Proof of Proposition, page 15:

For part (i): we have the following sequence of bijections ($\mathcal{U}$, D and M are fixed):

$$\mathcal{U} = \operatorname*{colim}_{(K,u)\in[\mathcal{U}]} \hat{K} \xrightarrow{\ \delta\ } D[M]$$

$$\left\langle \hat{K} \xrightarrow[\hat{\delta}_{(K,u)}]{} D[M] \right\rangle_{\text{natural in } (K,u)\in[\mathcal{U}]}$$

$$\left\langle \delta_{(K,u)} \in D[M](K) \right\rangle_{\text{natural in } (K,u)}$$

$$\left\langle \delta_{(K,u)} : D(K) \to M \right\rangle_{\text{natural in } (K,u)}$$

$$\dot{D}(\mathcal{U}) = \operatorname*{colim}_{(K,u)\in[\mathcal{U}]} D(K) \xrightarrow{\ \delta^{\#}\ } M$$

establishing a bijection $\delta \longleftrightarrow \delta^{\#}$.

Alternatively, one sees directly that, given δ, there is unique $\delta^{\#}$ satisfying the set of commutativities stated; and, with any $\delta^{\#}: \dot{D}(\mathcal{U}) \to M$, the system $\left\langle \delta_K(u) : D(K) \to M \right\rangle_{(K,u)}$ given as $\delta_K(u) \underset{\text{def}}{=} \delta^{\#} \circ \dot{D}(\hat{u})$ provides $\delta : \mathcal{U} \to D[M]$.

In both directions, one uses the appropriate colimits.

[A2]

For part (ii): Let: $(K, u) \in [U]$,

and consider the diagram

$$
\begin{tikzcd}
D(K) \arrow[rrrr, bend left=10, "\dot{D}(\hat{u})"] \arrow[ddrr, "\delta_K(u)"] \arrow[dddd, "\tau_K"'] & & & & \dot{D}(U) \arrow[ddll, "\delta^{\#}"'] \arrow[dddd, "\dot{\tau}_U"] \\
& & 5 & & \\
& 3 & M & 2 & \\
& & 4 & & \\
C(K) \arrow[uurr, "\gamma_K(u)"] \arrow[rrrr, bend right=10, "\dot{C}(\hat{u})"'] & & & & \dot{C}(U) \arrow[uull, "\gamma^{\#}"']
\end{tikzcd}
$$

same as 2 on prev. page

with outside quadrangle

$$
\begin{tikzcd}
D(K) \arrow[r, "\dot{D}(\hat{u})"] \arrow[d, "\tau_K"'] \arrow[dr, phantom, "6"] & \dot{D}(U) \arrow[d, "\dot{\tau}_U"] \\
C(K) \arrow[r, "\dot{C}(\hat{u})"'] & \dot{C}(U)
\end{tikzcd}
$$

A3

6 commutes by the naturality of $\mathring{\imath}$.

4 and 5 commute by the definitions of $\gamma^\#$ and $\delta^\#$ from γ and δ in part (i).

The commutativity of 1 is equivalent to the joint commutativity of all the instances of 3, one for each $(K,u) \in [\mathcal{U}]$, by the standard colimit representation of $\mathcal{U}$ by representables.

Assume 1 commutes. Then 2 commutes, by all the above, if precomposed with each of the colimit coprojections $\dot{D}(\hat{u}) : D(K) \longrightarrow \dot{D}(\mathcal{U})$ $((K,u) \in [\mathcal{U}])$. Since the latter are jointly epimorphic, it follows that 2 commutes.

Next, assume that 2 commutes. Then 3 commutes for each $(K,u) \in [\mathcal{U}]$, and thus 1 commutes

$\square$ Proposition

Proof of $(*)$, p $\boxed{17}$:

Now $\dot{Y}(U) = U$, $Y(K) = \hat{K}$, and, for $(K,u) \in [U]$ $\hat{u} : \hat{K} \to U$, we have $\dot{Y}(\hat{u}) = \hat{u}$.

The "defining" commutativity $\odot$ on p. $\boxed{15}$ is now:

$$\begin{tikzcd}
\hat{K} \arrow[rr, "\hat{u}"] \arrow[dr, "\delta_K^{(u)}"'] & \arrow[d, phantom, "2"] & U \arrow[dl, "\delta^{\#}"] \\
& M &
\end{tikzcd}$$

To prove $(*)$, called also 1 now, we precompose it with the arrow $\hat{K} \xrightarrow{\hat{u}} U$, and show the resulting commutativity. Since the cocone $\langle \hat{K} \xrightarrow{\hat{u}} U \rangle_{(K,u) \in [U]}$ is jointly epimorphic, this will be enough. Said precomposition gives:

$$\begin{tikzcd}
\hat{K} \arrow[r, "\hat{u}"] \arrow[dr, "\delta_K^{(u)}"'] \arrow[r, phantom, shift right=6ex, "2"] & U \arrow[r, "\delta"] \arrow[d, "\delta^{\#}"] \arrow[r, phantom, shift right=6ex, "1?"] & Y[M] \arrow[dl, "y[M]"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{K} \arrow[r, "\hat{u}"] \arrow[dr, "\delta_K^{(u)}"'] & U \arrow[r, "\delta"] \arrow[d, phantom, "3"] & Y[M] \arrow[dl, "y[M]"] \\
& M &
\end{tikzcd}$$

The commutativity of 3 is the equality of two arrows $r, s : \hat{K} \longrightarrow M$.

By Yoneda, $r = s$ follows once we know that

$$r_K(1_K) = s_K(1_K)$$

Checking this for 3 is left as an exercise.

$\square$ for $(*)$, p 17

Proof of the commutativity on p $\boxed{20}$:

Construct the following diagram:

$$\begin{tikzcd}
U \arrow[rr, bend left, "\gamma"] \arrow[dr, "\delta"'] \arrow[dd, bend right, "\dot{\imath}_U"'] \arrow[ddrr, bend right=20, "\delta^\#"'] \arrow[r, phantom, "{[1] = \text{def'n of } \delta}"] & & C[M] \arrow[dl, "\varepsilon[M]"'] \arrow[dd, bend left, "\tilde{\varepsilon}[M]"] \\
\arrow[r, phantom, "{[2]}"] & Y[M] \arrow[dr, "y[M]"] \arrow[r, phantom, "\text{def'n}"] & \\
\dot{C}(U) \arrow[rr, bend right, "\gamma^\#"'] & & M
\end{tikzcd}$$

$(*)$

[1] implies [2] : the case of

① implies ②, page 16, for $D = Y$.

□ comm. of p 20

Proof of Proposition, page 25 :

Take two commutative squares ①, ② and their composite ③ = ① $\square_m$ ② ('m' is the 'middle vertical'): in the category $\mathbb{S}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & E \arrow[d] \\ B \arrow[r] & D \arrow[r] & F \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & E \arrow[d] \\ B \arrow[r] & F \end{tikzcd}$$

(of course, $\overrightarrow{AE} = \overrightarrow{BE} \circ \overrightarrow{AE}$, etc., in the awkward algebraic notation). Construct pushouts as follows:

$$P = C \sqcup_A B, \quad Q = E \sqcup_C P, \quad R = Q \sqcup_P D$$

and fit them in the resulting commutative diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & E \arrow[d] \\
B \arrow[r] \arrow[dr] & P \arrow[r] \arrow[d] & Q \arrow[d] \\
& D \arrow[r] \arrow[dr] & R \arrow[d] \\
& & F
\end{tikzcd}
$$

Since pushouts compose, the squares

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "4"] & E \arrow[d] \\
B \arrow[r] & Q
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[dr, phantom, "5"] & E \arrow[d] \\
D \arrow[r] & R
\end{tikzcd}
$$

are pushouts.

<u>Suppose</u> 1 and 2 are EM squares. Then $P \to D$ and $R \to F$ are mono's. As a result, $Q \to R$ (as a pushout of a mono), and then the composite $Q \to F$ are mono's. Since 4 is a pushout and we have the commutative diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] & & E \arrow[dd] \arrow[dl] \\
& Q \arrow[dr] & \\
B \arrow[ur] \arrow[rr] & & F
\end{tikzcd}$$

we have shown that the "composite" ③ is EM;

① and ② are EM $\Rightarrow$ ① $\square$ ② is EM.

Now suppose that ① is, not only EM, but itself a pushout with $A \to B$ a mono; briefly, ① is PM. Then in the above composite diagram, $P \to D$, and therefore $Q \to R$ also, is an isomorphism, and

$$Q \to F \text{ is a mono iff } R \to F \text{ is a mono.}$$

Therefore

$$① \text{ is PM and } ① \underset{CD}{\square} ② \text{ is EM} \Rightarrow ② \text{ is EM.}$$

Now turn to the data $\mathbb{U} \overset{F}{\underset{G}{\underset{\downarrow\varphi}{\rightrightarrows}}} \mathbb{S}$.

For $f = \mathrm{id}_U : U \to U$, "$\varphi$ is EM at f" means just that $\varphi_U : FU \to GU$ is a mono. For U the initial object of $\mathbb{U}$, this is true (F and G are coconuntinuous).

Lemma 1 If φ is EM at $f \in \mathrm{Arr}(\mathbb{U})$, and g is a pushout of f, then φ is EM at g as well.

Reason:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "h"'] & W \arrow[d, "k"] \\ V \arrow[r, "g"'] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad \text{in } \mathbb{U} \text{ fed into } \quad \begin{tikzcd} \mathbb{U} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] & \mathbb{B} \arrow[from=F, to=G, Rightarrow, "\varphi"] \end{tikzcd}$$

generates: the diagrams

$$\begin{tikzcd}
GU \arrow[rrr, "Gf"] \arrow[ddd, "Gh"'] & & & GW \arrow[ddd, "Gk"] \\
& FU \arrow[ul, "\varphi_U"] \arrow[r, "Ff"] \arrow[d, "Fh"'] \arrow[dl, phantom, "3"] & FW \arrow[ur, "\varphi_V"'] \arrow[d, "Fk"] \arrow[u, phantom, "1"] \arrow[dr, phantom, "4"] & \\
& FV \arrow[dl, "\varphi_V"] \arrow[r, "Fg"'] \arrow[ur, phantom, "5"] & FX \arrow[dr, "\varphi_X"] \arrow[dl, phantom, "2"] & \\
GV \arrow[rrr, "Gg"'] & & & GX
\end{tikzcd}$$

and

$$\begin{tikzcd} GU \arrow[r, "Gf"] \arrow[d, "Gh"'] \arrow[dr, phantom, "6"] & GW \arrow[d, "Gk"] \\ GV \arrow[r, "Gg"'] & GX \end{tikzcd}$$

The quadrangles 1, 2, 3, 4 commute by naturality. 5 and 6 are pushouts as images of the coconinuous functors F and G, respectively, of the pushout square given in $\mathbb{M}$. The only essential hypothesis is that 1 is EM. Note that 6 is PM, not just PO, since, by 1 being EM, Gf is a mono. Composing squares:

$$\underset{PM}{\boxed{5}} \underset{Fg}{\square} \underset{?}{\boxed{2}} = \underset{EM}{\boxed{1}} \underset{Gf}{\square} \underset{PM}{\boxed{6}} = \begin{tikzcd} FU \arrow[r] \arrow[d] \arrow[dr, phantom, "7"] & FW \arrow[d] \\ GV \arrow[r] & GX \end{tikzcd}$$

Since $\boxed{7}$ is $\boxed{EM} \square \boxed{PM}$, $\boxed{7}$ is EM;

$\underset{PM}{\boxed{5}} \square \boxed{2} = \boxed{7}$ is EM, hence $\boxed{2}$ is EM

$\square$ Lemma]

Lemma 2 For $U \xrightarrow{f} V \xrightarrow{g} W$ in $\mathbb{U}$:

φ is EM at f and g $\Rightarrow$ φ is EM at gf.

This follows directly by "$\boxed{EM} \circ \boxed{EM}$ is $\boxed{EM}$".

I will now recall transfinite compositions / composites.. We work in an fixed (arbitrary) cocomplete category $\mathbb{S}$ (that could be either $\mathbb{U}$ or $\mathbb{S}$ of the starting context).

$\alpha, \beta, \gamma, \delta$ denote ordinals; $[\alpha]$ denotes the well-ordered set $\{\beta \mid \beta \le \alpha\}$ of ordinals; $[\alpha\rangle = \{\beta \mid \beta < \alpha\}$.

A diagram (functor) $\vec{A} : [\alpha] \longrightarrow \mathbb{S}$, denoted $\langle A_\beta \rangle_{\beta \le \alpha}$ for the objects, and $\langle a_{\gamma\beta} : A_\gamma \to A_\beta \rangle_{\gamma \le \beta \le \alpha}$ for the arrows — is a transfinite composition, TC for short, with (transfinite) composite $a_{0\alpha} : A_0 \to A_\alpha$. if for all limit ordinals β, $0 < \beta \le \alpha$, A_β is the colimit of the restriction

$$\vec{A} \upharpoonright [\beta\rangle : [\beta\rangle \longrightarrow \mathbb{S}$$

with coprojections $a_{\gamma\beta} : A_\gamma \to A_\beta$ $(\gamma < \beta)$.

We think of a TC $\vec{A}$ as a composition of its successor steps $a_{\beta,\beta+1} : A_\beta \to A_{\beta+1}$, $\beta < \alpha$, since all other items in $\vec{A}$ is "given by continuity".

Accordingly, for a class E of arrows of $\mathcal{G}$, to say that E is closed under transfinite compositions/composits is to say that if in a TC $\vec{A}$, each successor step $a_{\beta,\beta+1} : A_\beta \to A_{\beta+1}$ is in E, then so is the composite $a_{0\alpha} : A_0 \to A_\alpha$.

For $\vec{A}$ a TC, every restriction $\vec{A}\restriction[\beta]$, $\beta<\alpha$, is one too.

Now, for any diagram $\vec{A}:[\alpha]\to\mathcal{S}$ and arrow $f_0: A_0\to P_0$, we can define the pushout diagram $\vec{P}=f_0\underset{A_0}{\sqcup}\vec{A}:[\alpha]\to\mathcal{S}$ as follows:

For $0<\beta\leq\alpha$, the object P_β is defined by a pushout

$$\begin{tikzcd} A_0 \arrow[r, "a_{0\beta}"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\ulcorner" near end] & A_\beta \arrow[d, "f_\beta"] \\ P_0 \arrow[r, "p_{0\beta}"'] & P_\beta \end{tikzcd}.$$

For $\gamma\leq\beta\leq\alpha$, the arrow $p_{\gamma\beta}: P_\gamma\to P_\beta$

is defined as the unique arrow $p_{\gamma\beta} : P_\gamma \to P_\beta$

that makes parts ① and ② in

$$
\begin{tikzcd}
A_0 \arrow[r, "a_{0\gamma}"] \arrow[d, "f_0"'] & A_\gamma \arrow[r, "a_{\gamma\beta}"] \arrow[d, "f_\gamma"'] \arrow[dr, phantom, "②"] & A_\beta \arrow[d, "f_\beta"] \\
P_0 \arrow[r, "p_{0\gamma}"'] \arrow[rr, bend right=50, "p_{0\beta}"'] & P_\gamma \arrow[r, "p_{\gamma\beta}"'] \arrow[d, phantom, "①"] & P_\beta \\
 & {} &
\end{tikzcd}
$$

commute (using that P_γ is given as a pushout); it then follows that ② is a pushout square

The f_β, $\beta \le \alpha$, form a morphism (natural transformation) $\vec{f} : \vec{A} \to \vec{P}$. It is easy to see that if $\vec{A}$ is a TC, then so is the pushout $f_0 \underset{A_0}{\sqcup} \vec{A}$.

Still in the general case, $\vec{f} : \vec{A} \longrightarrow \vec{P} = f_0 \underset{A_0}{\sqcup} \vec{A}$ has the natural universal property: given arrows $A_0 \xrightarrow{f_0} P_0 \xrightarrow{g_0} C_0$,

diagram $\vec{B} : [\alpha] \longrightarrow \mathbb{S}$

and a morphism $\vec{h} : \vec{A} \longrightarrow \vec{B}$ such that

$h_0 : A_0 \longrightarrow B_0$ equals $h_0 = g_0 \circ f_0$

there is a unique $\vec{g} : \vec{P} \to \vec{C}$ starting with the given $g_0 : P_0 \to C_0$ such that $\vec{h} = \vec{g} \circ \vec{f}$.

I will apply the above in the context of the cocontinuous functors $F, G : \mathbb{U} \to \mathbb{S}$ and the natural transformation $\varphi : F \to G$.

Suppose that $\vec{X} : [\alpha] \longrightarrow \mathbb{U}$ is a TC in $\mathbb{U}$. Then $\vec{A} \underset{def}{=} F \circ \vec{X}$, $\vec{B} \underset{def}{=} G \circ \vec{X}$, both diagrams $[\alpha] \longrightarrow \mathbb{S}$, are also TC's, in $\mathbb{S}$; $\vec{h} \underset{def}{=} \varphi \circ \vec{X} : F \circ \vec{X} \to G \circ \vec{X}$ is a morphism between them. Take now the start of $\vec{X}$, the object X_0, and the induced items

$$\varphi_{X_0} : FX_0 \longrightarrow GX_0$$

written as $f_0 : A_0 \to P_0$, in $\mathbb{S}$, and consider the pushout

$$\vec{P} \underset{def}{=} f_0 \bigsqcup_{A_0} \vec{A}$$

with the induced factorization

$$\begin{tikzcd} \vec{A} \arrow[r, "\vec{f}"] \arrow[rr, bend right, "\vec{h}"'] & \vec{P} \arrow[r, "\vec{g}"] & \vec{B} \end{tikzcd}$$

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[rr, bend right, "h_0 = f_0"'] & P_0 \arrow[r, equal, "g_0 = 1_{P_0}"'] & C_0 \end{tikzcd}$$

These constructions and the notation will be used in proving the following lemma. We still have $F, G : \mathfrak{U} \to \mathbb{S}$ and $\varphi : F \to G$ as before.

Lemma 3 Assume, in addition to the previous assumptions, that $\mathcal{B}$ is a locally finitely presentable category ($\mathbf{Set}^{L}$ is certainly one such). Let $\vec{\mathcal{U}} = \langle U_\beta, u_{\gamma\beta}: U_\gamma \to U_\beta \rangle_{\gamma \le \beta \le \alpha}$ be a TC in $\mathcal{U}$. Suppose that $\varphi_{U_0}: FU_0 \to GU_0$ is a mono, and that φ is EM at $u_{\gamma,\gamma+1}: U_\gamma \to U_{\gamma+1}$ for all $\gamma < \alpha$. Then φ is EM at $u_{0\alpha}: U_0 \to U_\alpha$. In particular, $\varphi_{U_\alpha}: FU_\alpha \to GU_\alpha$ is a monomorphism.

Proof. The proof is by induction on the ordinal α.

For $\alpha = 0$, the assumption on φ_{U_0} being mono ensures that the assertion is true. When $\alpha = \beta+1$, then $u_{0\alpha} = u_{\beta,\beta+1} \circ u_{0\beta}$; $u_{0\beta} \in EM(\varphi)$ by the induction hypothesis, $u_{\beta,\beta+1} \in EM[\varphi]$ by assumption, so by Lemma 2, $u_{0\alpha} \in EM[\varphi]$.
Assume now that α is a limit ordinal. I use the notation introduced above. The inductive assumption is that, for all $\beta < \alpha$, in the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "a_{0\beta}"] \arrow[d, "f_0"'] & A_\beta \arrow[d, "f_\beta"] \\
P_0 \arrow[r, "p_{0\beta}"] \arrow[d, equal] & P_\beta \arrow[d, "g_\beta"] \\
B_0 \arrow[r, "b_{0\beta}"'] & B_\beta
\end{tikzcd}$$

g_β is a monomorphism. Since F and G preserve colimits, $\vec{P} = f_0 \sqcup_{A_0} \vec{A}$ is a TC, and P_α is the <u>directed</u> colimit of the diagram

$\langle P_\beta, p_{\gamma\beta} : P_\gamma \to P_\beta \rangle_{\gamma \leq \beta < \alpha}$, with coprojections

$p_{\beta\alpha} : P_\beta \to P_\alpha$ $(\beta < \alpha)$; similarly for the B's.

The diagram

$$\begin{tikzcd}
P_\beta \arrow[r, "p_{\beta\alpha}"] \arrow[d, "g_\beta"'] & P_\alpha \arrow[d, "g_\alpha"] \\
B_\beta = GU_\beta \arrow[r, "b_{\beta\alpha} = Gu_{\beta\alpha}"'] & B_\alpha = GU_\alpha
\end{tikzcd}$$

commutes for all $\beta < \alpha$, and g_β is a mono for all $\beta < \alpha$. It follows that g_α is a mono.

Indeed, let, first, X be a finitely presentable (fp) object, and let the arrows $r, s : X \to P_\alpha$ be such that $g_\alpha r = g_\alpha s : X \to B_\alpha$. Since the colimit for α is directed, and X is fp, there is $\beta < \alpha$ and there are $r', s' : X \to P_\beta$ such that $r = p_{\beta\alpha} r'$, $s = p_{\beta\alpha} s'$:

$$\begin{tikzcd}
X \arrow[d, shift right, "r'"'] \arrow[d, shift left, "s'"] \arrow[dr, shift right, "r"'] \arrow[dr, shift left, "s"] & \\
P_\beta \arrow[r, "p_{\beta\alpha}"'] \arrow[d, "g_\beta"'] & P_\alpha \arrow[d, "g_\alpha"] \\
B_\beta \arrow[r, "b_{\beta\alpha}"'] & B_\alpha
\end{tikzcd}$$

We have $b_{\beta\alpha}(g_\beta r') = b_{\beta\alpha}(g_\beta s') : X \to B_\alpha$.

Since the colimit that gives B_α is directed, and X is fp, it follows that there is β', $\beta \leq \beta' < \alpha$, such that

$$b_{\beta\beta'}(g_\beta r') = b_{\beta\beta'}(g_\beta s')$$

i.e.

$$g_{\beta'}(p_{\beta\beta'} r') = g_{\beta'}(p_{\beta\beta'} s') :$$

$$\begin{tikzcd}
X \arrow[d, shift right, "r'"'] \arrow[d, shift left, "s'"] \arrow[dr, shift right] \arrow[dr, shift left] & \\
P_\beta \arrow[r, "p_{\beta\beta'}"'] \arrow[d, "g_\beta"'] & P_{\beta'} \arrow[d, "g_{\beta'}"] \\
B_\beta \arrow[r, "b_{\beta\beta'}"'] & B_{\beta'}
\end{tikzcd}$$

Since $g_{\beta'}$ is a mono, it follows that $r' = s'$, hence $r = s$ as desired.

Now, let X be an arbitrary object in $\mathbb{S}$, and suppose $r, s : X \to P_\alpha$, $g_\alpha r = g_\alpha s : X \to B_\alpha$

$\mathfrak{B}$ is lfp; X can be represented as a (directed) colimit: $X = \operatorname{colim}_{i \in I} X_i$, of fp objects X_i, with coprojections $\varphi_i : X_i \to X$.

We have $g_\alpha(r\varphi_i) = g_\alpha(s\varphi_i) : X_i \to B_\alpha$. Since X_i is fp, we know that $r\varphi_i = s\varphi_i$ follows. Since the φ_i are collectively epimorphic, we conclude that $r = s$.

$\square$ Lemma 3

Lemma 4. $EM[\varphi]$ is closed under retract.

Suppose in $\mathbb{U}$:

$$\begin{tikzcd} & W \arrow[d, shift right, "p"'] \\ U \arrow[ur, "g"] \arrow[r, "f"'] & V \arrow[u, shift right, "i"'] \end{tikzcd} \qquad pi = 1_V, \; g = if;$$

Suppose: $g \in EM[\varphi]$

Want: ? $f \in EM[\varphi]$

Consider:

$$
\begin{tikzcd}
FU \arrow[r, "Ff"] \arrow[d, "\varphi_U"'] & FV \arrow[loop above, "1_{FV}"] \arrow[r, shift left, "Fi"] \arrow[d] & FW \arrow[l, shift left, "Fp"] \arrow[d] \\
GU \arrow[r] \arrow[dr, "Gf"'] & P \arrow[r, tail] \arrow[d, tail, "?"] \arrow[dr, dashed, tail] & Q \arrow[d, tail, "\checkmark"] \\
 & GV \arrow[r, "Gi"'] & GW
\end{tikzcd}
\qquad
\begin{tikzcd}
FV \arrow[dd, "\varphi_V"] \\
\\
GV
\end{tikzcd}
\qquad
\begin{tikzcd}
FU \arrow[r, "Fg"] \arrow[d, "\varphi_U"'] & FW \arrow[d, "\varphi_W"'] \arrow[dd, bend left] \\
GU \arrow[r] \arrow[dr, "Gg"'] & Q \arrow[d, tail, "\checkmark"] \\
 & GW
\end{tikzcd}
$$

Fi is a split mono. $P \to Q$ is a pushout of Fi;

$P \rightarrowtail Q$ is a mono; $P \to GW$ is a mono;

therefore, $P \to GV$ is a mono.

$\square$ Lemma 4.

Proof of Proposition, page $\boxed{25}$:

by Lemmas 1, 3 and 4

$\square$ Proposition, p $\boxed{25}$