§ 1.3 Miscellany

Let $\mathcal{S}$ be a cocomplete category in which pushouts of mono(morphism)s are mono's:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & C \arrow[d, "j"] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner po"] & D \end{tikzcd} \quad \& \ i \text{ mono} \implies j \text{ mono}$$

An effective monomorphic (EM) square (with left vertical i) in $\mathcal{S}$ is a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

such that 1) the left vertical i is a mono, and 2) for the pushout $P = B \sqcup_A C$: the canonical morphism $P \xrightarrow{m} D$ is a mono:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, tail, "i"'] & & C \arrow[dd, "j"] \arrow[dl, tail, "i'"'] \\ & P \arrow[dr, "m"] & \\ B \arrow[rr, "g"'] \arrow[ur, "f'"] & & D \end{tikzcd}$$

$m i' = j \ , \ m f' = g$

In particular, as a consequence, the right vertical j is also a mono.

Assume: $F, G : \mathfrak{U} \longrightarrow \mathfrak{B}$ functors

$$\varphi : F \to G \quad \in \mathfrak{B}^{\mathfrak{U}}(F, G)$$

Let $U \xrightarrow{f} V$ be an arrow in $\mathfrak{U}$. We say that φ is EM at f if the naturality square of φ at f is EM in $\mathfrak{B}$:

$$\begin{tikzcd} FU \arrow[r, "Ff"] \arrow[d, "\varphi_U"'] & FV \arrow[d, "\varphi_V"] \\ GU \arrow[r, "Gf"'] & GV \end{tikzcd} \quad \text{is EM.}$$

$EM[\varphi]$ denotes the class of all $f \in Arr(\mathfrak{U})$ such that φ is EM at f.

Assume, from now on, that both $\mathfrak{U}$ and $\mathfrak{B}$ are cocomplete categories, and both F and G are cocontinuous. For a class I of arrows in $\mathfrak{U}$, let $I^{\#}$ denote the (generalized) Gabriel/Zisman saturation of I, i.e the least class of arrows in $\mathcal{U}$ that contains I, and is closed under pushouts, transfinite compositions and retracts.

Proposition. Assume: $\mathcal{U}$, $\mathcal{S}$ cocomplete categories, $F, G : \mathcal{U} \to \mathcal{S}$ cocontinuous functors; $\varphi : F \to G$.
In addition, also assume that $\mathcal{S}$ is locally finitely presentable (in the sense of Gabriel / Ulmer) and that pushouts of mono's in $\mathcal{S}$ are mono's.
(Note that $\mathcal{S} = \mathrm{Set}^{L}$ meets all these conditions.)

Suppose $I \subseteq \mathrm{Arr}(\mathcal{U})$ such that $I \subseteq \mathrm{EM}(\varphi)$.
Then $I^{\#} \subseteq \mathrm{EM}(\varphi)$ as well.
In particular, with $\emptyset_{\mathcal{U}}$ the initial object of $\mathcal{U}$, if $\emptyset_{\mathcal{U}} \xrightarrow{!} U$ belongs to $I^{\#}$, then

$$\varphi_U : F(U) \longrightarrow G(U)$$

is a monomorphism.
(For proof: see Appendix)

Let L be a FOLDS signature. Then, for any $K \in L$, we have the "sphere" $\mathring{K} \in Set^L$; the subfunctor of $\hat{K} = L(K,-)$, with inclusion

$$i_K : \mathring{K} \longrightarrow \hat{K}$$

$\mathring{K}$ misses just one element of $el(\hat{K})$, the identity $(K, 1_K)$.

$$\mathring{K}(K') = \begin{cases} \hat{K}(K') & \text{if } K' \neq K \\ \emptyset & \text{if } K' = K \end{cases}$$

(The fact that L is a FOLDS signature ensures that $\mathring{K}$ is well-defined.)

Proposition When L is a FOLDS signature, the class $\{i_K : \mathring{K} \to \hat{K} \mid K \in Ob(L)\}^{\#}$, the saturation of the class of sphere inclusions, equals the class of all monomorphisms in Set^L

[For proof: see Appendix]

Part 2. Internal equivalences

§ 2.1 Internal equivalence generators (IEG's)

L is an arbitrary small category as in Part 1.

An IEG for L is a six-tuple

$$\underline{H} = (H, IH, \ell, r, q, m)$$

where H, IH are functors $L^{op} \longrightarrow Set^L$, and with Y the Yoneda functor $L^{op} \longrightarrow Set^L$, ℓ, r, q, m are natural transformations

$$\ell, r : \ Y \longrightarrow H$$

$$m : \ Y \longrightarrow IH$$

$$q : \ H \longrightarrow IH$$

Satisfying $m = q\ell = qr$.

Let $M \in Set^L$. Using the notation of Part 1, we have the following items:

In $(Set^L)^{L^{op}}$:

$$\begin{tikzcd} Y \arrow[d, shift right, "e"'] \arrow[d, shift left, "r"] \\ H \arrow[d, "q"'] \\ IH \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[d, "m"'] \\ IH \end{tikzcd}$$

In $(Set^L)^{Set^L}$:

$$\begin{tikzcd} Id_{Set^L} \arrow[d, shift right, "\mathring{q}"'] \arrow[d, shift left, "\mathring{r}"] \\ \mathring{H} \arrow[d, "\mathring{q}"] \\ I\mathring{H} \end{tikzcd} \qquad \begin{tikzcd} Id_{Set^L} \arrow[d, "\mathring{m}"] \\ I\mathring{H} \end{tikzcd}$$

The second diagram extends the first along $Y : L^{op} \to Set^L$

In Set^L :

$$\begin{tikzcd} IH[M] \arrow[d, "q[M]"'] \\ H[M] \arrow[d, shift right, "\bar{e}[M]"'] \arrow[d, shift left, "\bar{r}[M]"] \\ M \end{tikzcd} \qquad \begin{tikzcd} IH[M] \arrow[d, "\bar{m}[M]"] \\ M \end{tikzcd}$$

In $Set^{(Set^L)^{op}}$:

$$\begin{tikzcd} I\mathring{H}[M] \arrow[d, "\mathring{q}[M]"'] \\ \mathring{H}[M] \arrow[d, shift right, "\mathring{e}[M]"'] \arrow[d, shift left, "\mathring{r}[M]"] \\ Set^L(-,M) \end{tikzcd} \qquad \begin{tikzcd} I\mathring{H}[M] \arrow[d, "\mathring{m}[M]"'] \\ Set^L(-,M) \end{tikzcd}$$

The second diagram extends the first along $Y^{op} : L \to (Set^L)^{op}$

For $\mathcal{U} \in \mathrm{Set}^L$ and $\alpha, \beta : \mathcal{U} \to M$

we repeat the definition on p. $\boxed{22}$:

α and β are $\underline{H}$-related

$\Longleftrightarrow$

notation: $\alpha \underset{\underline{H}}{\sim} \beta$

$\Longleftrightarrow$

$\exists \gamma : \mathcal{U} \to H[M]$ such that

$\alpha = \bar{l}[M] \circ \gamma$ and $\beta = \bar{r}[M] \circ \gamma$

$\Longleftrightarrow$ (by p. $\boxed{22}$)

$\exists \sigma : \mathring{H}(\mathcal{U}) \to M$ such that

$\alpha = \sigma \circ \mathring{l}_{\mathcal{U}}$ and $\beta = \sigma \circ \mathring{r}_{\mathcal{U}}$

I make the following assumption on the IEG $\underline{H}$:

$\mathrm{Mon}(\underline{H})$: For all $U \in \mathrm{Set}^L$, the natural transformation $[\mathring{\ell}_U, \mathring{r}_U] : U \sqcup U \longrightarrow \mathring{H}(U)$ is a monomorphism.

Later $\mathrm{Mon}(\underline{H})$ will be deduced from a more easily verified condition.

Temporarily, we define "fiberwise surjective", FS, thus: a morphism $P \xrightarrow{p} M$ in Set^L is FS if it has the right lifting property with respect to all monomorphisms:

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\equiv"] & P \arrow[d, "p"] \\ V \arrow[r] & M \end{tikzcd} \Rightarrow \exists \begin{tikzcd} U \arrow[r] \arrow[d] \arrow[r, phantom, shift right=3, "\equiv"] & P \arrow[d] \\ V \arrow[r] \arrow[ur, dashed] \arrow[r, phantom, shift left=3, "\equiv"] & M \end{tikzcd}$$

For a FOLDS signature L, this will be deduced from a local condition. Recall that now L is still an arbitrary small category. A version is well-known for $L = \Delta^{op}$.

Main Argument

Suppose the IEG $\underline{H}$ satisfies the condition $Mon(\underline{H})$.

Suppose that

$$\bar{\ell}[M], \bar{r}[M] : H[M] \longrightarrow M$$

and $$\bar{m}[M] : IH[M] \longrightarrow M$$

are FS. (Diagram repeated:

$$\begin{tikzcd}
 & H[M] \arrow[dl, "{\bar{\ell}[M]}"'] \arrow[dr, "{\bar{r}[M]}"] & \\
M \arrow[dr, equal] & IH[M] \arrow[u, "{q[M]}"'] \arrow[d, "{\bar{m}[M]}"] & M \arrow[dl, equal] \\
 & M &
\end{tikzcd}$$

Then, for any $U \in Set^L$ and diagrams $U \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$

α and β are $\underline{H}$-related

if and only if

α and β are intrinsically equivalent

in symbols: $$\alpha \underset{\underline{H}}{\sim} \beta \iff \alpha \underset{int}{\sim} \beta \qquad (*)$$

Reminder:

$\alpha \underset{\text{int}}{\sim} \beta \underset{\text{def}}{\iff}$ there exists reflexive self-equivalence

$$\underline{P} : M \underset{L}{\sim} M$$

$$\left[\ \underline{P} = (P, Q, \ell^{\underline{P}}, r^{\underline{P}}, q^{\underline{P}}, m^{\underline{P}})\ \right]$$

$$\begin{tikzcd}
 & P \arrow[dl, "\ell^{\underline{P}}"'] \arrow[dr, "r^{\underline{P}}"] & \\
M \arrow[r, phantom, "1"] \arrow[dr, equal] & Q \arrow[u, "q^{\underline{P}}"'] \arrow[d, "m^{\underline{P}}"] \arrow[r, phantom, "2"] & M \arrow[dl, equal] \\
 & M &
\end{tikzcd}$$

$\ell^{\underline{P}}, r^{\underline{P}} : P \longrightarrow M$ both FS

$m^{\underline{P}} : Q \longrightarrow M$ FS

$\ell^{\underline{P}} q^{\underline{P}} = m^{\underline{P}}$ (1)

$r^{\underline{P}} q^{\underline{P}} = m^{\underline{P}}$ (2)

and there exists $\delta : U \to P$ such that

$$\begin{tikzcd}
 & P \arrow[dl, "\ell^{\underline{P}}"'] \arrow[dr, "r^{\underline{P}}"] & \\
M \arrow[r, phantom, "3"] & {} \arrow[r, phantom, "4"] & M \\
 & U \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, "\delta"] &
\end{tikzcd}$$

$\alpha = \ell^{\underline{P}} \circ \delta$ (3)

$\beta = r^{\underline{P}} \circ \delta$ (4)

in brief : $\underline{P} : (M, \alpha) \underset{L}{\sim} (M, \beta)$.

Proof of $(*)$, p $\boxed{30}$:

The assumption is that

$$\underline{H}[M] = (H[M],\ \bar{\ell}[M], \bar{r}[M], q[M], \bar{r}[M])$$

is a reflexive self equivalence $\underline{H}[M] : M \underset{L}{\sim} M$.

The condition on the left side of $(*)$, $\alpha \underset{\underline{H}}{\sim} \beta$

is the same as $\underline{H}[M] : (M, \alpha) \underset{L}{\sim} (M, \beta)$,

according to the first, equivalent, form of

the definition of $\alpha \underset{\underline{H}}{\sim} \beta$ (see p $\boxed{28.1}$)

Therefore, if we have $\alpha \underset{\underline{H}}{\sim} \beta$, then $\underline{H}$ is

a witness as $\underline{P}$ for $\alpha \underset{int}{\sim} \beta$;

in $(*)$, LHS implies RHS.

For the converse: assume $\alpha \underset{int}{\sim} \beta$.

We have the data and the commutativities numbered ①, ②, ③, ④ on page $\boxed{31}$.

Using the FS character of $\bar{m}[M]$ and that of $r^{\underline{P}}$, we have $\bar{\beta} : U \longrightarrow \dot{I\!H}[M]$ and

$$\tilde{\beta} : U \longrightarrow Q$$

such that $\bar{m}[M]\bar{\beta} = \beta$, $r^{\underline{P}}\tilde{\beta} = \beta$

$$\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] & I\!H[M] \arrow[d, "{\bar{m}[M]}"] \\
U \arrow[ur, "\bar{\beta}"] \arrow[r, "\beta"'] & M
\end{tikzcd} \quad ⑤ \qquad\qquad
\begin{tikzcd}
\emptyset \arrow[r, "!"] \arrow[d, "!"'] & Q \arrow[d, "r^{\underline{P}}"] \\
U \arrow[ur, "\tilde{\beta}"] \arrow[r, "\beta"'] & M
\end{tikzcd} \quad ⑥$$

The bijection $\Sigma[I\!H, M](U) : \mathrm{Set}^L(U, I\!H[M])$

$$\xrightarrow{\cong} \mathrm{Set}^L(\dot{I\!H}(U), M)$$

provides, from $\bar{\beta} : U \to I\!H[M]$, the morphism

$$\bar{\beta}^{\#} : \dot{I\!H}(U) \to M$$

such that the following commutes:

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "\bar{\beta}"] \arrow[d, "\mathring{m}_{\mathcal{U}}"'] \arrow[dr, phantom, "7"] & \mathbb{H}[M] \arrow[d, "\bar{m}[M]"] \\
\mathring{\mathbb{H}}(\mathcal{U}) \arrow[r, "\bar{\beta}^{\#}"'] & M
\end{tikzcd}
$$

(see p $\boxed{20}$). Consider

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "\bar{\beta}"] \arrow[d, "\mathring{m}_{\mathcal{U}}"'] \arrow[dr, "\beta"] & \mathbb{H}[M] \arrow[d, "\bar{m}[M]"] \arrow[dl, phantom, "5"] \\
\mathring{\mathbb{H}}(\mathcal{U}) \arrow[r, "\bar{\beta}^{\#}"] \arrow[ur, phantom, "8"] & M
\end{tikzcd}
$$

By the commutativity of 7 and 5, 8 also commutes.

Consider the diagram

$$\begin{tikzcd}
U \sqcup U \arrow[rr, "{[\delta,\, q^{\underline{P}}\tilde{\beta}]}"] \arrow[d, "{[\dot{l}_U,\, \dot{r}_U]}"'] \arrow[drr, phantom, "9\ ?"] & & P \arrow[d, "r^{\underline{P}}"] \\
\dot{H}(U) \arrow[dr, "\dot{q}_U"'] & & M \\
& \dot{IH}(U) \arrow[ur, "\bar{\beta}^{\#}"'] &
\end{tikzcd}$$

9 commutes because it commutes when precomposed with the coprojections $\iota_0 : U \to U \sqcup U$ and $\iota_1 : U \to U \sqcup U$: precomposing with ι_0 we get:

$$\begin{tikzcd}
U \arrow[rr, "\delta"] \arrow[d, "\dot{l}_U"'] \arrow[dr, "\dot{m}_U"] \arrow[drr, "\beta"] & & P \arrow[d, "r^{\underline{P}}"] \arrow[dl, phantom, "4"] \\
\dot{H}(U) \arrow[r, phantom, "\mathrm{IEG}"] \arrow[d, "\dot{q}_U"'] & \arrow[r, phantom, "8"] & M \\
\dot{IH}(U) \arrow[urr] & &
\end{tikzcd}$$

precomposing with ι_1 :

$$
\begin{tikzcd}
 & Q \arrow[r, dash, "q^{\underline{P}}"] \arrow[ddr, "m^{\underline{P}}"] & P \arrow[dd, "r^{\underline{P}}"] \\
U \arrow[ur, dash, "\tilde{\beta}"] \arrow[d, "\dot{r}_U"'] \arrow[drr, "\beta"] \arrow[ddr, "\mathring{m}_U"] \arrow[r, phantom, "6"] & & \arrow[l, phantom, "2"] \\
\mathring{H}(U) \arrow[dr, "\mathring{q}_U"'] \arrow[r, phantom, "\mathrm{IEG}"] & \arrow[r, phantom, "8"] & M \\
 & \mathring{\mathrm{IH}}(U) \arrow[ur] &
\end{tikzcd}
$$

So 9 ? is correct: 9 ✓.

$r^{\underline{P}}$ is FS, $[\dot{l}_U, \dot{r}_U]$ is a monomorphism ∴

there is $\Theta : \mathring{H}(U) \longrightarrow P$ such that in

the next diagram 10 and 11 commute:

$$
\begin{tikzcd}
U \sqcup U \arrow[rr, "{[\delta,\, q^{\underline{P}}\tilde{\beta}]}"] \arrow[d, "{[\dot{\ell}_U,\, \dot{r}_U]}"'] \arrow[drr, phantom, "10"] & & P \arrow[d, "r^{\underline{P}}"] \\
\dot{H}(U) \arrow[urr, "\Theta"] \arrow[dr, "\dot{q}_U"'] \arrow[rr, phantom, "11"] & & M \\
& I\dot{H}(U) \arrow[ur, "\tilde{\beta}^{\#}"'] &
\end{tikzcd}
$$

(We are interested in 10)

Define the morphism $\sigma : \dot{H}(U) \longrightarrow M$ as the composite $\sigma = \ell^{\underline{P}} \circ \Theta$:

$$
\begin{tikzcd}
\dot{H}(U) \arrow[rr, "\sigma"] \arrow[dr, "\Theta"'] \arrow[rr, phantom, shift right=3ex, "12"] & & M \\
& P \arrow[ur, "\ell^{\underline{P}}"'] &
\end{tikzcd}
$$

I claim the commutativities $\alpha = \sigma \dot{\ell}_U$, $\beta = \sigma \dot{r}_U$: 13? and 14?

$$
\begin{tikzcd}
& \dot{H}(U) \arrow[dd, "\sigma"] & \\
U \arrow[ur, "\dot{\ell}_U"] \arrow[dr, "\alpha"'] \arrow[r, phantom, "13?"] & & U \arrow[ul, "\dot{r}_U"'] \arrow[dl, "\beta"] \arrow[l, phantom, "14?"] \\
& M &
\end{tikzcd}
$$

Consider

$$
\begin{tikzcd}
\dot{H}(U) \arrow[r, "\theta"] & P \arrow[d, "\ell^{P}"] \\
U \arrow[u, "i_U"] \arrow[ur, "\delta"] \arrow[r, "\beta"'] \arrow[u, phantom, shift left=6ex, "15"] & M \arrow[ul, phantom, shift right=3ex, "3"]
\end{tikzcd}
$$

15 commutes: precompose 10 with $\iota_0 : U \to U \sqcup U$.

Thus

$$
\begin{tikzcd}
\dot{H}(U) \arrow[r, "\theta"] & P \arrow[d, "\ell^{P}"] \\
U \arrow[u, "i_U"] \arrow[r, "\beta"'] \arrow[ur, phantom, "16"] & M
\end{tikzcd}
$$

commutes, and

thus 13! as in

$$
\begin{tikzcd}
\dot{H}(U) \arrow[r, "\theta"] \arrow[dr, "\sigma"'] & P \arrow[d, "\ell^{P}"] \arrow[dl, phantom, shift right=3ex, "12"] \\
U \arrow[u, "i_U"] \arrow[r, "\alpha"'] \arrow[ur, phantom, shift left=3ex, "13!"] & M
\end{tikzcd}
$$

For (14)? : consider

$$\begin{tikzcd}
\mathring{H}(U) \arrow[rr, "\theta"] & & P \arrow[dd, "\ell^{\underline{P}}"] \arrow[ddl, phantom, "(1)"] \\
& Q \arrow[ur, "q^{\underline{P}}"] \arrow[dr, "m^{\underline{P}}"'] \arrow[ul, phantom, "(17)"] & \\
U \arrow[uu, "\mathring{r}_U"] \arrow[ur, "\tilde{\beta}"] \arrow[rr, "\beta"'] \arrow[r, phantom, "(6)"] & & M
\end{tikzcd}$$

(17) commutes : precompose (10) with $\iota_1 : U \to U \sqcup U$.

Hence

$$\begin{tikzcd}
\mathring{H}(U) \arrow[r, "\theta"] \arrow[dr, phantom, "(17)"] & P \arrow[d, "\ell^{\underline{P}}"] \\
U \arrow[u, "\mathring{r}_U"] \arrow[r, "\beta"'] & M
\end{tikzcd}$$

commutes and so does (14)!

$$\begin{tikzcd}
\mathring{H}(U) \arrow[r, "\theta"] \arrow[dr, "\sigma"] \arrow[dr, phantom, "(14)!"'] & P \arrow[d, "\ell^{\underline{P}}"] \arrow[dl, phantom, "(12)"] \\
U \arrow[u, "\mathring{r}_U"] \arrow[r, "\beta"'] & M
\end{tikzcd}$$

With (13) and (14), we have shown that σ witnesses $\alpha \underset{\underline{H}}{\sim} \beta$, according to the second, equivalent, definition : see page $\boxed{28.1}$

□ Main Argument

I now make the assumption on the category $\mathbb{L}$ that it is equipped with a "sphere-functor": we have a functor

$$\begin{aligned} (\overset{\circ}{-}) : \mathbb{L}^{op} &\longrightarrow Set^{\mathbb{L}} \\ K &\longmapsto \overset{\circ}{K} \end{aligned}$$

and a natural transformation

$$\begin{aligned} i_{(-)} : (\overset{\circ}{-}) &\longrightarrow Y \\ i_K : \overset{\circ}{K} &\to \hat{K} \end{aligned}$$

with the following properties: each $i_K : \overset{\circ}{K} \to \hat{K}$ is a monomorphism, and the Gabriel/Zisman saturation $\{ i_K \mid K \in \mathbb{L} \}^{\#}$ (see p $\boxed{25}$) of the set $\{ i_K \mid K \in \mathbb{L} \}$ equals the class of <u>all</u> monomorphisms in $Set^{\mathbb{L}}$. By a well-known, and very general, argument, this implies that if $p : P \to M$ in $Set^{\mathbb{L}}$ has the right lifting property relative to each i_K, $K \in \mathbb{L}$, then it has that property relative to every monomorphism, that is, it is fiberwise surjective, FS, in the sense used in the Main Argument.

Page $\boxed{26}$ explains that every FOLDS signature meets this requirement. It is well known that $L = \Delta^{op}$, Δ the abstract simplex category also does. In what follows, I take the sphere functor $(\mathring{(-)}, i_{(-)})$ to be given for the otherwise arbitrary small category L.

I now impose a new condition on the IEG $\underline{H}$ that I call "$\underline{H}$ is effectively monomorphic", in notation $EM(\underline{H})$. The condition is that for every object K of L, the commutative square

$$\begin{tikzcd} \mathring{K} \sqcup \mathring{K} \arrow[r, "i_K \sqcup i_K"] \arrow[d, "{[\dot{\ell}_{\mathring{K}}, \dot{r}_{\mathring{K}}]}"'] & \hat{K} \sqcup \hat{K} \arrow[d, "{[\dot{\ell}_{\hat{K}}, \dot{r}_{\hat{K}}]}"] \\ \dot{H}(\mathring{K}) \arrow[r, "\dot{H}(i_K)"] & \dot{H}(\hat{K}) (= H(K)) \end{tikzcd}$$

is an effectively monomorphis square (see p $\boxed{23}$). The square is the naturality square for the natural transformation

$$(*) \qquad [\dot{\ell}, \dot{r}] : \mathring{Y} (= \mathrm{Id}_{Set^L}) \sqcup \mathring{Y} \longrightarrow \dot{H}$$

at the morphism $i_K : \mathring{K} \to \hat{K}$.

Thus, the new condition is that we have:

$$\{ i_K \mid K \in L \} \subseteq EM([\dot{\ell}, \dot{r}])$$

(using the notation of pages $\boxed{24}$ and $\boxed{25}$).

Therefore, by the Proposition on p $\boxed{25}$, and the sphere-functor assumption on L, we have that every monomorphism in Set^L is in $EM([\dot{\ell}, \dot{r}])$, and in particular, each component

$$[\dot{\ell}_U, \dot{r}_U] : U \sqcup U \longrightarrow \dot{H}(U)$$

for any $U \in Set^L$, is a monomorphism, which is the condition $\mathrm{Mon}[\underline{H}]$ used for the

Main Argument. We conclude:

Theorem. Let L be a small category equipped with a sphere functor $(\overset{\circ}{(-)}, i_{(-)})$. Let $\underline{H}$ be an internal equivalence generator (IEG) for L satisfying the condition $EM(\underline{H})$.

Let $M \in Set^L$ be an L-structure.

Suppose that the morphisms

$$\bar{\ell}[M],\ \bar{r}[M] : H[M] \longrightarrow M$$

$$\bar{m}[M] : IH[M] \longrightarrow M$$

have the right lifting property relative to each morphism $i_K : \overset{\circ}{K} \to \hat{K}$, $K \in L$.

Then, for any $\mathcal{U} \in Set^L$, and diagrams $\alpha, \beta : \mathcal{U} \to M$ in M

$$\alpha \underset{\underline{H}}{\sim} \beta \iff \alpha \underset{int}{\sim} \beta.$$

Equivalently, intrinsic equivalence in M is induced by a single reflexive self equivalence of M, namely

$$\underline{H}[M] : M \underset{L}{\sim} M \text{; here:}$$

$$\underline{H}[M] = (H[M], IH[M], \bar{\ell}[M], \bar{r}[M], \bar{q}[M], \bar{m}[M]).$$

<u>Corollary</u> Let L be as in the Theorem, $\underline{H}_1$, $\underline{H}_2$ IEG's satisfying $EM[\underline{H}_1]$, $EM[\underline{H}_2]$. Suppose $M \in \mathrm{Set}^L$; $\underline{P}_1 \underset{def}{=} \underline{H}_1[M]$, $\underline{P}_2 \underset{def}{=} \underline{H}_2[M]$ are both reflexive self equivalences of M. Then $\underline{P}_1$ and $\underline{P}_2$ induce the same equivalence relations on each set $\mathrm{Set}^L(U, M)$:

$$\alpha \underset{\underline{P}_1}{\sim} \beta \iff \alpha \underset{\underline{P}_2}{\sim} \beta \qquad (\alpha, \beta : U \to M)$$