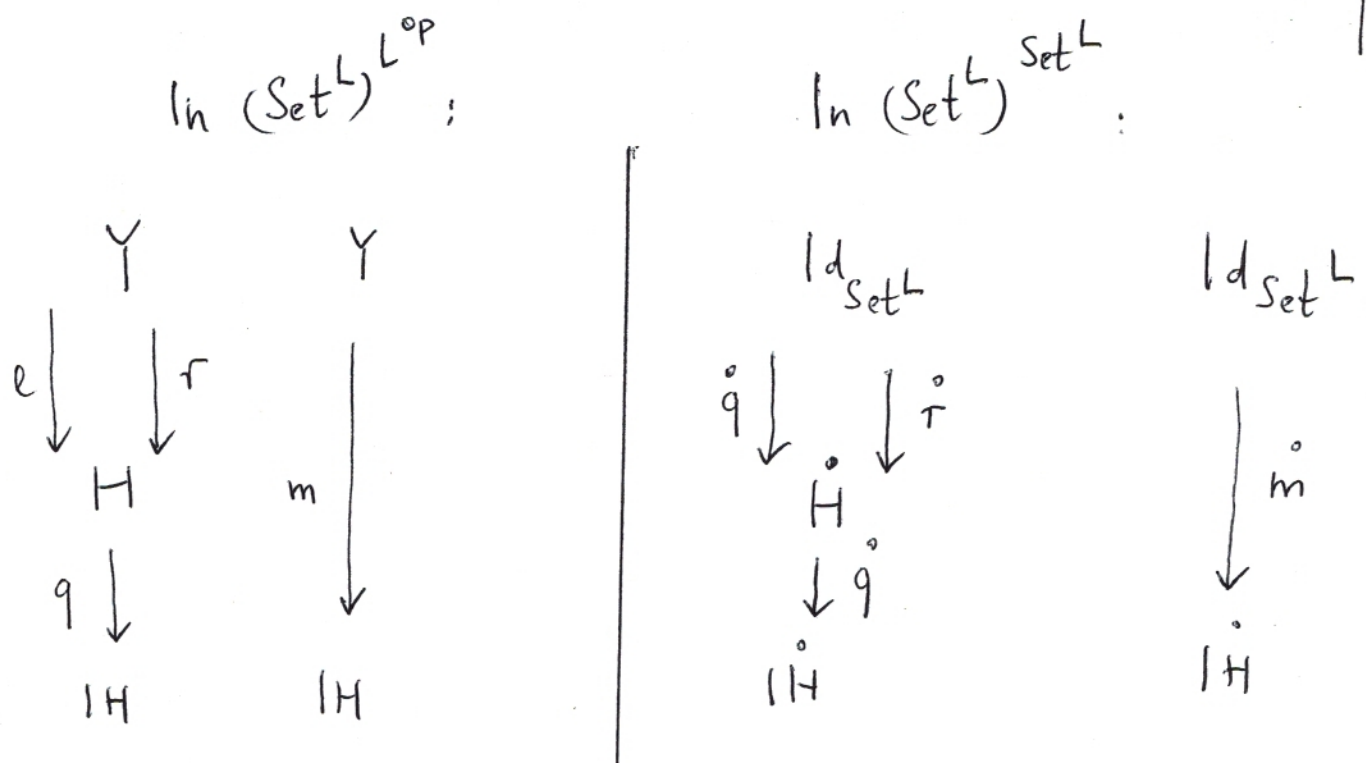
$$m: Y \rightarrow IH$$

$$q: H \rightarrow IH$$

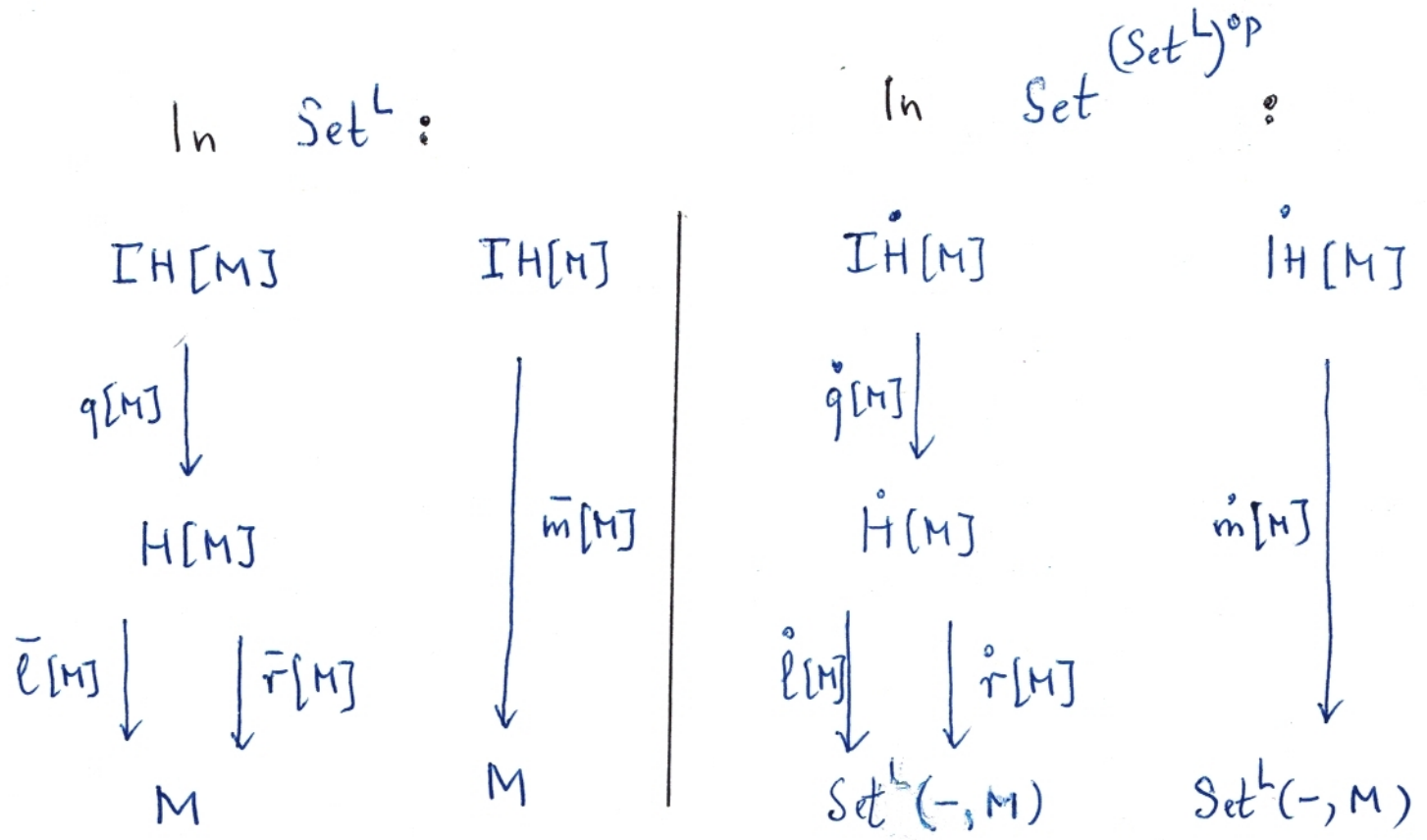
Satisfying  $m = q\ell = qr$ .

Let  $M \in Set^L$ . Using the notation of Part 1,

we have the following items:



The second diagram extends the first along  $Y: L^{op} \rightarrow \text{Set}^L$



The second diagram extends the first along  $Y^{op}: L \rightarrow (\text{Set}^L)^{op}$

For  $U \in \text{Set}^L$  and  $\alpha, \beta : U \rightarrow M$

We repeat the definition on p. 22:

$\alpha$  and  $\beta$  are  $H$ -related

$\Leftrightarrow$

notation:  $\alpha \underset{H}{\sim} \beta$

$\Leftrightarrow$

$\exists \gamma : U \rightarrow H[M]$  such that

$$\alpha = \bar{l}[M] \circ \gamma \quad \text{and} \quad \beta = \bar{r}[M] \circ \gamma$$

$\Leftrightarrow$  (by p. 22)

$\exists \sigma : \dot{H}(U) \rightarrow M$  such that

$$\alpha = \sigma \circ \dot{l}_U \quad \text{and} \quad \beta = \sigma \circ \dot{r}_U$$

I make the following assumption on the IEG  $\underline{H}$ :

$\text{Mon}(\underline{H})$ : For all  $U \in \text{Set}^L$ , the natural transformation  $[\dot{\ell}_U, \dot{r}_U] : U \sqcup U \rightarrow \dot{H}(U)$  is a monomorphism.

Later  $\text{Mon}(\underline{H})$  will be deduced from a more easily verified condition.

Temporarily, we define "fiberwise surjective", FS, thus: a morphism  $P \xrightarrow{p} M$  in  $\text{Set}^L$  is FS if it has the right lifting property with respect to all monomorphisms:

$$\begin{array}{ccc} U & \rightarrow & P \\ \downarrow & \cong & \downarrow \\ V & \rightarrow & M \end{array} \Rightarrow \exists \begin{array}{ccc} U & \rightarrow & P \\ \downarrow & \cong & \downarrow \\ V & \rightarrow & M \end{array}$$

For a FOLDS signature  $L$ , this will be deduced from a local condition. Recall that now  $L$  is still an arbitrary small category. A version is well-known for  $L = \Delta^{\text{op}}$ .

Main Argument

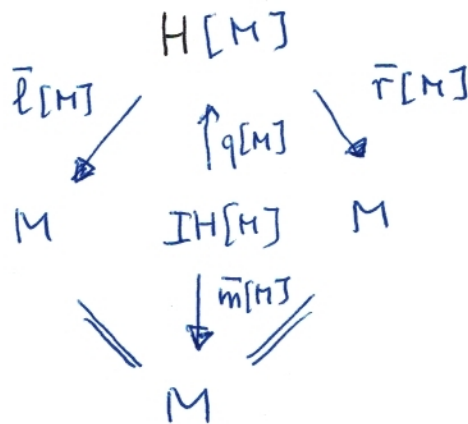
Suppose the IEG  $\underline{H}$  satisfies the condition  $\text{Mon}(\underline{H})$ .

Suppose that

$$\bar{\ell}[M], \bar{r}[M] : H[M] \rightarrow M$$

and  $\bar{m}[M] : IH[M] \rightarrow M$

are FS. (Diagram repeated:



Then, for any  $U \in \text{Set}^L$  and diagrams  $U \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} M$

$\alpha$  and  $\beta$  are  $\underline{H}$ -related  
if and only if

$\alpha$  and  $\beta$  are intrinsically equivalent

in symbols:

$$\alpha \underset{\underline{H}}{\sim} \beta \iff \alpha \underset{\text{int}}{\sim} \beta \quad (*)$$

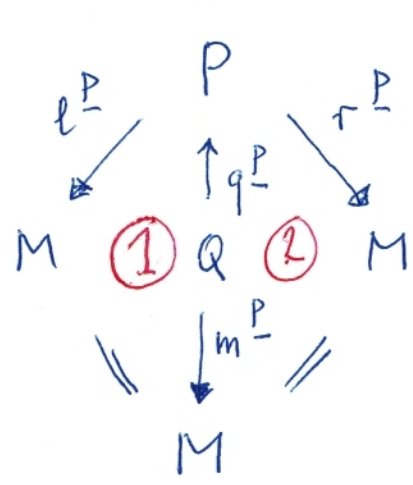


Reminder:

$\alpha \underset{\text{int}}{\sim} \beta \iff \text{there exists reflexive self-equivalence}$

$$\underline{P} : M \underset{L}{\sim} M$$

$$\underline{P} = (P, Q, l^P, r^P, q^P, m^P)$$



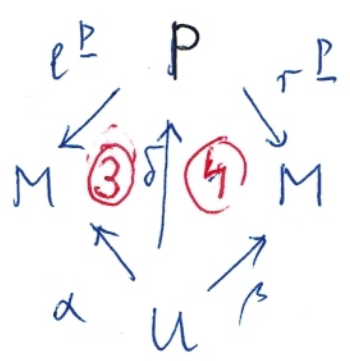
$l^P, r^P : P \rightarrow M$   
both FS

$m^P : Q \rightarrow M$  FS

$$l^P \circ q^P = m^P \quad (1)$$

$$r^P \circ q^P = m^P \quad (2)$$

and there exists  $\delta : U \rightarrow P$  such that



$$\alpha = l^P \circ \delta \quad (3)$$

$$\beta = r^P \circ \delta \quad (4)$$

in brief :  $\underline{P} : (M, \alpha) \underset{L}{\sim} (M, \beta)$ .

Proof of (\*), p 30:

The assumption is that

$$\underline{H}[M] = (H[M], \bar{e}[M], \bar{r}[M], q[M], \bar{r}[M])$$

is a reflexive self equivalence  $\underline{H}[M]: M \underset{L}{\sim} M$ .

The condition on the left side of (\*),  $\alpha \underset{H}{\sim} \beta$

is the same as  $\underline{H}[M]: (M, \alpha) \underset{L}{\sim} (M, \beta)$ ,

according to the first, equivalent, form of

the definition of  $\alpha \underset{H}{\sim} \beta$  (see p 28.1)

Therefore, if we have  $\alpha \underset{H}{\sim} \beta$ , then  $\underline{H}$  is

a witness as  $\underline{P}$  for  $\alpha \underset{int}{\sim} \beta$ ;

in (\*), LHS implies RHS.

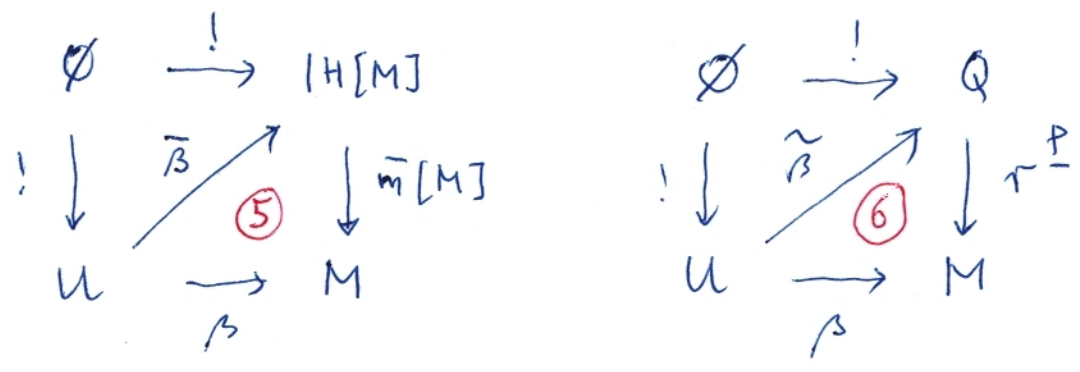
For the converse: assume  $\alpha \underset{int}{\sim} \beta$ .

We have the data and the commutativities numbered ①, ②, ③, ④ on page 31.

Using the FS character of  $\bar{m}[M]$  and that of  $\tau^P$ , we have  $\bar{\beta} : U \rightarrow IH[M]$  and

$$\tilde{\beta} : U \rightarrow Q$$

such that  $\bar{m}[M] \bar{\beta} = \beta$ ,  $\tau^P \tilde{\beta} = \beta$



The bijection  $\Sigma[IH, M](U) = \text{Set}^L(U, IH[M]) \xrightarrow{\cong} \text{Set}^L(IH(U), M)$

provides, from  $\bar{\beta} : U \rightarrow IH[M]$ , the morphism

$$\bar{\beta}^\# : IH(U) \rightarrow M$$

Such that the following commutes:

$$\begin{array}{ccc}
 U & \xrightarrow{\bar{\beta}} & IH[M] \\
 \downarrow \dot{m}_u & \textcircled{7} & \downarrow \bar{m}[M] \\
 IH(u) & \xrightarrow{\bar{\beta}^\#} & M
 \end{array}$$

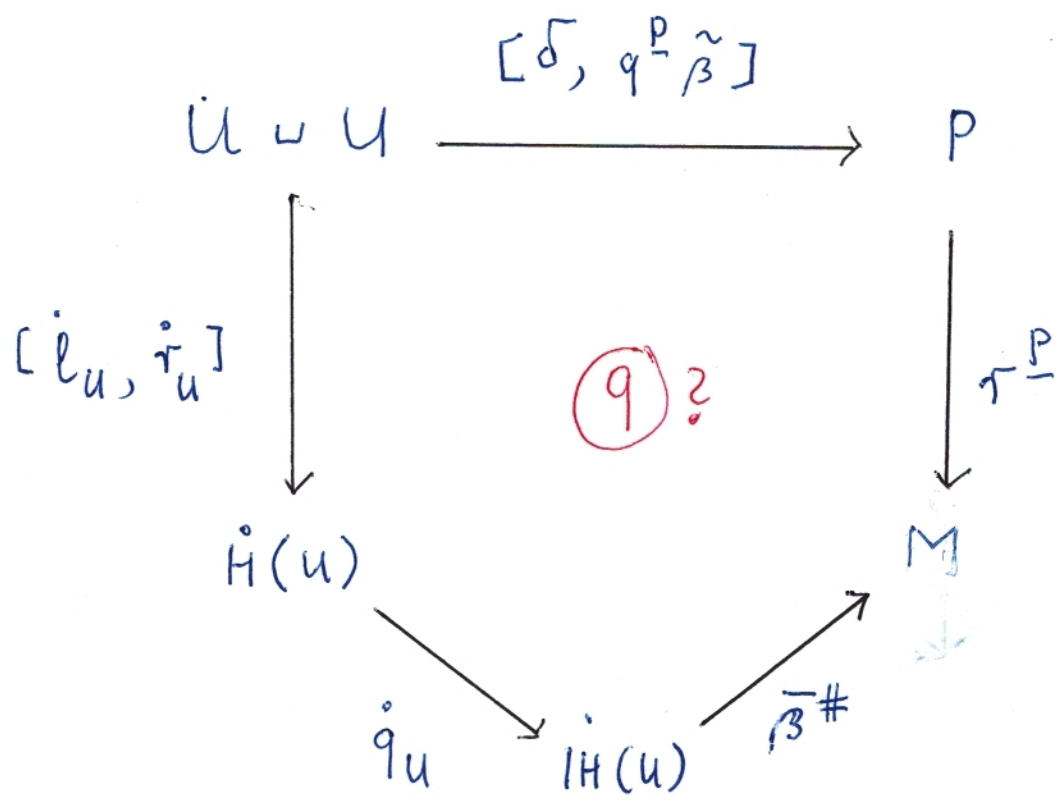
(see p 20). Consider

$$\begin{array}{ccc}
 U & \xrightarrow{\bar{\beta}} & IH[M] \\
 \downarrow \dot{m}_u & \searrow \beta & \downarrow \bar{m}[M] \\
 IH(u) & \xrightarrow{\bar{\beta}^\#} & M
 \end{array}$$

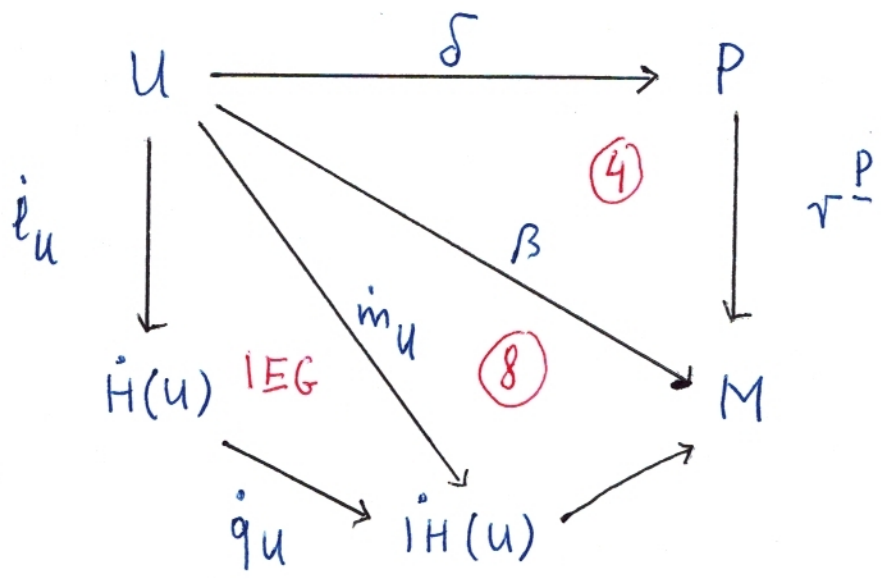
\textcircled{8}
\textcircled{5}

By the commutativity of 7 and 5, 8 also commutes.

Consider the diagram

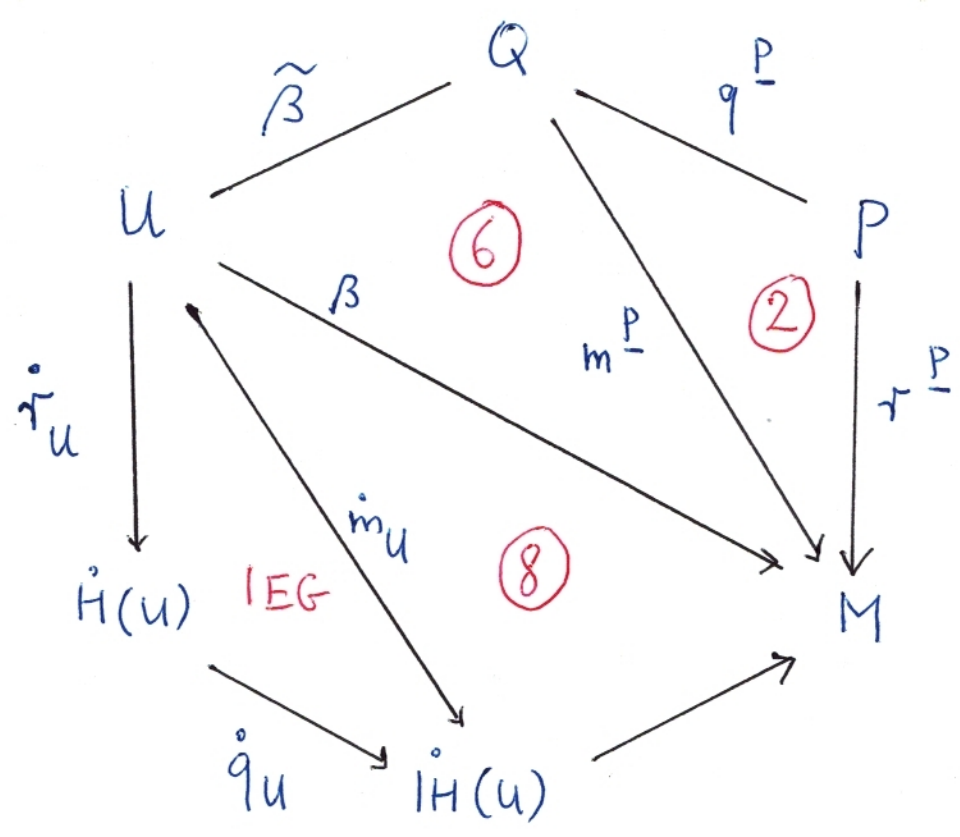


$(9)$  commutes because it commutes when precomposed with the coprojections  $\iota_0 : U \rightarrow U \sqcup U$  and  $\iota_1 : U \rightarrow U \sqcup U$ :  
 precomposing with  $\iota_0$  we get:





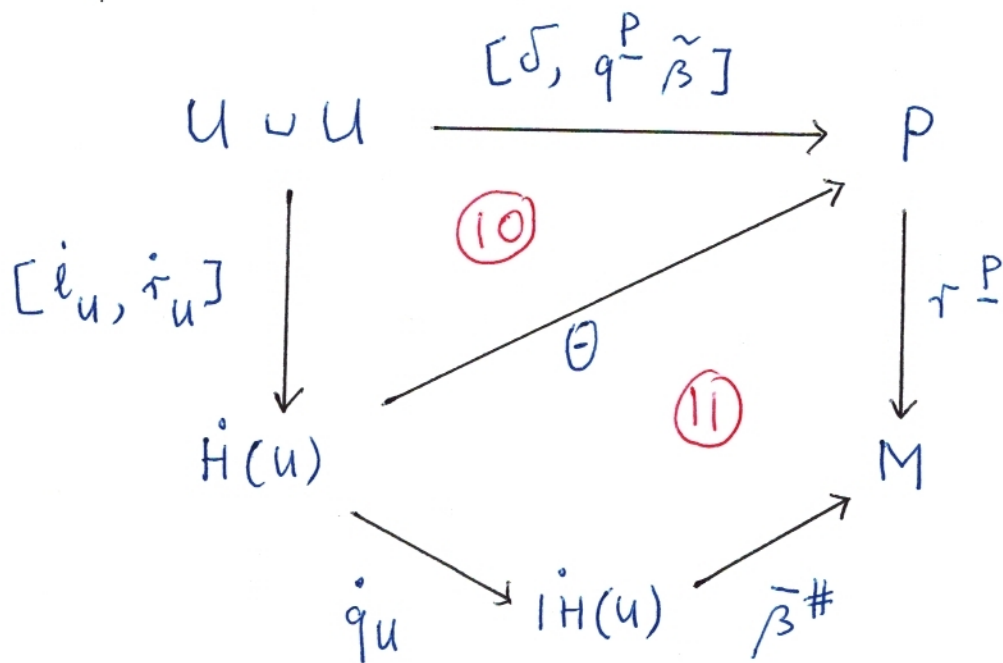
precomposing with  $\iota_1$ :



So 9 is correct: 9 ✓.

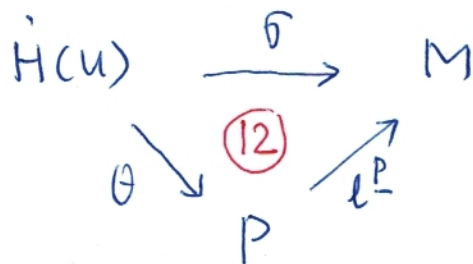
$r^P$  is FS,  $[i_u, r_u]$  is a monomorphism.

there is  $\Theta: H(U) \rightarrow P$  such that in the next diagram 10 and 11 commute:

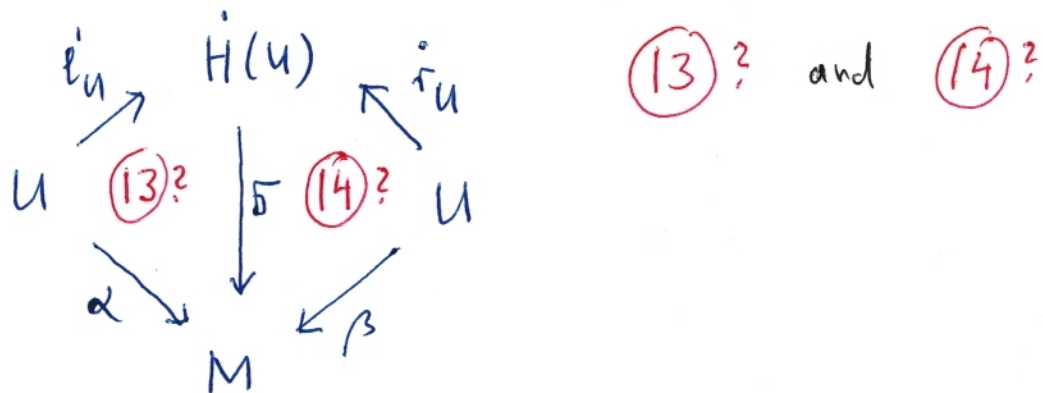


( We are interested in (10) )

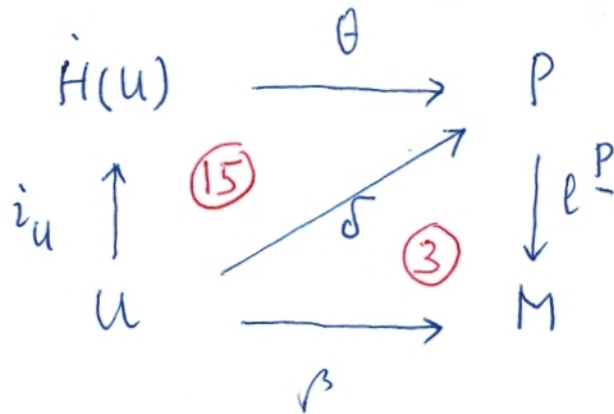
Define the morphism  $\sigma : \dot{H}(U) \rightarrow M$  as the composite  $\sigma = \ell^P \circ \theta$ :



I claim the commutativities  $\alpha = \sigma \dot{l}_u$ ,  $\beta = \sigma \dot{r}_u$ :

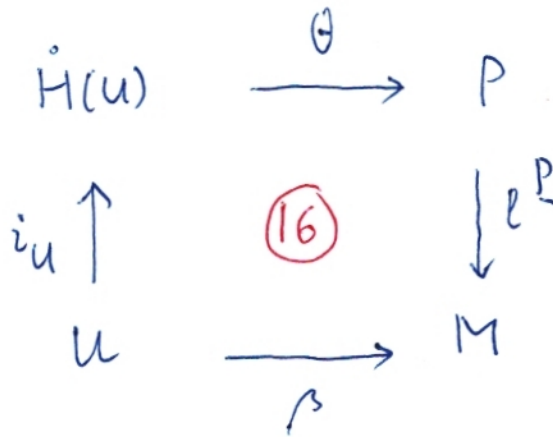


Consider



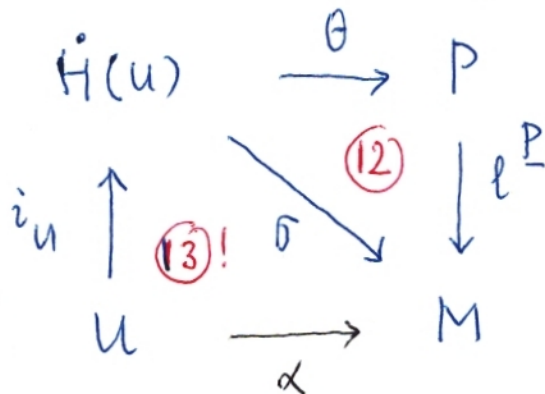
$\textcircled{15}$  commutes: pre compose  $\textcircled{10}$  with  $c_0: U \rightarrow U \cup U$ .

Thus

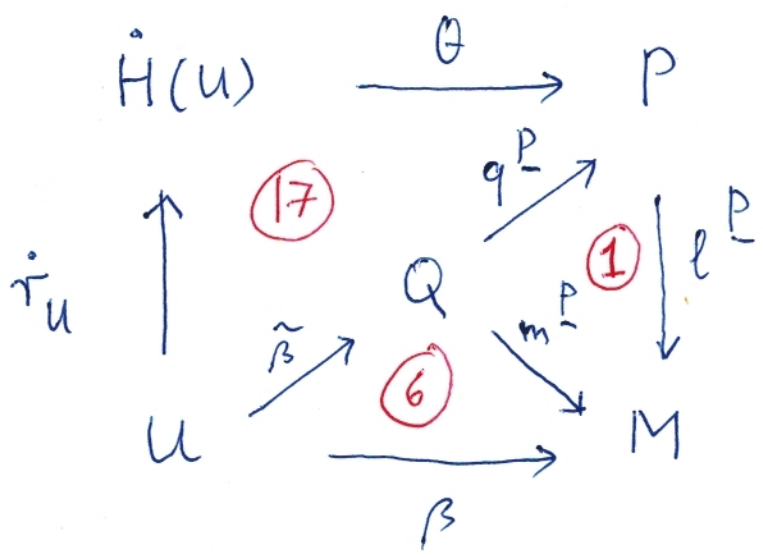


commutes, and

thus  $\textcircled{13}!$  as in

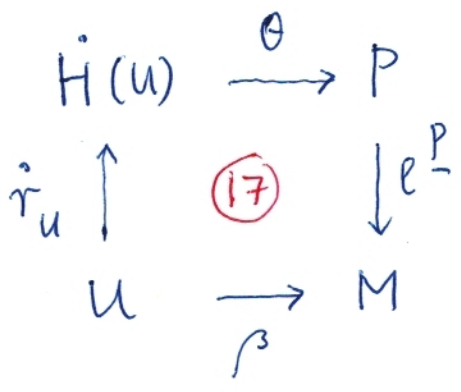


For (14)? : consider

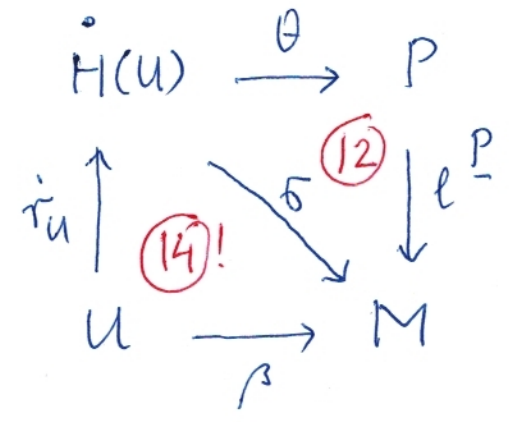


(17) commutes : pre compose (10) with  $c_1 : U \rightarrow U \sqcup U$ .

Hence



Commutates and so does (14)!



With (13) and (14), we have shown that  $\sigma$  witnesses  $\alpha \underset{H}{\sim} \beta$ , according to the second, equivalent, definition : see page 28.1

□ Main Argument

I now make the assumption on the category  $L$  that it is equipped with a "sphere-functor":

We have a functor

$$\begin{array}{ccc} (-)^\circ : L^{op} & \longrightarrow & \text{Set}^L \\ K & \longmapsto & K^\circ \end{array}$$

and a natural transformation

$$\begin{array}{ccc} i_{(-)} : (-)^\circ & \longrightarrow & Y \\ i_K : K^\circ & \longrightarrow & \hat{K} \end{array}$$

with the following properties: each  $i_K : K^\circ \rightarrow \hat{K}$  is a monomorphism, and the Gabriel/Zisman saturation  $\{i_K | K \in L\}^\#$  (see p [25]) of the set  $\{i_K | K \in L\}$  equals the class of all monomorphisms in  $\text{Set}^L$ .  
 By a well-known, and very general, argument, this implies that if  $p: P \rightarrow M$  in  $\text{Set}^L$  has the right lifting property relative to each  $i_K, K \in L$ , then it has that property relative to every monomorphism, that is, it is fiberwise surjective, FS, in the sense used



in the Main Argument.

Page 26 explains that every FOLDS signature meets this requirement. It is well known that  $L = \Delta^{op}$ ,  $\Delta$  the abstract simplex category also does. In what follows, I take the sphere functor  $(\overset{\circ}{(-)}, i_{(-)})$  to be given for the otherwise arbitrary category  $L$ .

I now impose a new condition on the IEG  $\underline{H}$  that I call " $\underline{H}$  is effectively monomorphic", in notation  $EM(\underline{H})$ . The condition is that for every object  $K$  of  $L$ , the commutative square

$$\begin{array}{ccc}
 \overset{\circ}{K} \sqcup \overset{\circ}{K} & \xrightarrow{i_K \sqcup i_K} & \hat{K} \sqcup \hat{K} \\
 \downarrow [i_{\overset{\circ}{K}}, r_{\overset{\circ}{K}}] & & \downarrow [i_{\hat{K}}, r_{\hat{K}}] \\
 \overset{\circ}{H}(K) & \xrightarrow{\overset{\circ}{H}(i_K)} & \overset{\circ}{H}(\hat{K}) (= H(K))
 \end{array}$$

is an effectively monomorphic square

(see p [23]). The square is the

naturality square for the natural transformation

$$(X) \quad [\dot{\ell}, \dot{r}] : \dot{Y} (= \text{Id}_{\text{Set}^L}) \sqcup \dot{Y} \longrightarrow \dot{H}$$

at the morphism  $\dot{r}_K : K \rightarrow K$ .

Thus, the new condition is that we have:

$$\{\dot{r}_K \mid K \in L\} \subseteq \text{EM}([\dot{\ell}, \dot{r}])$$

(using the notation of pages [24] and [25]).

Therefore, by the Proposition on p [25], and the sphere-functor assumption on  $L$ , we have

that every monomorphism in  $\text{Set}^L$  is in

$\text{EM}([\dot{\ell}, \dot{r}])$ , and in particular, each

component

$$[\dot{\ell}_U, \dot{r}_U] : U \sqcup U \longrightarrow \dot{H}(U)$$

for any  $U \in \text{Set}^L$ , is a monomorphism, which

is the condition  $\text{Mon}[\underline{H}]$  used for the

Main Argument. We conclude:

Theorem. Let  $L$  be a small category equipped with a sphere functor  $(\overset{\circ}{(-)}, i_{(-)})$ . Let  $\underline{H}$  be an internal equivalence generator (IEG) for  $L$  satisfying the condition  $\text{EM}(\underline{H})$ . Let  $M \in \text{Set}^L$  be an  $L$ -structure.

Suppose that the morphisms

$$\bar{\ell}[M], \bar{r}[M] : H[M] \rightarrow M$$

$$\bar{m}[M] : H[M] \rightarrow M$$

have the right lifting property relative to each morphism  $i_K : \overset{\circ}{K} \rightarrow \hat{K}$ ,  $K \in L$ .

Then, for any  $U \in \text{Set}^L$ , and

diagrams  $\alpha, \beta : U \rightarrow M$  in  $M$

$$\alpha \underset{\underline{H}}{\sim} \beta \iff \alpha \underset{\text{int}}{\sim} \beta.$$

Equivalently, intrinsic equivalence in  $M$  is induced by a single reflexive self equivalence of  $M$ , namely

$$H(M) = (H[M]) : M \underset{L}{\sim} M; \text{ here:}$$

$$H[M] = (H[M], IH[M], \bar{r}[M], \bar{r}[M], \bar{q}[M], \bar{m}[M]).$$

Corollary Let  $L$  be as in the Theorem,

$H_1, H_2 \in \mathcal{E}L$ 's satisfying  $EM[H_1], EM[H_2]$ .

Suppose  $M \in \text{Set}^L$ ,  $\underline{P}_1 \stackrel{\text{def}}{=} H_1[M], \underline{P}_2 \stackrel{\text{def}}{=} H_2[M]$  are both reflexive self equivalences of  $M$ . Then

$\underline{P}_1$  and  $\underline{P}_2$  induce the same equivalence relations on each set  $\text{Set}^L(u, M)$ :

$$\alpha \underset{\underline{P}_1}{\sim} \beta \iff \alpha \underset{\underline{P}_2}{\sim} \beta \quad (\alpha, \beta : u \rightarrow M)$$