§ 2.2 Internal equivalence generators constrained by logic

We continue with a fixed small category L, and a fixed 1EG $\underline{H} = (H, 1H, \ldots)$ as before. We 'constrain' $\underline{H}$ by "logical conditions", collectively denoted by $\underline{\Phi}$, resulting in constrained versions of the entities $H[M]$, etc, for any given L-structure $M \in Set^L$. We will get the items $(H \restriction \Phi)[M]$, etc, such that, for instance, $(H \restriction \Phi)[M]$ is a subfunctor of $H[M] \in Set^L$. We summarize the new entities with their relations to the old ones in Diagram 1 and Diagram 2 below. Diagram 1 takes place in the category Set^L, Diagram 2 in $Set^{(Set^L)^{op}}$.

Diagram 1:

$$\begin{tikzcd}
(\mathrm{I}H\upharpoonright\Phi)[M] \arrow[rr, "\text{inclusion}"] \arrow[d, "(q\upharpoonright\Phi)[M]"'] \arrow[drr, phantom, "\lrcorner\ \text{pullback}", very near start] & & \mathrm{I}H[M] \arrow[d, "q[M]"] \\
(H\upharpoonright\Phi)[M] \arrow[rr, "\text{inclusion}"] \arrow[dr, shift right, "(\bar{\ell}\upharpoonright\Phi)[M]"'] \arrow[dr, shift left, "(\bar{r}\upharpoonright\Phi)[M]"] & & H[M] \arrow[dl, shift right, "\bar{\ell}[M]"'] \arrow[dl, shift left, "\bar{r}[M]"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{I}H\upharpoonright\Phi)[M] \arrow[rr, "\text{incl}"] \arrow[dr, "(\bar{m}\upharpoonright\Phi)[M]"'] & & \mathrm{I}H[M] \arrow[dl, "\bar{m}[M]"] \\
& M &
\end{tikzcd}$$

Diagram 2:

$$\begin{tikzcd}
(\mathrm{I}\dot{H}\upharpoonright\Phi)[M] \arrow[rr, "\text{incl}"] \arrow[d, "(\dot{q}\upharpoonright\Phi)[M]"'] \arrow[drr, phantom, "\lrcorner\ \text{pb}", very near start] & & \mathrm{I}\dot{H}[M] \arrow[d, "\dot{q}[M]"] \\
(\dot{H}\upharpoonright\Phi)[M] \arrow[rr, "\text{incl}"] \arrow[dr, shift right, "(\dot{\ell}\upharpoonright\Phi)[M]"'] \arrow[dr, shift left, "(\dot{r}\upharpoonright\Phi)[M]"] & & \dot{H}[M] \arrow[dl, shift right, "\dot{\ell}[M]"'] \arrow[dl, shift left, "\dot{r}[M]"] \\
& \mathrm{Set}^{L}(-,M) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{I}\dot{H}\upharpoonright\Phi)[M] \arrow[rr, "\text{incl}"] \arrow[dr, "(\dot{m}\upharpoonright\Phi)[M]"'] & & \mathrm{I}\dot{H}[M] \arrow[dl, "\dot{m}[M]"] \\
& \mathrm{Set}^{L}(-,M) &
\end{tikzcd}$$

$\bar{\Phi}$ is a system $\langle \Phi_K \rangle_{K \in L}$ indexed by the objects of L, where each Φ_K is a class of $(L, H(K))$-type augmented structures: Φ_K is a class of pairs (N, λ) where $N \in Set^L$ and $\lambda : H(K) \to N$. Usually, Φ_K is the class of all pairs (N, λ) such that

$$N \models \tilde{\Phi}_K [\lambda / H[K]]$$

with $\tilde{\Phi}_K$ is a formula in a possibly infinitary language $L_{\infty\infty}$ with free variables the set of objects in $el(H(K))$, the category of elements of the functor $H(K) : L \to Set$.

We impose a functoriality condition 1), and an invariance condition 2) on the system $\langle \Phi_K \rangle_{K \in L}$

1) For any $K \xrightarrow{p} K_p$ in L

$(N, \lambda) \in \Phi_K$ implies $(N, \lambda \circ H(p)) \in \Phi_{K_p}$

(note:
$$\begin{tikzcd} H(K_p) \arrow[rr, "H(p)"] \arrow[dr, "\lambda \circ H(p)"'] & & H(K) \arrow[dl, "\lambda"] \\ & N & \end{tikzcd}$$
)

2) Each class Φ_K is invariant under fiberwise surjective (FS) maps: If $P \xrightarrow{p} N$ in Set^L is FS:

$$(N, \lambda) \in \Phi_K \iff (P, \lambda \circ p) \in \Phi_K$$

(Recall: p is called fiberwise surjective, for a general category L, if p has the right lifting property relative to all monomorphisms in Set^L. When L is a FOLDS signature, this is equivalent to its own special cases for monomorphisms $i_K : \mathring{K} \to \hat{K}$.)

The subfunctor $(H \upharpoonright \Phi)[M]$ of $H[M]$ is defined thus: for $K \in L$ and $\lambda \in H[M](K)$

$$\lambda \in (H \upharpoonright \Phi)[M](K) \underset{\text{def}}{\Longleftrightarrow} (M, \lambda) \in \Phi_K .$$

The functoriality condition 1) ensures that we have a consistent definition of a subfunctor $(H \upharpoonright \Phi)[M] \subseteq H[M]$.

The definitions of the rest of the items in Diagram 1 are self-explanatory – including, of course, the definition of $(IH \upharpoonright \Phi)[M]$ as a pullback.

In Diagram 2, we define the subfunctor $(\dot{H} \upharpoonright \Phi)[M]$ of $\dot{H}[M] \in \mathbf{Set}^{(\mathbf{Set}^L)^{op}}$ as follows: for any $\mathcal{U} \in \mathbf{Set}^L$, $\sigma \in \dot{H}[M](\mathcal{U}) = \mathbf{Set}^L(\dot{H}(\mathcal{U}), M)$ we let

$$\sigma \in (\dot{H} \upharpoonright \Phi)[M](\mathcal{U}) \overset{\text{def}}{\Longleftrightarrow}$$

$$\sigma \circ \dot{H}(\hat{u}) \in (H \upharpoonright \Phi)[M](K)$$

for all $(K, u) \in [\mathcal{U}]$.

The action, for $U \xrightarrow{f} V$:

$$\begin{aligned} \dot{H}[M](f) : \dot{H}[M](V) &\longrightarrow \dot{H}[M](U) \\ \theta &\longmapsto \theta \circ \dot{H}(f) \end{aligned}$$

restricts to an action

$$(\dot{H} \upharpoonright \Phi)[M](f) : (\dot{H} \upharpoonright \Phi)[M](V) \longrightarrow (\dot{H} \upharpoonright \Phi)[M](U)$$

because of the commutativity of

$$\begin{tikzcd} \dot{H}(U) \arrow[rr, "\dot{H}(f)"] & & \dot{H}(V) \arrow[r] & M \\ & H(K) \arrow[ul, "\dot{H}(\hat{u})"] \arrow[ur, "\dot{H}(f_K(u))"'] & & \end{tikzcd}$$

for all $(K, u) \in [U]$.

This completes the definitions of the items in Diagram 2, including that of $(\dot{H} \upharpoonright \Phi)[M]$ by the pullback indicated.

For any given $M, U \in \mathbf{Set}^L$, we have the bijection

$$\Sigma = \Sigma[H,M](U) : \mathbf{Set}^L(U, H[M]) \xrightarrow{\cong} \dot{H}[M](U)$$

(see page $\boxed{15}$; recall that $\dot{H}[M](U) = \mathbf{Set}^L(\dot{H}(U), M)$)

Σ restricts to a bijection

$$\Sigma\restriction\Phi = \Sigma[H\restriction\Phi, M](U) :$$
$$\mathbf{Set}^L(U, (H\restriction\Phi)[M]) \xrightarrow{\cong} \mathbf{Set}^L((\dot{H}\restriction\Phi)[M], U)$$

defined by the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Set}^L(U, (H\restriction\Phi)[M] \arrow[r, dashed, "\Sigma\restriction\Phi", "\cong"'] \arrow[d, "\mathrm{incl}\circ(-)"'] & (\dot{H}\restriction\Phi)[M](U) \arrow[d, "\mathrm{incl}"] \\
\mathbf{Set}^L(U, H[M]) \arrow[r, "\cong", "\Sigma"'] & \dot{H}[M](U)
\end{tikzcd}$$

(for details, see Appendix)

For us, the significance of this fact is that we have the desired generalization of the dual description of internal equivalence, the fact displayed on p $\boxed{22}$:

For any pair $U \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$ of U-diagrams in M

the following conditions (1) and (2) are equivalent:

(1) the "external" condition for $\alpha \underset{\underline{H}\restriction\Phi}{\sim} \beta$:

there exists $\gamma : U \to (H\restriction\Phi)[M]$

such that the following commutes:

$$\begin{tikzcd}
& (H\restriction\Phi)[M] \arrow[dl, "(\bar{\ell}\restriction\Phi)[M]"'] \arrow[dr, "(\bar{r}\restriction\Phi)[M]"] & \\
M & & M \\
& U \arrow[uu, "\gamma"] \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

(2) the "internal" condition for $\alpha \underset{\underline{H}\restriction\Phi}{\sim} \beta$:

there exists $\sigma \in (\dot{H}\restriction\Phi)[M](U)$ such that:

the following commutes:

$$\begin{tikzcd}
& (\dot{H}\restriction\Phi)[M] \arrow[dl, "(\dot{\ell}\restriction\Phi)_U"'] \arrow[dr, "(\dot{r}\restriction\Phi)_U"] \arrow[dd, "\sigma"'] & \\
U \arrow[dr, "\alpha"'] & & U \arrow[dl, "\beta"] \\
& M &
\end{tikzcd}$$

We have the analogous situation for $\mathbb{H}$ in place of H; the bijection $\Sigma[\mathbb{H}] = \Sigma[\mathbb{H}, M](U)$ restricts to the bijection $(\Sigma \upharpoonright \underline{\Phi})[\mathbb{H}]$ shown in the commutative diagram

$$
\begin{tikzcd}
\mathrm{Set}^{L}(U, (\mathbb{H} \upharpoonright \underline{\Phi})[M]) \arrow[r, dashed, "(\Sigma \upharpoonright \underline{\Phi})[\mathbb{H}]", "\cong"'] \arrow[d, "\mathrm{incl} \circ (-)"'] & (\dot{\mathbb{H}} \upharpoonright \underline{\Phi})[M](U) \arrow[d, "\mathrm{incl}"] \\
\mathrm{Set}^{L}(U, \mathbb{H}[M]) \arrow[r, "\cong", "\Sigma[\mathbb{H}]"'] & \dot{\mathbb{H}}[M](U)
\end{tikzcd}
$$

For details, see Appendix.

The invariance assumption, condition 2) on page $\boxed{48}$, on the classes $\underline{\Phi}_K$ will be used as follows:

for any $U \in \mathrm{Set}^{L}$, any FS map $p : P \to M$ in Set^{L}

$$
\begin{tikzcd}
\dot{H}(U) \arrow[r, "\theta"] \arrow[dr, "\sigma"'] & P \arrow[d, "p"] \\
& M
\end{tikzcd}
$$

, i.e. $\sigma = p\theta$, implies that

$$\theta \in (\dot{H} \upharpoonright \underline{\Phi})[P](U) \iff \sigma \in (\dot{H} \upharpoonright \underline{\Phi})[M](U)$$

For details, see Appendix

We have the "constrained" version of the Main Argument as follows:

Suppose the IEG $\underline{H}$ satisfies the condition $\mathrm{Mon}(\underline{H})$. Suppose $\underline{\Phi} = \langle \Phi_K \rangle_{K \in L}$ is a constraint on $\underline{H}$ satisfying the functoriality and invariance conditions, 1) and 2) on page $\boxed{48}$. In brief, we suppose the constrained IEG $\underline{H} \restriction \underline{\Phi}$ (with $\mathrm{Mon}(\underline{H})$).

Let $M \in \mathrm{Set}^L$. Assume that M satisfies the following conditions: the morphisms

$$(\bar{\ell} \restriction \underline{\Phi})[M],\ (\bar{r} \restriction \underline{\Phi})[M] : H \restriction \underline{\Phi} \longrightarrow M$$

and

$$(\bar{m} \restriction \underline{\Phi})[M] : (IH \restriction \underline{\Phi})[M] \longrightarrow M$$

are FS. Then, for any $\mathcal{U} \in \mathrm{Set}^L$ and diagrams $\alpha, \beta : \mathcal{U} \to M$, we have

$$\alpha \underset{\underline{H}\restriction\Phi}{\sim} \beta \quad \text{if and only if} \quad \alpha \underset{\text{int}}{\sim} \beta$$

where the relation $\underset{\underline{H}\restriction\Phi}{\sim}$ is meant in either of the two equivalent senses given on page $\boxed{52}$, and $\underset{\text{int}}{\sim}$ is the usual (unchanged!) notion of intrinsic equivalence.

The <u>proof</u> is a quick modification of the original proof for the Main Argument.

We see that the assumptions amount to saying that

$$(\underline{H}\restriction\Phi)[M] = \big((H\restriction\Phi)[M],\ (\bar{\ell}\restriction\Phi)[M],\ (\bar{r}\restriction\Phi)[M]\big)$$

is a reflexive self-equivalence of M, with reflexivity

$$\begin{tikzcd}
(IH\restriction\Phi)[M] \arrow[r, "(\bar{q}\restriction\Phi){[M]}"] \arrow[rr, bend right, "(\bar{m}\restriction\Phi){[M]}"'] & (H\restriction\Phi)[M] \arrow[r, shift left, "(\bar{\ell}\restriction\Phi){[M]}"] \arrow[r, shift right, "(\bar{r}\restriction\Phi){[M]}"'] & M
\end{tikzcd}$$

Therefore $\alpha \underset{\underline{H} \upharpoonright \Phi}{\sim} \beta$, read in the external sense 1), means that $\underline{H} \upharpoonright \Phi$ is a witness for $\alpha \underset{int}{\sim} \beta$, defined as the existence of a RSE P such that $\alpha \underset{P}{\sim} \beta$.

In brief $\alpha \underset{\underline{H} \upharpoonright \Phi}{\sim} \beta$ implies $\alpha \underset{int}{\sim} \beta$.

For the converse direction, we stay in the context of the previous proof (starting on page $\boxed{32}$) and put appropriate additional conditions on some of the items constructed. We assume $\alpha \underset{int}{\sim} \beta$ with the witnessing data $P : M \underset{L}{\sim} M$ and all the rest as displayed on p. $\boxed{31}$.

We choose $\bar{\bar{\beta}} : U \longrightarrow (\mathrm{I}H \upharpoonright \Phi)[M]$ such that $(\bar{m} \upharpoonright \Phi)[M] \circ \bar{\bar{\beta}} = \beta$ – which we can do since $(\bar{m} \upharpoonright \Phi)[M]$ is assumed to be FS:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "!"'] & (\mathrm{I}H \upharpoonright \Phi)[M] \arrow[d, "(\bar{m} \upharpoonright \Phi)[M]"] \\
U \arrow[r, "\beta"'] \arrow[ur, dashed, "\bar{\bar{\beta}}"] & M
\end{tikzcd}$$

To consolidate the present notation with the earlier one, we write $\bar{\beta} : \mathcal{U} \longrightarrow \mathbb{H}[M]$ for the composite

$$\mathcal{U} \xrightarrow{\bar{\bar{\beta}}} (\mathbb{H} \upharpoonright \Phi)[M] \xrightarrow{\text{incl}} \mathbb{H}[M]; \quad \bar{\beta} = \text{incl} \circ \bar{\bar{\beta}};$$

thus, we have $\bar{\beta}$ as before, but with the additional property that it factors through $(\mathbb{H} \upharpoonright \Phi)[M] \xrightarrow{\text{incl}} \mathbb{H}[M]$.

We have the bijection

$$(\Sigma \upharpoonright \Phi)[\mathbb{H}] : \mathbf{Set}^{L}(\mathcal{U}, (\mathbb{H} \upharpoonright \Phi)[M]) \xrightarrow{\cong} (\dot{\mathbb{H}} \upharpoonright \Phi)[M](\mathcal{U})$$

(see p. $\boxed{53}$); let $\bar{\bar{\beta}}^{\#} \underset{\text{def}}{=} (\Sigma \upharpoonright \Phi)[\mathbb{H}](\bar{\bar{\beta}}) \in (\dot{\mathbb{H}} \upharpoonright \Phi)[M](\mathcal{U})$

Also, $\bar{\beta}^{\#} \underset{\text{def}}{=} \Sigma[\mathbb{H}](\mathcal{U})(\bar{\beta}) \in \dot{\mathbb{H}}[M](\mathcal{U})$.

The commutative diagram on page $\boxed{53}$ tells us that

$$\bar{\beta}^{\#} = \bar{\bar{\beta}}^{\#}$$

from which we conclude that

$$\bar{\beta}^{\#} \in (\dot{\mathbb{H}} \upharpoonright \Phi)[M](\mathcal{U});$$

with $\bar{\beta}^{\#}$ we are back in the notation of the previous proof. Following that proof, we get, eventually,

$$\sigma : \dot{H}(U) \to M$$

such that the following commutes:

$$\begin{tikzcd}
 & \dot{H}(U) \arrow[dd, "\sigma"] & \\
U \arrow[ur, "\dot{e}_U"] \arrow[dr, "\alpha"'] & & U \arrow[ul, "\mathring{r}_U"'] \arrow[dl, "\beta"] \\
 & M &
\end{tikzcd}$$

.

I claim that $\sigma \in \dot{H}[M](U)$ belongs to the subset $(\dot{H}\restriction\Phi)[M](U)$ of $\dot{H}[M](U)$. Once we see this, we have proved that $\alpha \underset{\underline{H}\restriction\Phi}{\sim} \beta$ by the second, "internal", definition of this relation – and this is what we want.

For the proof of the claim, we return to the diagram ⑨ on page $\boxed{35}$, and take the composite

$$\dot{H}(U) \xrightarrow{\vec{\beta}^{\#} \circ \mathring{q}_U} M .$$

The pullback-definition of $(\dot{\mathbb{H}}\restriction\Phi)[M]$ (see page $\boxed{46}$) says that for any $z \in \dot{\mathbb{H}}[M](U)$

$$z \in (\dot{\mathbb{H}})[M](U) \iff z\,\dot{q}_U \in (\dot{\mathbb{H}}\restriction\Phi)[M](U)$$
$$(\subseteq \dot{\mathbb{H}}[M](U))$$

Since $z = \bar{\beta}^{\#}$ does belong to $\dot{\mathbb{H}}[M](U)$, we have that

$$\bar{\beta}^{\#} \circ \dot{q}_U \in (\dot{\mathbb{H}}\restriction\Phi)[M](U).$$

Now, looking at diagram (11) on page $\boxed{37}$:

$$\begin{tikzcd} & & P \arrow[d, "r^{\underline{P}}"] \\ \dot{H}(U) \arrow[urr, "\theta"] \arrow[rr, "\bar{\beta}^{\#}\circ\dot{q}_U"'] & \equiv & M \end{tikzcd}$$

and using (again!) that $r^{\underline{P}}$ is FS, we obtain that

$$\theta \in (\dot{\mathbb{H}}\restriction\Phi)[P](U) :$$

see page $\boxed{53}$ for this inference.

Finally, by diagram (12) on page $\boxed{37}$:

$$\begin{tikzcd} & P \arrow[d, "\ell^{\underline{P}}"] \\ \dot{H}(U) \arrow[ur, "\theta"] \arrow[r, phantom, "\equiv"] \arrow[dr, "\sigma"'] & M \end{tikzcd}$$

and $\ell^{\underline{P}}$ being FS (which fact was not used previously!), we obtain that $\sigma \in (\dot{H}\lceil\Phi)[M](U)$ as desired

$\square$ Main Argument
constrained version

The following theorem, generalizing the theorem on page $\boxed{43}$, is a consequence of what we have:

Theorem Let: $\mathbb{L}$ be a small category equipped with a sphere functor $((\overset{\circ}{-}), i_{(-)})$ (if $\mathbb{L}$ is a FOLDS signature, this is satisfied).

Let: $\underline{H}$ be an internal equivalence generator (IEG) for $\mathbb{L}$ satisfying the condition

$EM[\underline{H}]$.

Let: $\bar{\Phi} = \langle \bar{\Phi}_K \rangle_{K \in L}$ be a constraint on $\underline{H}$ satisfying the functoriality and invariance conditions (see page $\boxed{48}$). Let: $M \in Set^L$.

Suppose: the morphisms

$$(\bar{\ell} \upharpoonright \bar{\Phi})[M],\ (\bar{r} \upharpoonright \bar{\Phi})[M] : (H \upharpoonright \bar{\Phi})[M] \to M$$

$$(\bar{m} \upharpoonright \bar{\Phi})[M] : (IH \upharpoonright \bar{\Phi})[M] \to M$$

have the right lifting property relative to each of the morphisms $i_K : \mathring{K} \to \hat{K}$, $K \in L$.

Then, for any $U \in Set^L$ and $\alpha, \beta : U \to M$

$$\alpha \underset{\underline{H} \upharpoonright \bar{\Phi}}{\sim} \beta \iff \alpha \underset{int}{\sim} \beta.$$

Equivalently: intrinsic equivalence for diagrams in M is induced by a single reflexive self equivalence of M, namly

$$(\underline{H}\restriction\underline{\Phi})[M] : M \underset{L}{\sim} M$$

where $(\underline{H}\restriction\underline{\Phi})[M] = ((H\restriction\Phi)[M], (\bar{\ell}\restriction\Phi)[M], (\bar{r}\restriction\Phi)[M])$

with reflexivity given by $(\bar{q}\restriction\Phi)[M]$ and $(\bar{m}\restriction\Phi)[M]$.

Let me call a reflective self equivalence of M a <u>schematic</u> RSE if it is of the form $(\underline{H}\restriction\underline{\Phi})[M]$ obtained from a constrained LEG $\underline{H}\restriction\underline{\Phi}$ as above.

<u>Corollary</u> If $P_1, P_2 : M \underset{L}{\sim} M$ are both schematic reflexive self equivalences of M, then they induce, for every $U \in Set^L$, the same <u>equivalence</u> relation $\overset{U}{\underset{P_1}{\sim}}$, $\overset{U}{\underset{P_2}{\sim}}$ on the set $Set^L(U,M)$:

$$\alpha \underset{P_1}{\sim} \beta \iff \alpha \underset{P_2}{\sim} \beta \quad (\alpha, \beta : U \to M)$$

and this relation is the same as intrinsic equivalence.