

Proof of Proposition, page 15:

A1

For part (i): we have the following sequence of bijections (U , D and M are fixed):

$$U = \operatorname{colim}_{(K,u) \in [U]} \hat{K} \xrightarrow{\delta} D[M]$$

$$\langle \hat{K} \xrightarrow{\hat{\delta}_{(K,u)}} D[M] \rangle_{\text{natural in } (K,u) \in [U]}$$

$$\langle \delta_{(K,u)} \in D[M](K) \rangle_{\text{natural in } (K,u)}$$

$$\langle \delta_{(K,u)} : D(K) \rightarrow M \rangle_{\text{natural in } (K,u)}$$

$$\dot{D}(U) = \operatorname{colim}_{(K,u) \in [U]} D(K) \xrightarrow{\delta^\#} M$$

establishing a bijection $\delta \leftrightarrow \delta^\#$.

Alternatively, one sees directly that, given δ , there is unique $\delta^\#$ satisfying the set of commutativities stated; and, with any $\delta^\# : \dot{D}(U) \rightarrow M$,

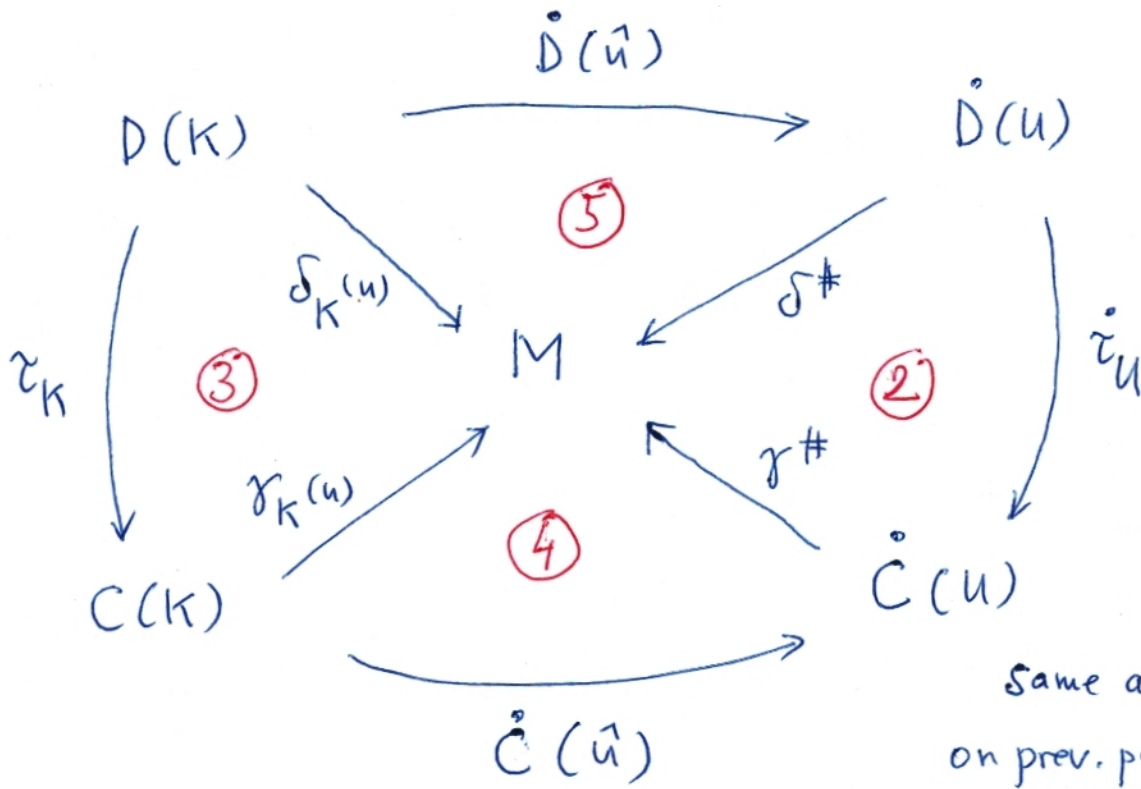
the system $\langle \delta_K(u) : D(K) \rightarrow M \rangle_{(K,u)}$ given

as $\delta_K(u) \stackrel{\text{def}}{=} \delta^\# \circ \dot{D}(u)$ provides $\delta : U \rightarrow D[M]$.

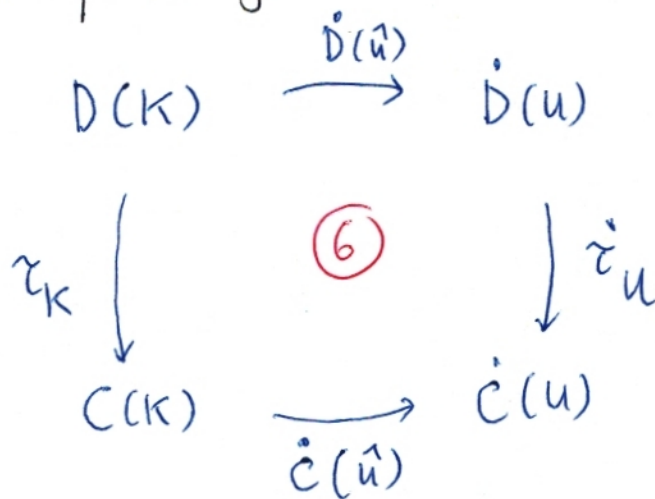
In both directions, one uses ^{the} appropriate colimits.

A2

For part (ii): Let $(K, u) \in [U]$,
and consider the diagram



with outside quadrangle



⑥ commutes by the naturality of $\tilde{\tau}$.

A3

④ and ⑤ commute by the definitions

of $\gamma^\#$ and $\delta^\#$ from γ and δ in part (i).

The commutativity of ① is equivalent to the

joint commutativity of all the instances of ③,

one for each $(K, u) \in [U]$, by the standard colimit representation of U by representables.

Assume ① commutes. Then ② commutes, by all the above, if precomposed with each of

the colimit coprojections $\dot{D}(\tilde{u}) : D(K) \rightarrow \dot{D}(U)$

$((K, u) \in [U])$. Since the latter are jointly

epimorphic, it follows that ② commutes.

Next, assume that ② commutes. Then ③

commutes for each $(K, u) \in [U]$, and thus ①

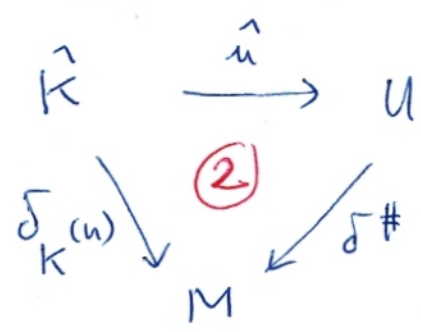
commutes

□ Proposition page 15

Proof of $(*)$, p. 17:

Now $\dot{Y}(U) = U$, $Y(K) = \hat{K}$, and, for $(K, u) \in [U]$
 $\hat{u} : \hat{K} \rightarrow U$, we have $\dot{Y}(\hat{u}) = \hat{u}$.

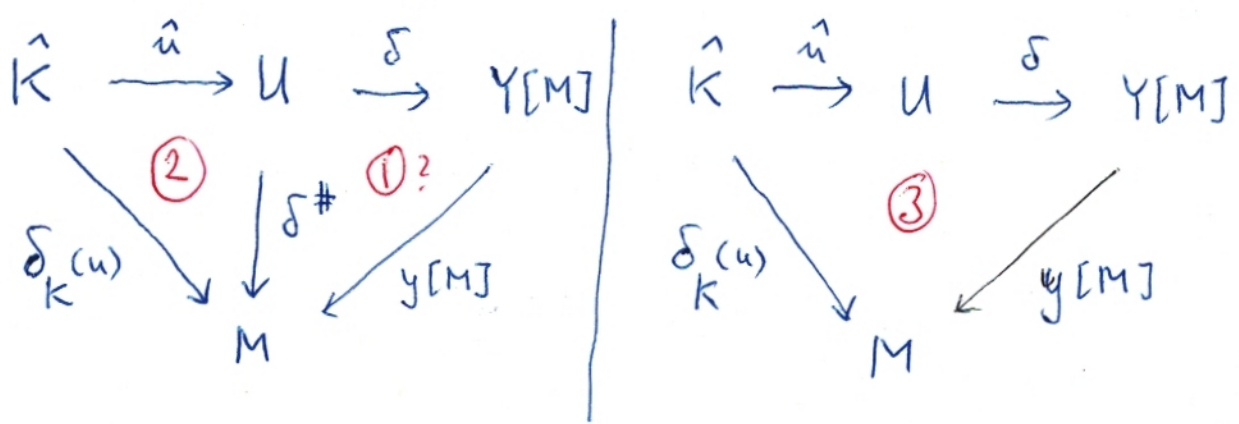
The "defining" commutativity \odot on p. 15 is now:



To prove $(*)$, called also $\text{\textcircled{1}}$ now, we precompose it with the arrow $\hat{K} \xrightarrow{\hat{u}} U$, and show the resulting commutativity. Since the cocone

$\langle \hat{K} \xrightarrow{\hat{u}} U \rangle_{(K, u) \in [U]}$ is jointly epimorphic,

this will be enough. Said precomposition gives:



The commutativity of (3) is the equality of two arrows $\tau, s: \hat{K} \rightarrow M$.

By Yoneda, $\tau = s$ follows once we know that

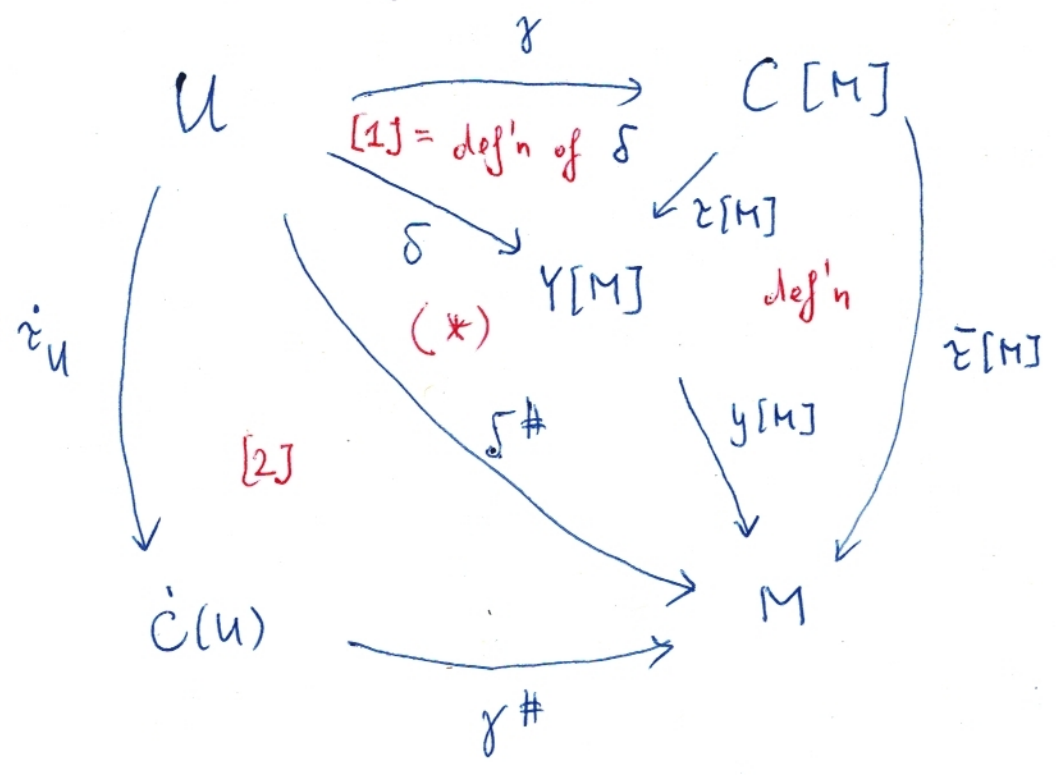
$$\tau_K(1_K) = s_K(1_K)$$

Checking this for (3) is left as an exercise

□ for (*), p 17

Proof of the commutativity on p [20]:

Construct the following diagram:



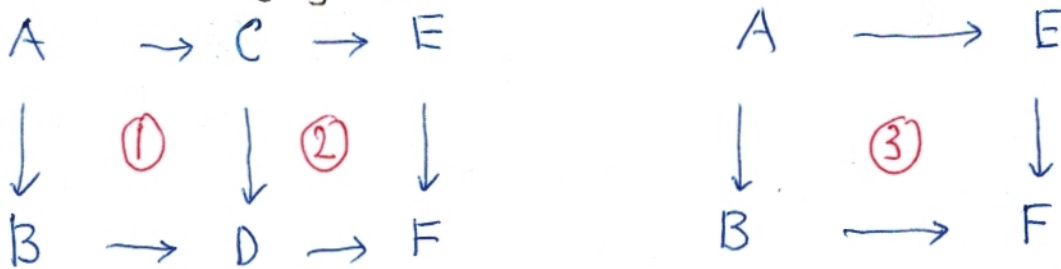
[1] implies [2] : the case of

① implies ②, page 16, for $D = Y$.

□ comm. of p [20]

Proof of Proposition, page [25]:

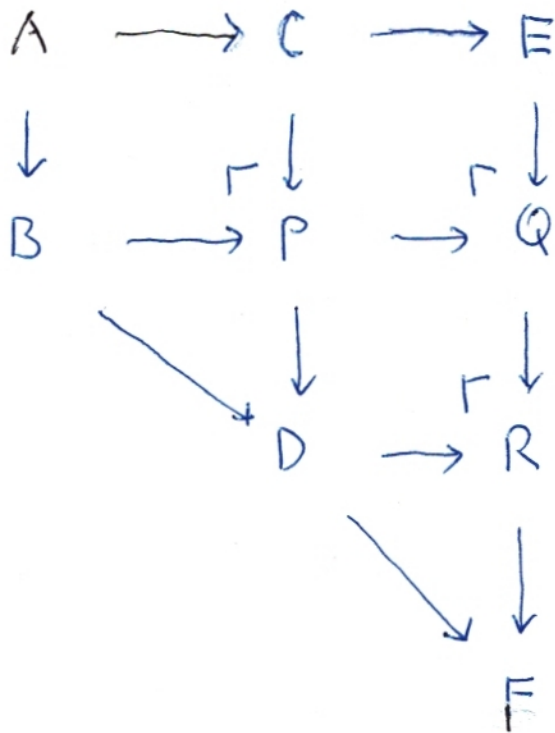
Take two commutative squares ①, ② and their composite ③ = ① \square_m ② ('m' is the 'middle vertical'):
in the category \mathcal{S} :



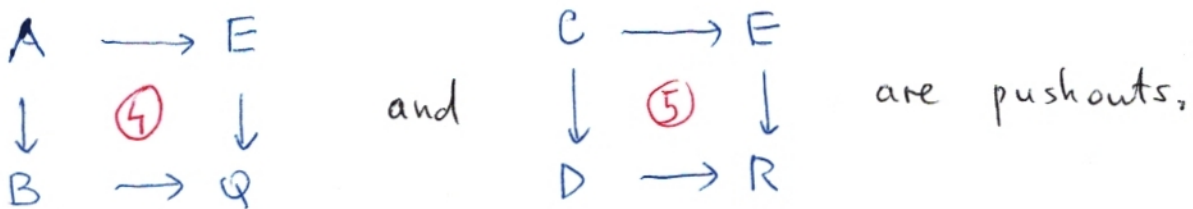
(of course, $\vec{AE} = \vec{BE} \circ \vec{AE}$, etc., in the awkward algebraic notation). Construct pushouts as follows:

$$P = \underset{A}{C \cup B}, \quad Q = \underset{C}{E \cup P}, \quad R = \underset{P}{Q \cup D}$$

and fit them in the resulting commutative diagram:

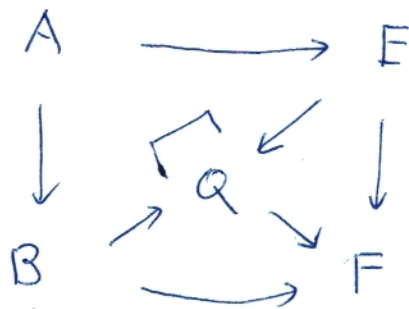


Since pushouts compose, the squares



Suppose $\textcircled{1}$ and $\textcircled{2}$ are EM squares. Then

$P \rightarrow D$ and $R \rightarrow F$ are mono's. As a result, $Q \rightarrow R$ (as a pushout of a mono), and then the composite $Q \rightarrow F$ are mono's. Since $\textcircled{4}$ is a pushout and we have the commutative diagram



We have shown that the "composite" ③ is EM;

① and ② are EM \Rightarrow ① \square ② is EM.

Now suppose that ① is, not only EM, but itself a pushout with $A \rightarrow B$ a mono; briefly, ① is PM.

Then in the above composite diagram,

$P \rightarrow D$, and therefore $Q \rightarrow R$ also, is an isomorphism, and

$Q \rightarrow F$ is a mono iff $R \rightarrow F$ is a mono.

Therefore

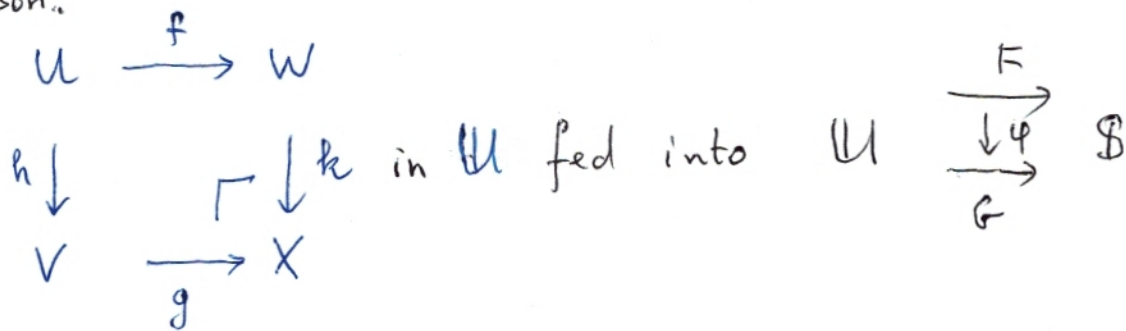
① is PM and ① \square ② is EM \Rightarrow ② is EM.

Now turn to the data $\mathcal{U} \begin{array}{c} \xrightarrow{F} \\ \downarrow \varphi \\ \xrightarrow{\quad} \\ \downarrow \\ G \end{array} \mathcal{S}.$

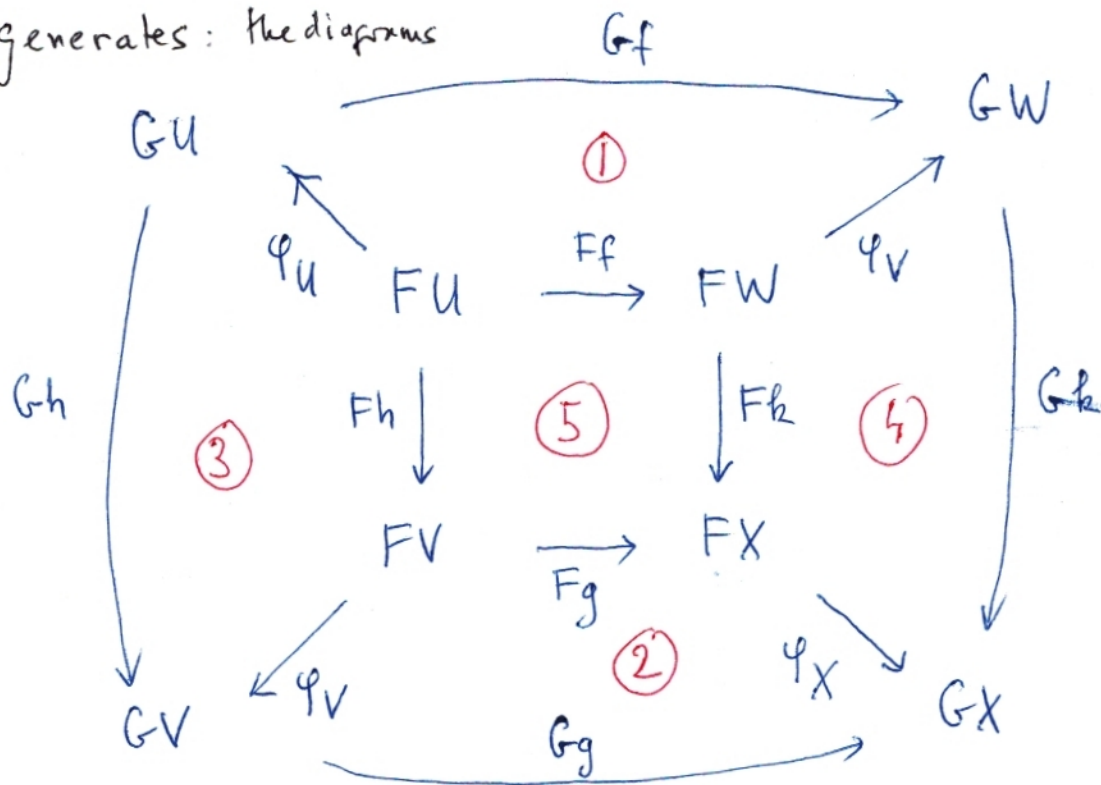
For $f = \text{id}_U: U \rightarrow U$, " φ is EM at f " means just that $\varphi_U: FU \rightarrow GU$ is a mono. For U the initial object of \mathcal{U} , this is true (F and G are cocontinuous).

Lemma 1 If φ is EM at $f \in \text{Arr}(\mathcal{U})$, and g is a pushout of f , then φ is EM at g as well.

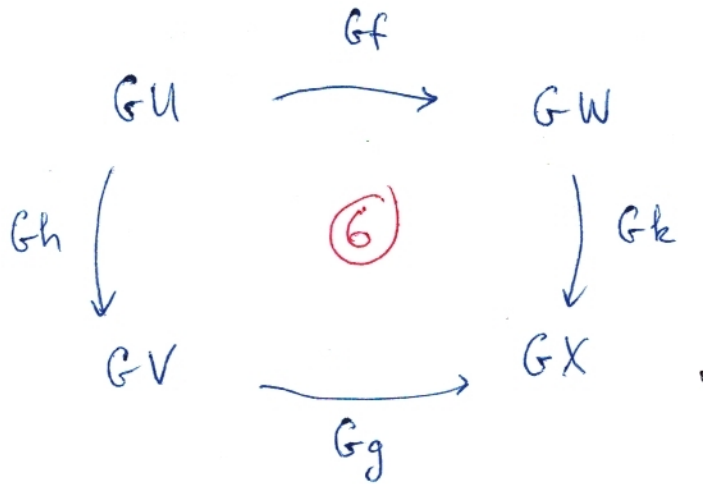
Reason:



generates: the diagrams



and



The quadrangles ①, ②, ③, ④ commute by naturality. ⑤ and ⑥ are pushouts as images of the cocontinuous functors F and G , respectively, of the pushout square given in \mathcal{U} . The only essential hypothesis is that ① is EM. Note that ⑥ is PM, not just PO, since, by ① being EM, Gf is a mono. Composing squares:

$$\begin{array}{ccccc}
 \boxed{5} & \square & \boxed{2} & = & \boxed{1} & \square & \boxed{6} & = & \begin{array}{ccc} Fu & \rightarrow & Fw \\ \downarrow & \textcircled{7} & \downarrow \\ Gv & \rightarrow & Gx \end{array} \\
 \text{PM} & & Fg & & \text{EM} & & Gf & & \text{PM}
 \end{array}$$

Since $\boxed{7}$ is $\boxed{\text{EM}} \circ \boxed{\text{PM}}$, $\boxed{7}$ is EM;

$$\boxed{5} \circ \boxed{2} = \boxed{7} \text{ is EM, hence } \boxed{2} \text{ is EM}$$

□ Lemma 1

Lemma 2 For $U \xrightarrow{f} V \xrightarrow{g} W$ in \mathcal{U} ;

A11

φ is EM at f and $g \Rightarrow \varphi$ is EM at gf .

This follows directly by " $\boxed{\text{EM}} \circ \boxed{\text{EM}}$ is $\boxed{\text{EM}}$ ".

I will now recall transfinite compositions/composites.

We work in an fixed cocomplete category \mathcal{S} (that
(arbitrary)

could be either \mathcal{U} or \mathcal{S} of the starting context).

$\alpha, \beta, \gamma, \delta$ denote ordinals; $[\alpha]$ denotes the well-ordered set $\{\beta \mid \beta \leq \alpha\}$ of ordinals; $[\alpha >) = \{\beta \mid \beta < \alpha\}$.

A diagram (functor) $\vec{A} : [\alpha] \rightarrow \mathcal{S}$, denoted

$\langle A_\beta \rangle_{\beta \leq \alpha}$ for the objects, and $\langle a_{\gamma\beta} : A_\gamma \rightarrow A_\beta \rangle_{\gamma \leq \beta \leq \alpha}$

for the arrows — is a transfinite composition, TC

for short, with (transfinite) composite $a_{0\alpha} : A_0 \rightarrow A_\alpha$.

if for all limit ordinals β , $0 < \beta \leq \alpha$,

A_β is the colimit of the restriction

$$\vec{A} \upharpoonright [\beta >) : [\beta >) \rightarrow \mathcal{S}$$

with coprojections $a_{\gamma\beta} : A_\gamma \rightarrow A_\beta$ ($\gamma < \beta$).

A12

We think of a TC \vec{A} as a composition of its successor steps $a_{\beta, \beta+1} : A_\beta \rightarrow A_{\beta+1}$, $\beta < \alpha$, since all other items in \vec{A} is "given by continuity".

Accordingly, for a class E of arrows of \mathcal{G} , to say that E is closed under transfinite compositions/composites is to say that if in a TC \vec{A} , each successor step $a_{\beta, \beta+1} : A_\beta \rightarrow A_{\beta+1}$ is in E , then so is the composite $a_{0\alpha} : A_0 \rightarrow A_\alpha$.

For \vec{A} a TC, every restriction $\vec{A} \upharpoonright [\beta]$, $\beta < \alpha$, is one too.

Now, for any diagram $\vec{A} : [\alpha] \rightarrow \mathcal{S}$ and arrow $f_0 : A_0 \rightarrow P_0$, we can define the pushout diagram $\vec{P} = f_0 \sqcup_{A_0} \vec{A} : [\alpha] \rightarrow \mathcal{S}$ as follows:

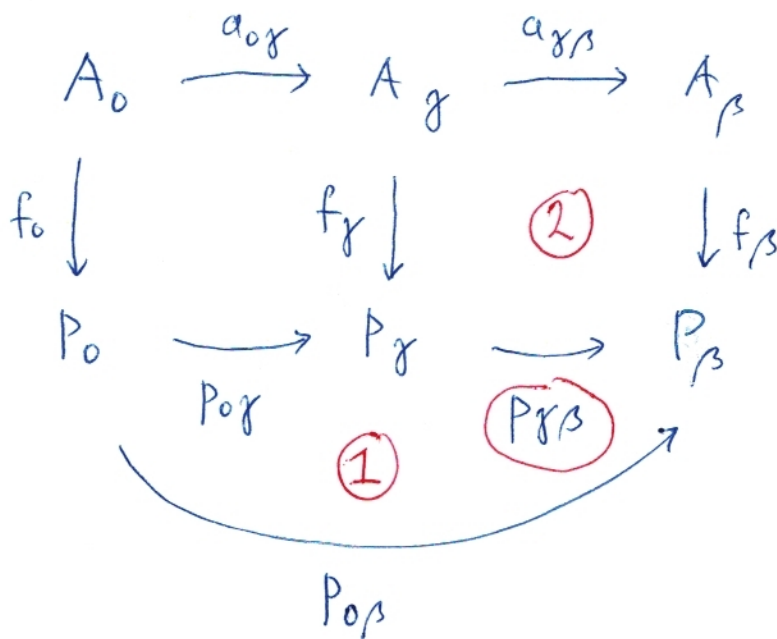
For $0 < \beta \leq \alpha$, the object P_β is defined by a pushout

$$\begin{array}{ccc} A_0 & \xrightarrow{a_{0\beta}} & A_\beta \\ f_0 \downarrow & \lrcorner & \downarrow f_\beta \\ P_0 & \xrightarrow{p_{0\beta}} & P_\beta \end{array}$$

For $\gamma \leq \beta \leq \alpha$, the arrow $p_{\gamma\beta} : P_\gamma \rightarrow P_\beta$

is defined as the unique arrow $P_{\gamma\beta} = P_\gamma \rightarrow P_\beta$

that makes parts ① and ② in



commute (using that P_γ is given as a pushout); it then follows that ② is a pushout square

The f_β , $\beta \leq \alpha$, form a morphism (natural transformation) $\vec{f}: \vec{A} \rightarrow \vec{P}$. It is easy to see that if \vec{A} is a TC, then so is the pushout $f_0 \sqcup_{A_0} \vec{A}$.

Still in the general case, $\vec{f}: \vec{A} \rightarrow \vec{P} = f_0 \sqcup_{A_0} \vec{A}$

has the natural universal property: given

arrows $A_0 \xrightarrow{f_0} P_0 \xrightarrow{g_0} C_0$,

diagram $\vec{B} : [\alpha] \rightarrow \mathcal{S}$

and a morphism $\vec{h} : \vec{A} \rightarrow \vec{B}$ such that

$$h_0 : A_0 \rightarrow B_0 \text{ equals } h_0 = g_0 \circ f_0$$

there is a unique $\vec{g} : \vec{P} \rightarrow \vec{C}$ starting with
 the given $g_0 : P_0 \rightarrow C_0$ such that $\vec{h} = \vec{g} \circ \vec{f}$.

I will apply the above in the context of
 the cocontinuous functors $F, G : \mathcal{U} \rightarrow \mathcal{S}$ and
 the natural transformation $\varphi : F \rightarrow G$.

Suppose that $\vec{X} : [\alpha] \rightarrow \mathcal{U}$ is a TC in \mathcal{U} .

Then $\vec{A} \stackrel{\text{def}}{=} F \circ \vec{X}$, $\vec{B} \stackrel{\text{def}}{=} G \circ \vec{X}$, both

diagrams $[\alpha] \rightarrow \mathcal{S}$, are also TC's, in \mathcal{S} ;

$\vec{h} \stackrel{\text{def}}{=} \varphi \circ \vec{X} : F \circ \vec{X} \rightarrow G \circ \vec{X}$ is a morphism

between them. Take now the start of \vec{X} ,

the object X_0 , and the induced items

$$\varphi_{X_0} : FX_0 \rightarrow GX_0$$

written as $f_0 : A_0 \rightarrow P_0$, in \mathcal{S} , and considers the pushout

$$\vec{P} \stackrel{\text{def}}{=} f_0 \sqcup_{A_0} \vec{A}$$

with the induced factorization

$$\vec{A} \xrightarrow{\vec{f}} \vec{P} \xrightarrow{\vec{g}} \vec{B}$$

$\underbrace{\hspace{10em}}_{\vec{h}}$

$$A_0 \xrightarrow{f_0} P_0 = C_0$$

$\underbrace{\hspace{10em}}_{h_0 = f_0} \xrightarrow{g_0 = 1_{P_0}}$

These constructions and the notation will be used in proving the following lemma. We still have $F, G : \mathcal{U} \rightarrow \mathcal{S}$ and $\varphi : F \rightarrow G$ as before.

Lemma 3 Assume, in addition to the previous assumptions, that \mathcal{B} is a locally finitely presentable category (Set^L is certainly one such).

Let $\vec{U} = \langle U_\beta, u_{\gamma\beta}: U_\gamma \rightarrow U_\beta \rangle_{\gamma \leq \beta < \alpha}$ be a TC in \mathcal{U} . Suppose that $\varphi_{U_0}: FU_0 \rightarrow GU_0$ is a mono,

and that φ is EM at $u_{\gamma, \gamma+1}: U_\gamma \rightarrow U_{\gamma+1}$

for all $\gamma < \alpha$. Then φ is EM at $u_{0\alpha}: U_0 \rightarrow U_\alpha$.

In particular, $\varphi_{U_\alpha}: FU_\alpha \rightarrow GU_\alpha$ is a monomorphism.

Proof. The proof is by induction on the ordinal α .

For $\alpha = 0$, the assumption on φ_{U_0} being mono ensures that the assertion is true. When $\alpha = \beta + 1$,

then $u_{0\alpha} = u_{\beta, \beta+1} \circ u_{0\beta}$; $u_{0\beta} \in \text{EM}(\varphi)$ by

the induction hypothesis, $u_{\beta, \beta+1} \in \text{EM}[\varphi]$ by

assumption, so by Lemma 2, $u_{0\alpha} \in \text{EM}[\varphi]$.

Assume now that α is a limit ordinal. I use

the notation introduced above. The inductive

assumption is that, for all $\beta < \alpha$, in the diagram

$$\begin{array}{ccc}
 A_0 & \xrightarrow{a_{0\beta}} & A_\beta \\
 f_0 \downarrow & & \downarrow f_\beta \\
 P_0 & \xrightarrow{p_{0\beta}} & P_\beta \\
 \parallel & & \downarrow g_\beta \\
 B_0 & \xrightarrow{b_{0\beta}} & B_\beta
 \end{array}$$

g_β is a monomorphism. Since F and G preserve colimits, $\vec{P} = f_0 \sqcup_{A_0} \vec{A}$ is a TC, and P_α

is the directed colimit of the diagram

$$\langle P_\beta, p_{\gamma\beta}: P_\gamma \rightarrow P_\beta \rangle_{\gamma \in \beta < \alpha}, \text{ with coprojections}$$

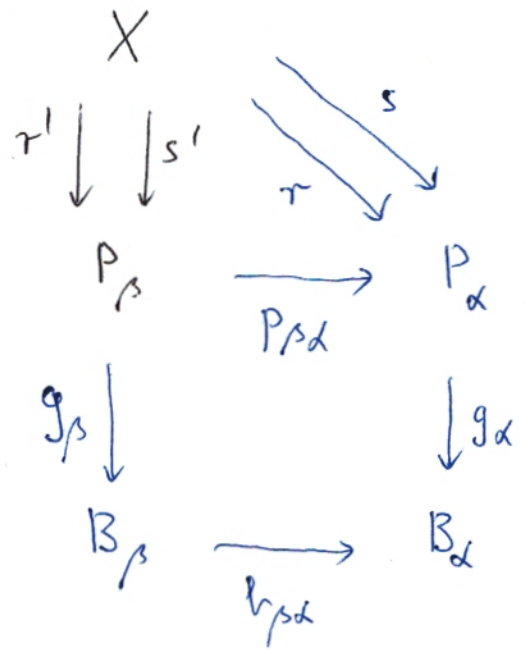
$$p_{\beta\alpha}: P_\beta \rightarrow P_\alpha \quad (\beta < \alpha); \text{ similarly for the } B'_s.$$

The diagram

$$\begin{array}{ccc}
 P_\beta & \xrightarrow{p_{\beta\alpha}} & P_\alpha \\
 g_\beta \downarrow & & \downarrow g_\alpha \\
 B_\beta = Gu_\beta & \longrightarrow & B_\alpha = Gu_\alpha \\
 & & b_{\beta\alpha} = Gu_{\beta\alpha}
 \end{array}$$

commutes for all $\beta < \alpha$, and g_β is a mono for all $\beta < \alpha$. It follows that g_α is a mono.

Indeed, let, first, X be a finitely presentable (fp) object, and let the arrows $r, s : X \rightarrow P_\alpha$ be such that $g_\alpha r = g_\alpha s : X \rightarrow B_\alpha$. Since the colimit for α is directed, and X is fp, there is $\beta < \alpha$ and there are $r', s' : X \rightarrow P_\beta$ such that $r = P_{\beta\alpha} r', s = P_{\beta\alpha} s'$:



We have $b_{\beta\alpha} (g_\beta r') = b_{\beta\alpha} (g_\beta s') : X \rightarrow B_\alpha$. Since the colimit that gives B_α is directed,

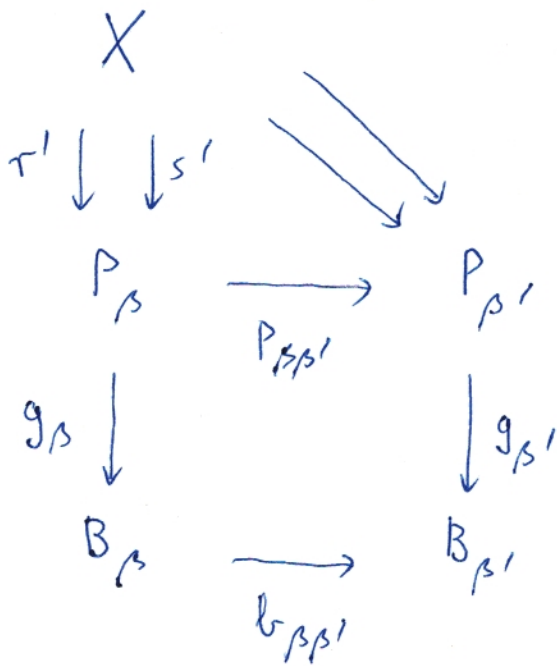
and X is fp, it follows that there is

β' , $\beta \leq \beta' < \alpha$, such that

$$b_{\beta\beta'}(g_\beta \tau') = b_{\beta\beta'}(g_\beta s')$$

i.e.

$$g_{\beta'}(p_{\beta\beta'} \tau') = g_{\beta'}(p_{\beta\beta'} s') :$$



Since $g_{\beta'}$ is a mono, it follows that $\tau' = s'$, hence $\tau = s$ as desired.

Now, let X be an arbitrary object in \mathcal{S} , and

Suppose $\tau, s: X \rightarrow P_\alpha$, $g_\alpha \tau = g_\alpha s: X \rightarrow B_\alpha$

\mathcal{S} is lfp; X can be represented as

a (directed) colimit: $X = \text{colim}_{i \in I} X_i$, of fp objects

X_i , with coprojections $\psi_i: X_i \rightarrow X$.

We have $g_\alpha(\tau\psi_i) = g_\alpha(s\psi_i): X_i \rightarrow B_\alpha$. Since X_i is fp,

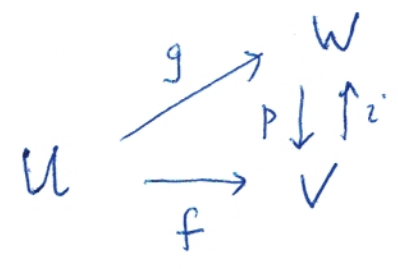
we know that $\tau\psi_i = s\psi_i$ follows. Since the ψ_i

are collectively epimorphic, we conclude that $\tau = s$.

□ Lemma 3

Lemma 4. $EM[\varphi]$ is closed under retract.

Suppose in \mathcal{U} :

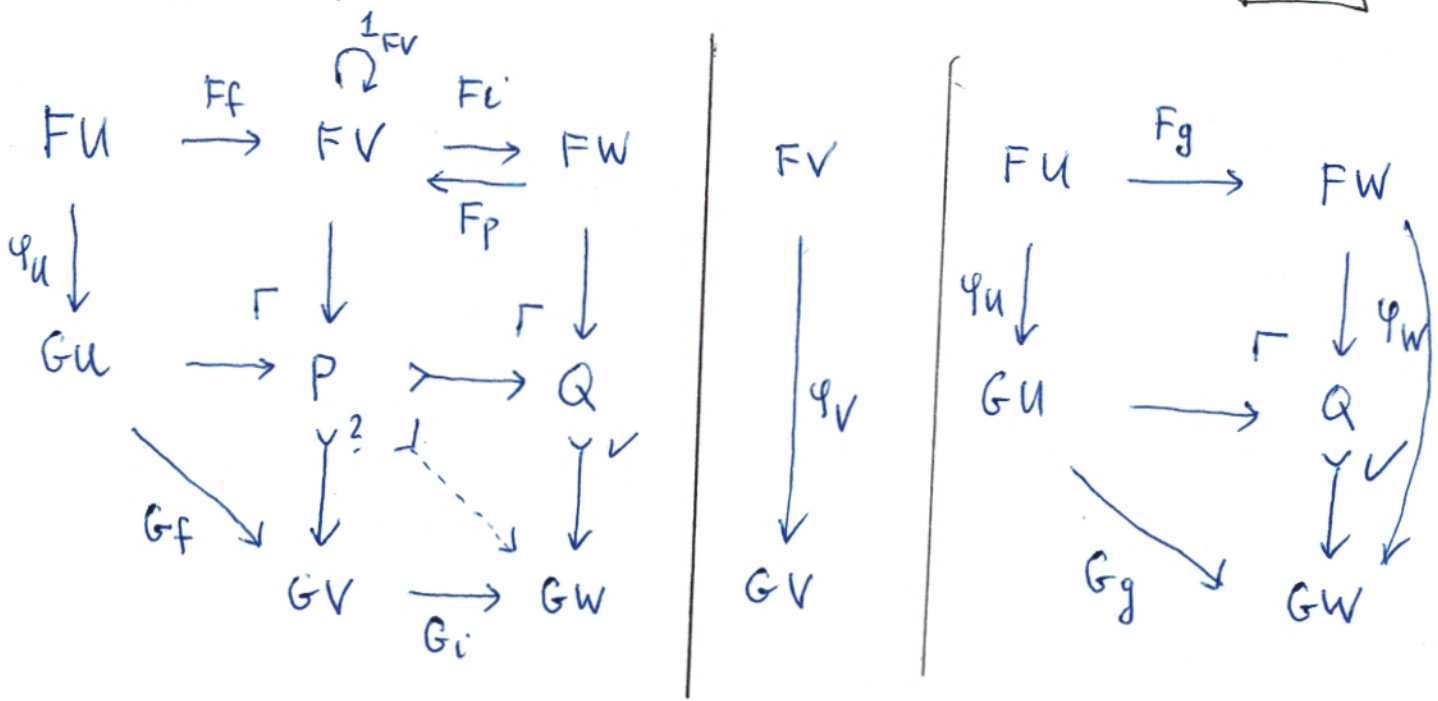


$p^i = 1_V, g = if;$

Suppose: $g \in EM[\varphi]$

Want: ? $f \in EM[\varphi]$

Consider:



F_i is a split mono., $P \rightarrow Q$ is a pushout of F_i ;

$P \twoheadrightarrow Q$ is a mono; $P \rightarrow GW$ is a mono;

therefore, $P \rightarrow GV$ is a mono.

□ Lemma 4.

Proof of Proposition, page [25]:

by Lemmas 1, 3 and 4

□ Proposition, p [25]