

§ 2.2 Internal equivalence generators constrained by logic

We continue with a fixed small category L ,
and a fixed $1 \in G$ $\underline{H} = (H_1, H_2, \dots)$ as before.
We 'constrain' \underline{H} by "logical conditions", collectively
denoted by Φ , resulting in constrained
versions of the entities $H[M]$, etc, for any
given L -structure $M \in \text{Set}^L$. We will get
the items $(H \wedge \Phi)[M]$, etc, such that, for instance,
 $(H \wedge \Phi)[M]$ is a subfunctor of $H[M] \in \text{Set}^L$.

We summarize the new entities with their relations
to the old ones in Diagram 1 and Diagram 2 below.

Diagram 1 takes place in the category Set^L ,

Diagram 2 in $\text{Set}^{(\text{Set}^L)^{\text{op}}}$.

Diagram 1:

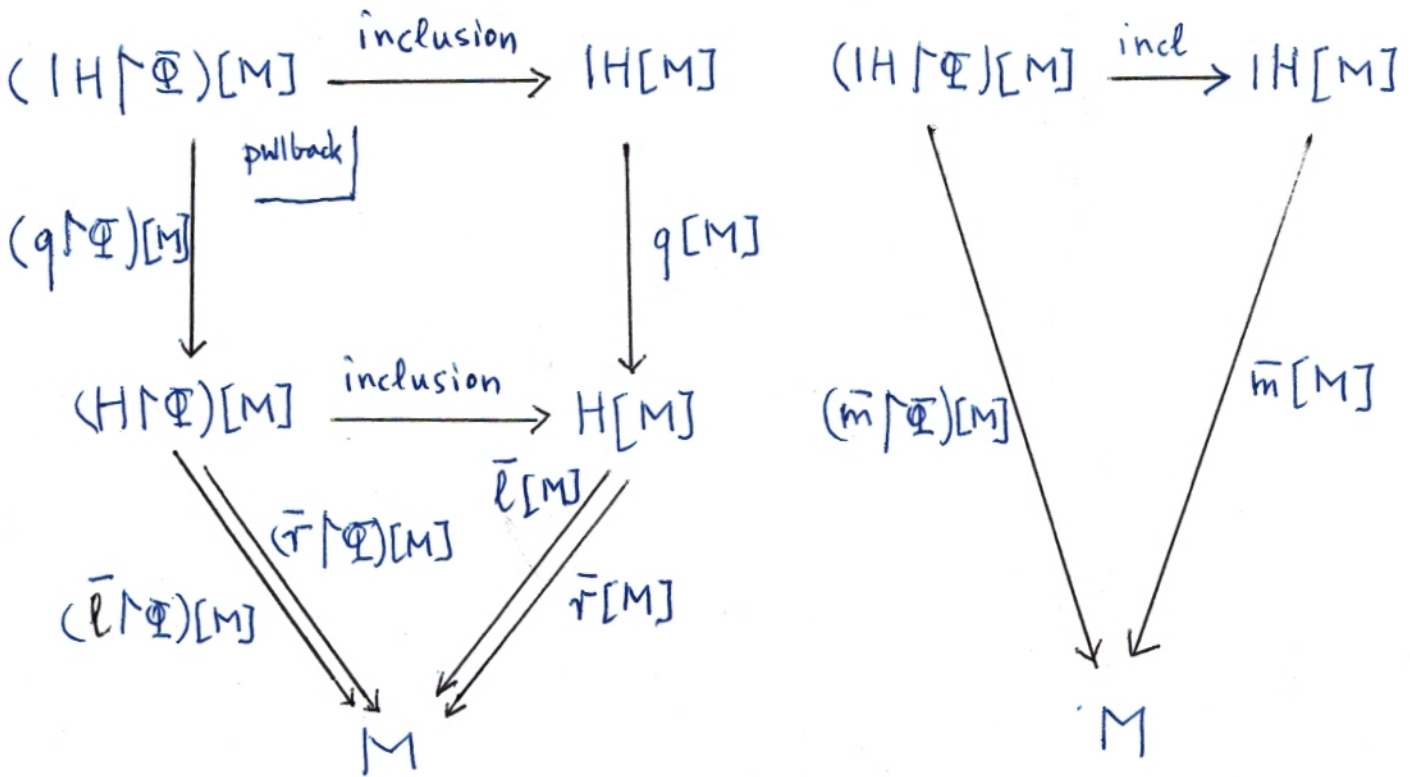
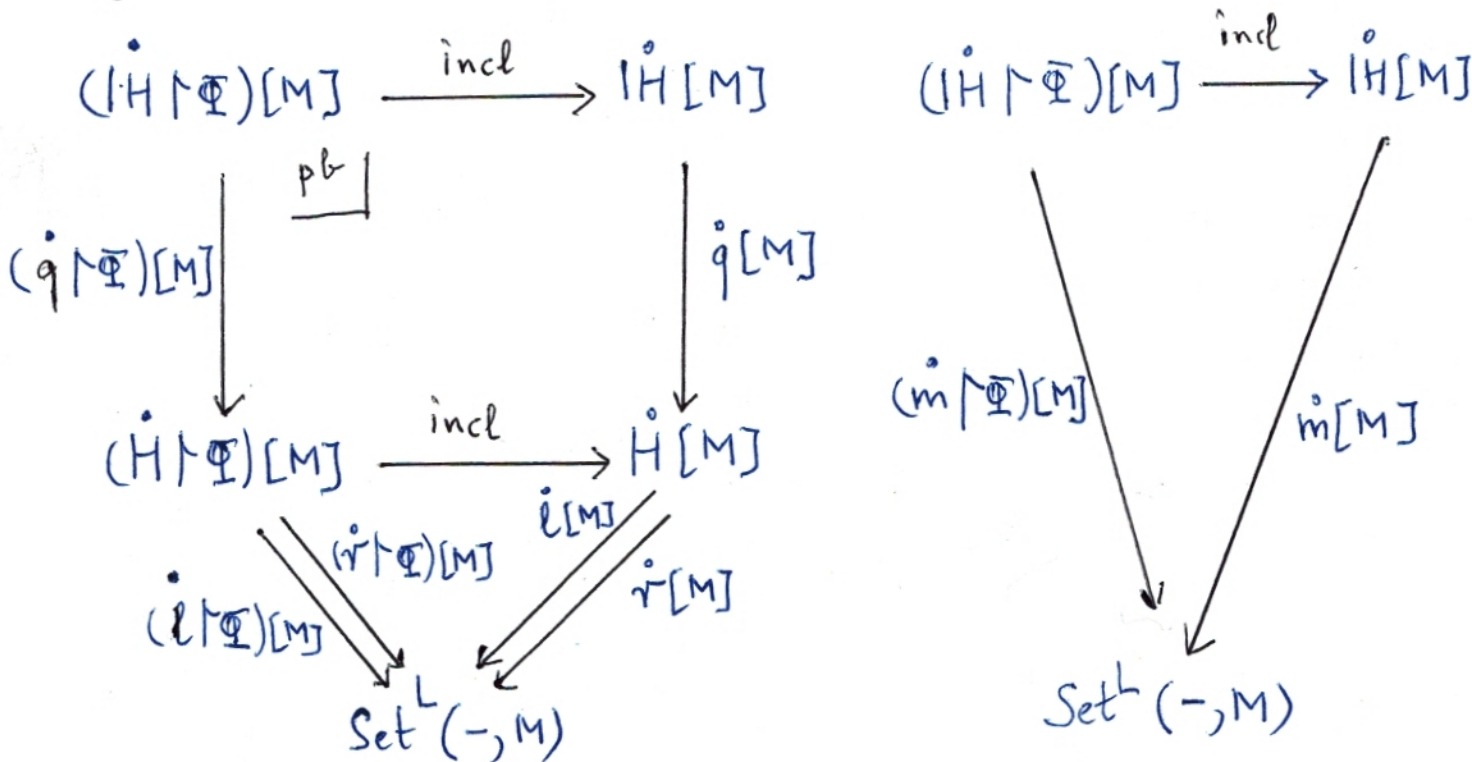


Diagram 2:



$\bar{\Phi}$ is a system $\langle \bar{\Phi}_K \rangle_{K \in L}$ indexed

by the objects of L , where each $\bar{\Phi}_K$ is

a class of $(L, H(K))$ -type augmented

structures: $\bar{\Phi}_K$ is a class of pairs (N, λ) where

$N \in \text{Set}^L$ and $\lambda: H(K) \rightarrow N$. Usually, $\bar{\Phi}_K$ is the

class of all pairs (N, λ) such that

$$N \models \tilde{\Phi}_K[\lambda / H[K]]$$

with $\tilde{\Phi}_K$ is a formula in a possibly

infinitary language $L_{\omega\omega}$ with free variables

the set of objects in $\text{el}(H(K))$, the category

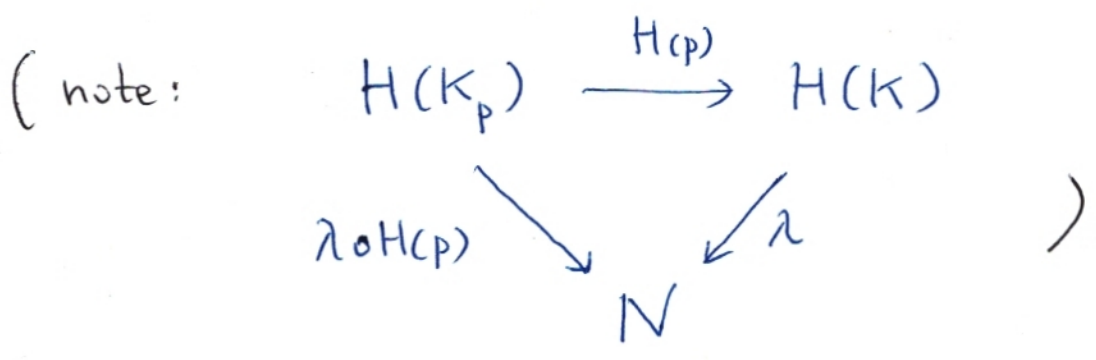
of elements of the functor $H(K): L \rightarrow \text{Set}$.

We impose a functoriality condition 1), and

an invariance condition 2) on the system $\langle \bar{\Phi}_K \rangle_{K \in L}$

1) For any $K \xrightarrow{p} K_p$ in L

$(N, \lambda) \in \bar{\Phi}_K$ implies $(N, \lambda \circ H(p)) \in \bar{\Phi}_{K_p}$



2) Each class $\bar{\Phi}_K$ is invariant under fiberwise surjective (FS) maps: If $P \xrightarrow{p} N$ in Set^L is FS:

$$(N, \lambda) \in \bar{\Phi}_K \iff (P, \lambda \circ p) \in \bar{\Phi}_K$$

(Recall: p is called fiberwise surjective, for a general category L , if p has the right lifting property relative to all monomorphisms in Set^L .)

When L is a FOLDS signature, this is equivalent to its own special cases for monomorphisms $i_K: \hat{K} \rightarrow \hat{K}$.)

The subfunctor $(H \uparrow \Phi)[M]$ of $H[M]$

is defined thus: for $K \in L$ and $\lambda \in H[M](K)$

$$\lambda \in (H \uparrow \Phi)[M](K) \stackrel{\text{def}}{\iff} (M, \lambda) \in \bar{\Phi}_K.$$

The functoriality condition 1) ensures that we have a consistent definition of a subfunctor

$$(H \uparrow \Phi)[M] \subseteq H[M].$$

The definitions of the rest of the items in Diagram 1 are self-explanatory - including, of course, the definition of $(\downarrow H \uparrow \Phi)[M]$ as a pullback.

In Diagram 2, we define the subfunctor $(\dot{H} \uparrow \Phi)[M]$ of $\dot{H}[M] \in \text{Set}^{(\text{Set}^L)^{\text{op}}}$ as follows:

for any $U \in \text{Set}^L$, $\sigma \in \dot{H}[M](U) = \text{Set}^L(\dot{H}(U), M)$

we let

$$\sigma \in (\dot{H} \uparrow \Phi)[M](U) \stackrel{\text{def}}{\iff}$$

$$\sigma \circ \dot{H}(\hat{u}) \in (H \uparrow \Phi)[M](K)$$

for all $(K, u) \in [U]$.

The action, for $u \xrightarrow{f} v$:

$$\begin{aligned} \dot{H}[M](f) : \dot{H}[M](V) &\longrightarrow \dot{H}[M](U) \\ \theta &\longmapsto \theta \circ \dot{H}(f) \end{aligned}$$

restricts to an action

$$(\dot{H} \upharpoonright \Phi)[M](f) : (\dot{H} \upharpoonright \Phi)[M](V) \longrightarrow (\dot{H} \upharpoonright \Phi)[M](U)$$

because of the commutativity of

$$\begin{array}{ccccc} \dot{H}(u) & \xrightarrow{\dot{H}(f)} & \dot{H}(v) & \longrightarrow & M \\ & \nwarrow \dot{H}(\hat{u}) & \nearrow \dot{H}(f_K(u)) & & \\ & & H(K) & & \end{array}$$

for all $(K, u) \in [U]$.

This completes the definitions of the items in Diagram 2, including that of $(\dot{H} \upharpoonright \Phi)[M]$ by the pullback indicated.

For any given $M, U \in \text{Set}^L$, we have the bijection

$$\Sigma = \Sigma [H, M](U): \text{Set}^L(U, H[M]) \xrightarrow{\cong} \dot{H}[M](U)$$

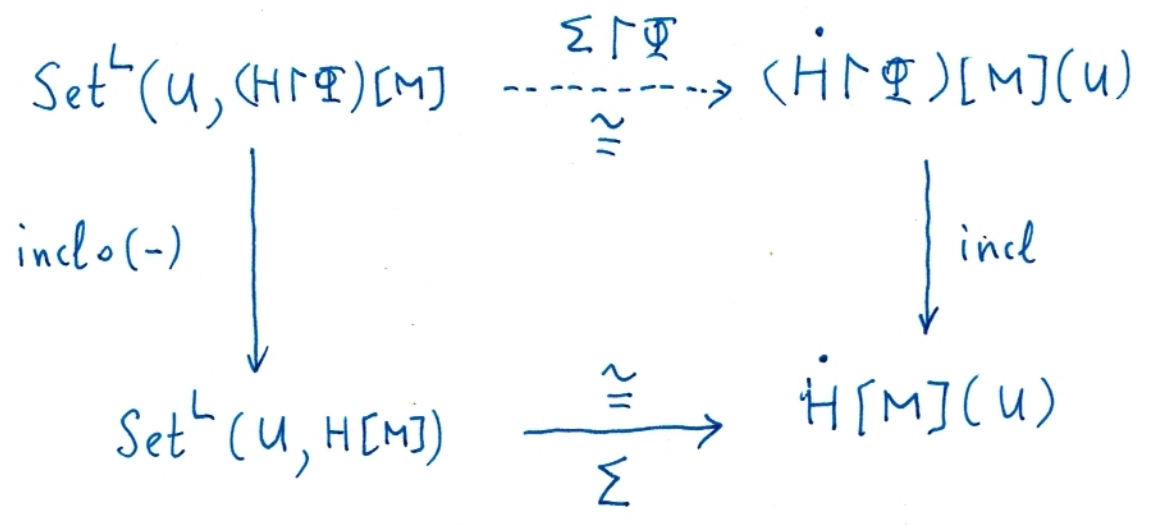
(see page 115); recall that $\dot{H}[M](U) = \text{Set}^L(\dot{H}(U), M)$

Σ restricts to a bijection

$$\Sigma \upharpoonright \Phi = \Sigma [H \upharpoonright \Phi, M](U):$$

$$\text{Set}^L(U, (H \upharpoonright \Phi)[M]) \xrightarrow{\cong} \text{Set}^L((\dot{H} \upharpoonright \Phi)[M], U)$$

defined by the following commutative diagram:



(for details, see Appendix)

For us, the significance of this fact is that we have the desired generalization of the dual description of internal equivalence, the fact displayed on p 122:

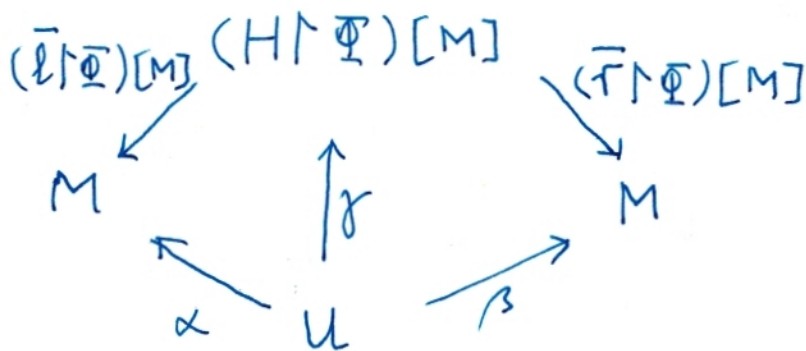
For any pair $U \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} M$ of U -diagrams in M

the following conditions (1) and (2) are equivalent:

(1) the "external" condition for $\alpha \sim \beta$:
 $\underline{H} \uparrow \Phi$

there exists $\gamma: U \rightarrow (H \uparrow \Phi)[M]$

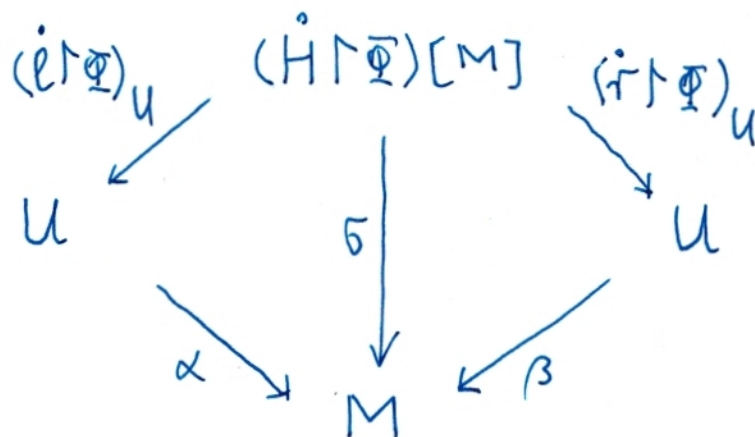
such that the following commutes:



(2) the "internal" condition for $\alpha \sim \beta$:
 $\underline{H} \uparrow \Phi$

there exists $\sigma \in (\dot{H} \uparrow \Phi)[M](U)$ such that:

the following commutes:



We have the analogous situation for \dot{H} in place of H ; the bijection $\Sigma[H] = \Sigma[H, M](u)$ restricts to the bijection $(\Sigma \uparrow \Phi)[H]$ shown in the commutative diagram

$$\begin{array}{ccc}
 \text{Set}^L(u, (H \uparrow \Phi)[M]) & \xrightarrow[\cong]{(\Sigma \uparrow \Phi)[H]} & (\dot{H} \uparrow \Phi)[M](u) \\
 \text{incl} \circ (-) \downarrow & & \downarrow \text{incl} \\
 \text{Set}^L(u, H[M]) & \xrightarrow[\Sigma[H]]{\cong} & \dot{H}[M](u)
 \end{array}$$

For details, see Appendix.

The invariance assumption, condition 2) on page 48, on the classes Φ_K will be used as follows:

for any $u \in \text{Set}^L$, any FS map $p: P \rightarrow M$ in Set^L

$$\begin{array}{ccc}
 \dot{H}(u) & \xrightarrow{\theta} & P \\
 & \cong & \downarrow p \\
 & \searrow \sigma & M
 \end{array}
 , \text{ i.e. } \sigma = p \circ \theta, \text{ implies that}$$

$$\theta \in (\dot{H} \uparrow \Phi)[P](u) \Leftrightarrow \sigma \in (\dot{H} \uparrow \Phi)[M](u)$$

For details, see Appendix

We have the "constrained" version of the Main Argument as follows:

Suppose the IEG \underline{H} satisfies the condition $\text{Mon}(\underline{H})$. Suppose $\bar{\Phi} = \langle \bar{\Phi}_K \rangle_{K \in L}$ is a constraint on \underline{H} satisfying the functoriality and invariance conditions, 1) and 2) on page [48]. In brief, we suppose the constrained IEG $\underline{H} \uparrow \bar{\Phi}$ (with $\text{Mon}(\underline{H})$).

Let $M \in \text{Set}^L$. Assume that M satisfies the following conditions:
the morphisms

$$(\bar{l} \uparrow \bar{\Phi})[M], (\bar{f} \uparrow \bar{\Phi})[M] : \underline{H} \uparrow \bar{\Phi} \longrightarrow M$$

and

$$(\bar{m} \uparrow \bar{\Phi})[M] : (\underline{H} \uparrow \bar{\Phi})[M] \longrightarrow M$$

are FS. Then, for any $U \in \text{Set}^L$ and diagrams $\alpha, \beta : U \rightarrow M$, we have

$$\alpha \underset{H \uparrow \Phi}{\sim} \beta \quad \text{if and only if} \quad \alpha \underset{\text{int}}{\sim} \beta$$

where the relation $\underset{H \uparrow \Phi}{\sim}$ is meant in either of

the two equivalent senses given on page 52,

and $\underset{\text{int}}{\sim}$ is the usual (unchanged!) notion of

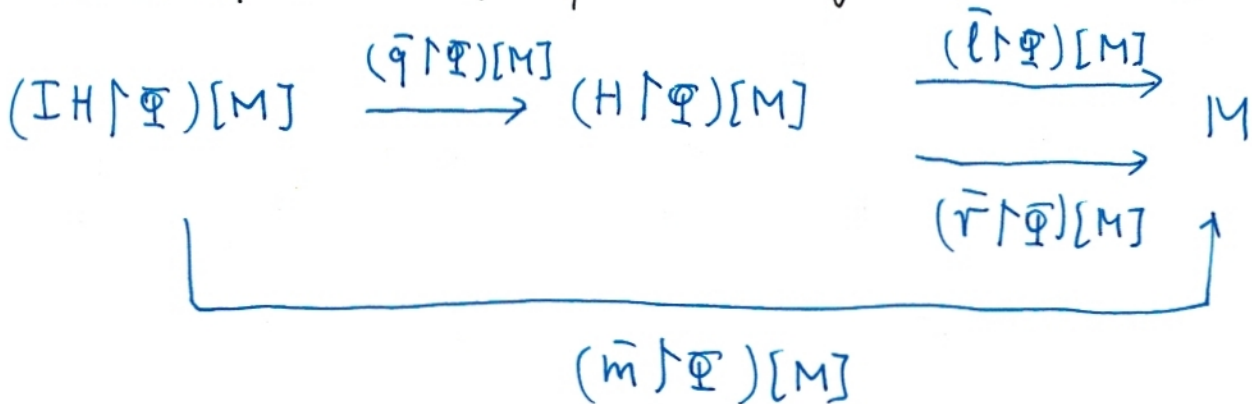
intrinsic equivalence.

The proof is a quick modification of the original proof for the Main Argument.

We see that the assumptions amount to saying that

$$(H \uparrow \Phi)[M] = ((H \uparrow \Phi)[M], (\bar{L} \uparrow \Phi)[M], (\bar{r} \uparrow \Phi)[M])$$

is a reflexive self-equivalence of M , with reflexivity



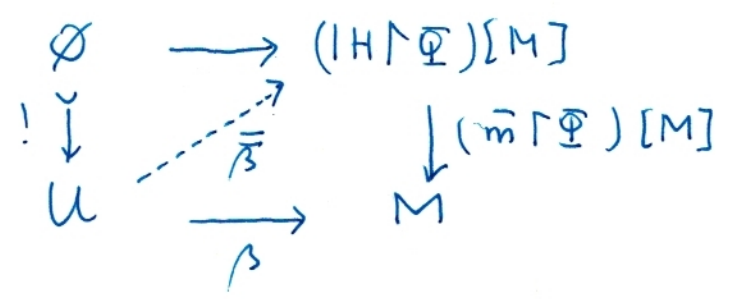
Therefore $\alpha \underset{H \uparrow \Phi}{\sim} \beta$, read in the

external sense 1), means that $H \uparrow \Phi$ is a witness for $\alpha \underset{int}{\sim} \beta$, defined as the existence of a RSE P such that $\alpha \underset{P}{\sim} \beta$.

In brief $\alpha \underset{H \uparrow \Phi}{\sim} \beta$ implies $\alpha \underset{int}{\sim} \beta$.

For the converse direction, we stay in the context of the previous proof (starting on page 32) and put appropriate additional conditions on some of the items constructed. We assume $\alpha \underset{int}{\sim} \beta$ with the witnessing data $P: M \underset{L}{\sim} M$ and all the rest as displayed on p. 31.

We choose $\bar{\beta}: U \rightarrow (H \uparrow \Phi)[M]$ such that $(\bar{m} \uparrow \Phi)[M] \circ \bar{\beta} = \beta$ - which we can do since $(\bar{m} \uparrow \Phi)[M]$ is assumed to be FS:



To consolidate the present notation with the earlier one, we write

$\bar{\beta} : \mathcal{U} \rightarrow \text{IH}[M]$ for the composite

$$\mathcal{U} \xrightarrow{\bar{\beta}} (\text{IH} \uparrow \Phi)[M] \xrightarrow{\text{incl}} \text{IH}[M]; \bar{\beta} = \text{incl} \circ \bar{\beta};$$

thus, we have $\bar{\beta}$ as before, but with the additional property that it factors through

$$(\text{IH} \uparrow \Phi)[M] \xrightarrow{\text{incl}} \text{IH}[M].$$

We have the bijection

$$(\Sigma \uparrow \Phi)[\text{IH}] : \text{Set}^L(\mathcal{U}, (\text{IH} \uparrow \Phi)[M]) \xrightarrow{\cong} (\text{IH} \uparrow \Phi)[M](\mathcal{U}).$$

(see p. 53); let $\bar{\beta}^\# \stackrel{\text{def}}{=} (\Sigma \uparrow \Phi)[\text{IH}](\bar{\beta}) \in (\text{IH} \uparrow \Phi)[M](\mathcal{U})$

Also, $\bar{\beta}^\# \stackrel{\text{def}}{=} \Sigma[\text{IH}](\mathcal{U})(\bar{\beta}) \in \text{IH}[M](\mathcal{U})$.

The commutative diagram on page 53 tells us

that
$$\bar{\beta}^\# = \bar{\beta}^\#$$

from which we conclude that

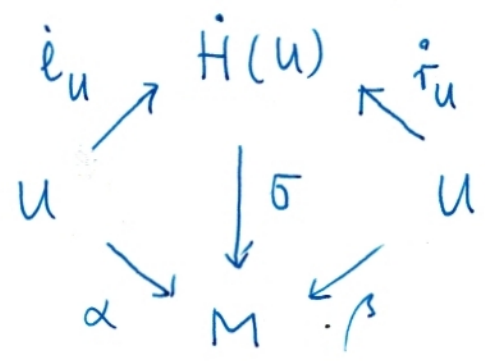
$$\bar{\beta}^\# \in (\text{IH} \uparrow \Phi)[M](\mathcal{U});$$

with $\bar{\beta}^\#$ we are back in the notation of the previous proof. Following that proof, we

get, eventually,

$$\sigma: \dot{H}(U) \rightarrow M$$

such that the following commutes:



I claim that $\sigma \in \dot{H}[M](U)$ belongs to the subset $(\dot{H} \upharpoonright \Phi)[M](U)$ of $\dot{H}[M](U)$. Once

we see this, we have proved that $\alpha \underset{\underline{H} \upharpoonright \Phi}{\sim} \beta$

by the second, "internal", definition of this relation - and this is what we want.

For the proof of the claim, we return to the diagram (9) on page 35, and take the

composite

$$\dot{H}(U) \xrightarrow{\bar{\beta}^\# \circ \dot{q}_U} M$$

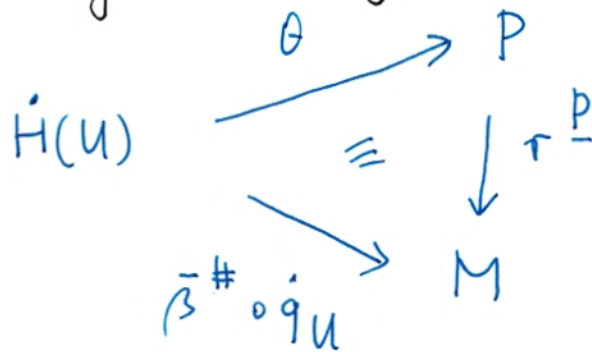
The pullback - definition of $(\dot{H} \uparrow \Phi)[M]$ (see page 46) says that for any $z \in \dot{H}[M](u)$

$$z \in (\dot{H})[M](u) \Leftrightarrow z \hat{q}_u \in (\dot{H} \uparrow \Phi)[M](u) \quad (\subseteq \dot{H}[M](u))$$

Since $z = \bar{\beta}^\#$ does belong to $\dot{H}[M](u)$, we have that

$$\bar{\beta}^\# \circ \hat{q}_u \in (\dot{H} \uparrow \Phi)[M](u)$$

Now, looking at diagram (11) on page 37:

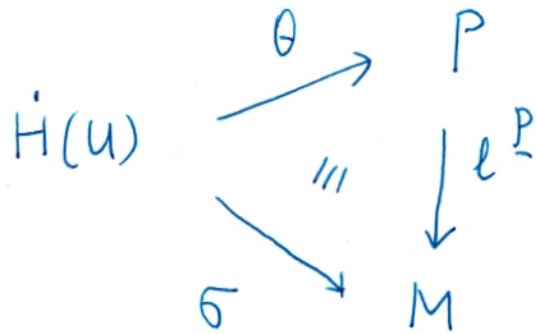


and using (again!) that τ^P is FS, we obtain that

$$\theta \in (\dot{H} \uparrow \Phi)[P](u) :$$

see page 53 for this inference.

Finally, by diagram (12) on page 37: 60



and e^P being FS (which fact was not used previously!), we obtain that $\sigma \in (\dot{H} \upharpoonright \Phi)[M](u)$ as desired

□ Main Argument

constrained version

The following theorem, generalizing the theorem on page 43, is a consequence of what we have:

Theorem Let: L be a small category equipped with a sphere functor $(\overset{\circ}{(-)}, i_{(-)})$ (if L is a FOLDS signature, this is satisfied).

Let: \underline{H} be an internal equivalence generator (IEG) for L satisfying the condition

$EM[\underline{H}]$.

Let: $\bar{\Phi} = \langle \bar{\Phi}_K \rangle_{K \in L}$ be a constraint on \underline{H} satisfying the functoriality and invariance conditions (see page [48]). Let: $M \in \text{Set}^L$.
Suppose: the morphisms

$$(\bar{e} \uparrow \bar{\Phi})[M], (\bar{r} \uparrow \bar{\Phi})[M]: (H \uparrow \bar{\Phi})[M] \rightarrow M$$

$$(\bar{m} \uparrow \bar{\Phi})[M]: (IH \uparrow \bar{\Phi})[M] \rightarrow M$$

have the right lifting property relative to each of the morphisms $i_K: \hat{K} \rightarrow \hat{K}^\circ$, $K \in L$.

Then, for any $U \in \text{Set}^L$ and $\alpha, \beta: U \rightarrow M$

$$\alpha \underset{H \uparrow \bar{\Phi}}{\sim} \beta \iff \alpha \underset{\text{int}}{\sim} \beta.$$

Equivalently: intrinsic equivalence for diagrams in M is induced by a single

reflexive self equivalence of M , namely

$$(\underline{H} \uparrow \bar{\Phi})[M] : M \underset{L}{\sim} M$$

where $(\underline{H} \uparrow \bar{\Phi})[M] = ((\underline{H} \uparrow \bar{\Phi})[M], (\bar{\ell} \uparrow \bar{\Phi})[M], (\bar{\tau} \uparrow \bar{\Phi})[M])$
with reflexivity given by $(\bar{q} \uparrow \bar{\Phi})[M]$ and $(\bar{m} \uparrow \bar{\Phi})[M]$.

Let me call a reflexive self equivalence of M
a schematic RSE if it is of the form

$(\underline{H} \uparrow \bar{\Phi})[M]$ obtained from a constrained LEG
 $\underline{H} \uparrow \bar{\Phi}$ as above.

Corollary If $P_1, P_2 : M \underset{L}{\sim} M$ are both

schematic reflexive self equivalences of M ,

then they induce, for every $U \in \text{Set}^L$,

the same equivalence relation $\underset{P_1}{\sim}^U, \underset{P_2}{\sim}^U$

on the set $\text{Set}^L(U, M)$:

$$\alpha \underset{P_1}{\sim} \beta \Leftrightarrow \alpha \underset{P_2}{\sim} \beta \quad (\alpha, \beta : U \rightarrow M)$$

and this relation is the same as intrinsic equivalence.