

Proof of Proposition, p. 26

(Many readers will find this superfluous since the proof is the same as that for the corresponding fact for simplicial sets.)

For any functor $V: L \rightarrow \text{Set}$, let us write

$|V|$ for the set $\{(K, v) \mid K \in L, v \in V(K)\}$

(the set of objects of the category $[V]$ we used earlier). Call a subset X of $|V|$ closed

if for all arrows $p: K \rightarrow K_p$ in L ,

$(K, v) \in X$ implies $(K_p, V(p)(v)) \in X$.

The mapping

$$\text{Sub}(V) \longrightarrow \mathcal{P}(|V|)$$

$$U \longmapsto |U|$$

of subfunctors U of V to subsets of $|V|$ is a bijection onto the closed subsets of $|V|$; the mapping is a complete-lattice embedding of

$\text{Sub}(V)$ into $\mathcal{P}(|V|)$; in particular,

it preserves arbitrary unions. It will be remembered that directed unions of subfunctors are colimits.

Furthermore: if X and Z are closed subsets of $|V|$, and Z is a single-element extension of X : $Z = X \dot{\cup} \{z\}$, then we have a pushout diagram: in Set^L :

$$\begin{array}{ccc} U & \xrightarrow{\text{Incl.}} & W \\ \uparrow & & \uparrow \hat{w} \\ \overset{\circ}{K} & \xrightarrow{i_K} & \hat{K} \end{array}$$

where U and W are the subfunctors of V corresponding to X and Z , resp. ($X = |U|$, $Z = |W|$), $z = (K, w)$ ($w \in W(K) \subseteq V(K)$) and \hat{w} , as usual, given by $\hat{w}_K(1_K) = w$. The reason is that the composite $\hat{w} \circ i_K: \overset{\circ}{K} \rightarrow W$ factors through $U \xrightarrow{\text{incl}} W$, resulting in the above

diagram of functors and natural transformations, which, when evaluated at any object $K' \in L$, becomes a pushout diagram in Set : for $K' \neq K$, we get

$$\begin{array}{ccc}
 U(K') & = & W(K') \\
 \uparrow & & \uparrow \\
 L(K, K') & = & L(K, K')
 \end{array}$$

and for $K' = K$:

$$\begin{array}{ccccc}
 U(K) & \xrightarrow{\text{incl.}} & U(K) \cup \{w\} & & w \\
 \uparrow & & \uparrow \hat{w}_K & & \uparrow \\
 \emptyset & \longrightarrow & \{1_K\} & & 1_K
 \end{array}$$

Define the binary relation \prec on the set $|V|$ as follows:

$$(K', w) \prec (K, v)$$

$$\stackrel{\text{def}}{\iff}$$

There is a proper (non-identity) arrow $p: K \rightarrow K'$ such that $w = V(p)(v)$

Note the obvious fact that a subset X of $|V|$ is closed in the sense defined above if and only if it is downward closed under \prec :

for $x, y \in |V|$,

$$x \in X \text{ \& } y \prec x \implies y \in X.$$

The relation \prec is well-founded: with any \prec -descending chain

$$(K, v) = (K_0, v_0) \succ (K_1, v_1) \succ \dots$$

We have at least one corresponding sequence of proper arrows

$$K = K_0 \xrightarrow[\neq 1]{p_1} K_1 \xrightarrow[\neq 1]{p_2} \dots$$

and, as part of the definition of L being a FOLDS signature, any such sequence has a maximal finite length that we called the dimension of K .

Now, let $f: U \rightarrow V$ be any monomorphism in Set^L . Let

$$X_0 = \{ (K, f_K(u)) \mid (K, u) \in |U| \}$$

X_0 equals $|\bar{u}|$ for the subfunctor \bar{u} of V that is the image of U under f .

It is a general fact that every well-founded relation can be extended to a (total) well-ordering.

Therefore, there is a one-to-one ordinal indexing

$\langle x_\beta \rangle_{\beta < \alpha}$ of the set $|V| - X_0$:

$$|V| - X_0 = \{ x_\beta \mid \beta < \alpha \}$$

and $\gamma \neq \beta \Rightarrow x_\gamma \neq x_\beta$.

such that $x_\gamma < x_\beta$ implies $\gamma < \beta$ ($\gamma, \beta < \alpha$)

Define the sets X_β for $\beta \leq \alpha$
as follows:

$$X_0 = X_0 \text{ as before;}$$

for limit ordinal $\beta \leq \alpha$:

$$X_\beta = \bigcup_{\gamma < \beta} X_\gamma;$$

for $\beta = \gamma + 1 \leq \alpha$:

$$X_{\gamma+1} = X_\gamma \cup \{x_\gamma\}$$

More simply: $X_\beta = X_0 \cup \{x_\gamma \mid \gamma < \beta\}$

The sets X_β are closed:

Suppose $y < x$ and $x \in X_\beta$: then

either $x \in X_0$ or $x = x_\gamma$ for some $\gamma < \beta$;

in the first case, $y \in X_0 \subseteq X_\beta$, since X_0 is closed

in the second case: either $y \in X_0$, or $y = x_\delta$

for some $\delta < \alpha$; in the first sub-case $y \in X_\beta$;

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in the second sub-case, we have

$$x_\delta < x_\gamma, \text{ hence } \delta < \gamma < \beta,$$

and so $y = x_\delta \in X_\beta$ as desired.

$$\text{Also: } X_\alpha = |V|.$$

We have the transfinite diagram

$$\vec{U} = \langle U_\beta; u_{\gamma\beta} : U_\gamma \rightarrow U_\beta \rangle_{\gamma \leq \beta \leq \alpha}$$

where $U_0 = U$;

for $0 < \beta \leq \alpha$: U_β is the subfunctor of V
corresponding to the closed set X_β : $X_\beta = |U_\beta|$;

in particular, $U_\alpha = V$;

for $0 < \gamma \leq \beta \leq \alpha$: $u_{\gamma\beta} : U_\gamma \rightarrow U_\beta$ is the
inclusion (corresponding to the inclusion $X_\gamma \subseteq X_\beta$)

$u_{0\beta} : U_0 \rightarrow U_\beta$ ($0 < \beta \leq \alpha$) is the composite

$$U_0 = U \xrightarrow{\hat{g}} \vec{U} \rightarrow U_\beta$$

where g is the isomorphism of U onto

its f -image \bar{U} , and the inclusion A 29

$\bar{U} \rightarrow U_\beta$ corresponds to the

inclusion $X_0 \subseteq X_\beta$.

By what we saw above, \vec{U} is a

transfinite composition in which each

successor step $u_{\gamma, \gamma+1} : U_\gamma \rightarrow U_{\gamma+1}$ is

a pushout of a sphere inclusion $K^0 \xrightarrow{i_K} K^1$;

and $u_{0\alpha} : U_0 \rightarrow U_\alpha$ equals the

given monomorphism $f : U \rightarrow V$.

This proves that $f : U \rightarrow V$ belongs

to the saturation of the set $\{K \xrightarrow{i_K} \hat{K} \mid K \in L\}$.

□ Proposition, p 26.

Explanations to page 151, for the
 bijection $\Sigma \uparrow \Phi$:

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The fact that the bijection $\Sigma \uparrow \Phi$ is well-defined
 is to say that: if we have

$$U \xrightarrow{\gamma} H[M] \quad \text{and} \quad \dot{H}(u) \xrightarrow{\sigma} M$$

and the given bijection Σ relates the two: $\gamma \mapsto \sigma$
then the following conditions (1) and (2) are
 equivalent:

(1) γ factors through $(H \uparrow \Phi)[M] \xrightarrow{\text{incl}} H[M]$

(2) σ belongs to the subset $(\dot{H} \uparrow \Phi)[M](u)$ of $\dot{H}[M](u)$.

Condition (1) on any $U \xrightarrow{\gamma} H[M]$ is the same as:

for all $K \in L$, $\gamma_K : U(K) \rightarrow H[M](K) (= \text{Set}^L(H(K), M))$

factors through the subset $(H \uparrow \Phi)[M](K)$ of $H[M](K)$;

hence, the same as

for all $K \in L$ and for all $u \in U(K)$

$$\gamma_K(u) \in (H \uparrow \Phi)[M](K).$$

On the other hand, (2) is to say that, for all $(K, u) \in [U]$

the composite $\sigma \circ \dot{H}(\hat{u}) : H(K) \rightarrow M$ belongs to $(H \uparrow \Phi)[M](K)$

$$\sigma \circ \dot{H}(\hat{u}) \in (H \uparrow \Phi)[M](K).$$

But, if $\Sigma : \gamma \mapsto \sigma$, then

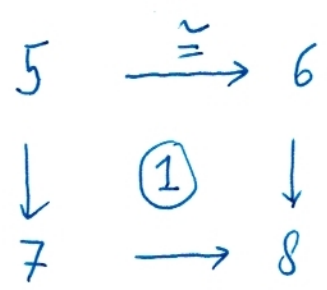
$$\gamma_K(u) = \sigma \circ H(\hat{u})$$

as shown on p. [15] as the relation \odot (now, $D = H, \delta = \gamma, \delta^\# = \sigma$). □

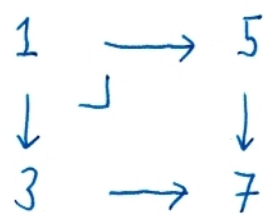
Explanations to page [53], for the bijection $(\Sigma \Gamma \Phi) [M]$:

This is purely diagrammatic, based on three facts:

One: the diagram on p. [51]; we abbreviate it as

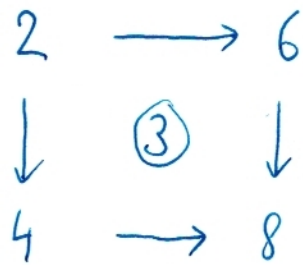


Two: the pullback square in Diagram 1, p. [46], with the horizontal turned vertical and vice versa and with the prefix $\text{Set}^L(u, \dots)$ prefixed - it remains a pullback:

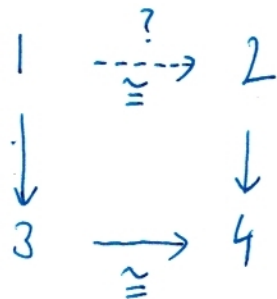


(the verticals $\begin{array}{c} 5 \\ \downarrow \\ 7 \end{array}$ in the two are the same).

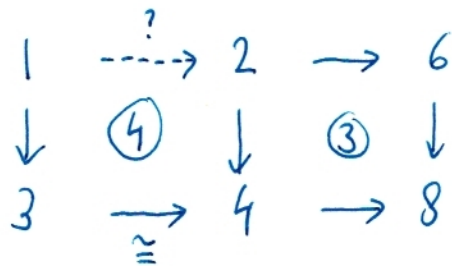
Three: similar to Two, with Diagram 2 in place of Diagram 1:



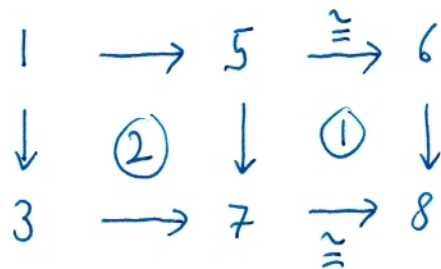
The square we seek, on page 51, is abbreviated as:



The four fit together in a cube - spread out as:



=



The "composite" $(2) \square (1) = \begin{array}{ccc} 1 & \rightarrow & 6 \\ \downarrow & & \downarrow \\ 3 & \rightarrow & 8 \end{array}$ is a pullback as a "composite" of two pullbacks; there is an arrow $1 \dashrightarrow 2$ that makes (4) commute by using the pullback (3); $(4) \square (3) = (2) \square (1)$ is a pullback; it follows that (4) is a pullback; the pullback of the isomorphism $3 \xrightarrow{\cong} 4$ is an isomorphism: $1 \xrightarrow{\cong} 2$ is an isomorphism. \square

Explanation on invariance, bottom page [53]:

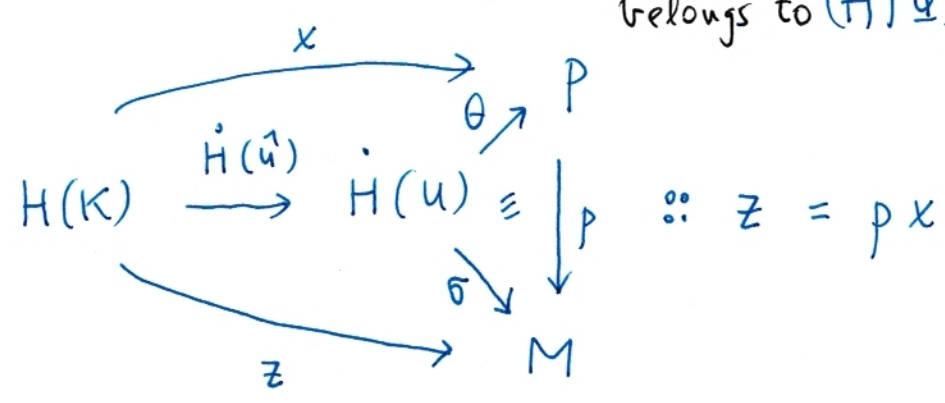
The LHS of the bi-implication is equivalent to saying:

for all $(K, u) \in [U]$, $x \stackrel{\text{def}}{=} \theta \circ \dot{H}(\vec{u})$ belongs to $(H \Gamma \Phi)[P](K)$

and the RHS:

for all $(K, u) \in [U]$, $z \stackrel{\text{def}}{=} \sigma \circ \dot{H}(\vec{u})$ belongs to $(H \Gamma \Phi)[M](K)$

We have:



Also:

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$$x \in (H \upharpoonright \Phi)[P](K) \Leftrightarrow (P, x) \in \overline{\Phi}_K$$

$$z \in (H \upharpoonright \Phi)[M](K) \Leftrightarrow (M, z) \in \overline{\Phi}_K$$

and the invariance property is that

$$(P, x) \in \overline{\Phi}_K \text{ iff } (M, z) \in \overline{\Phi}_K$$

provided p is FS and $z = px$

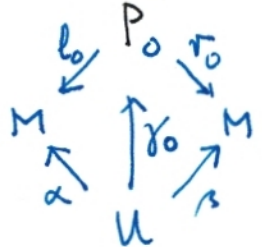
as it is the case now.

□

Explanations for the Introduction

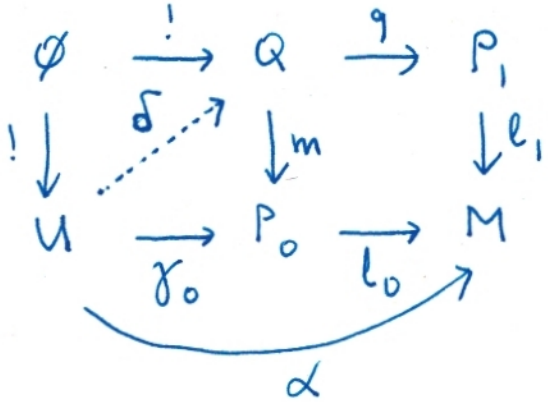
① " If \underline{P}_1 extends \underline{P}_0 , then $E_q[\underline{P}_0, U] \subseteq E_q[\underline{P}_1, U]$

Suppose



witnessing $\alpha \underset{\underline{P}_0}{\sim} \beta$ With the

diagonal $\delta: U \rightarrow Q$ given by m being FS, we have



and similarly with τ 's in place of l 's,

$\gamma_1 \stackrel{\text{def}}{=} q\delta: U \rightarrow P_1, \alpha = l_1 \gamma_1, \beta = \tau_1 \gamma_1$

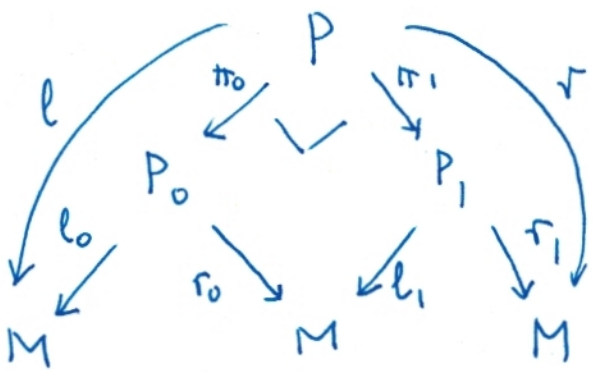
witnessing $\alpha \underset{\underline{P}_1}{\sim} \beta$

① " \underline{P}_0 and \underline{P}_1 can both be extended to $\underline{P}_0 \otimes \underline{P}_1$ "

First of all, note that the composite of two

FS maps is FS, and a pullback of an FS map is FS. These are cases of the general elementary facts that the class of all maps that have the right lifting property relative to a fixed map has the said closure properties.

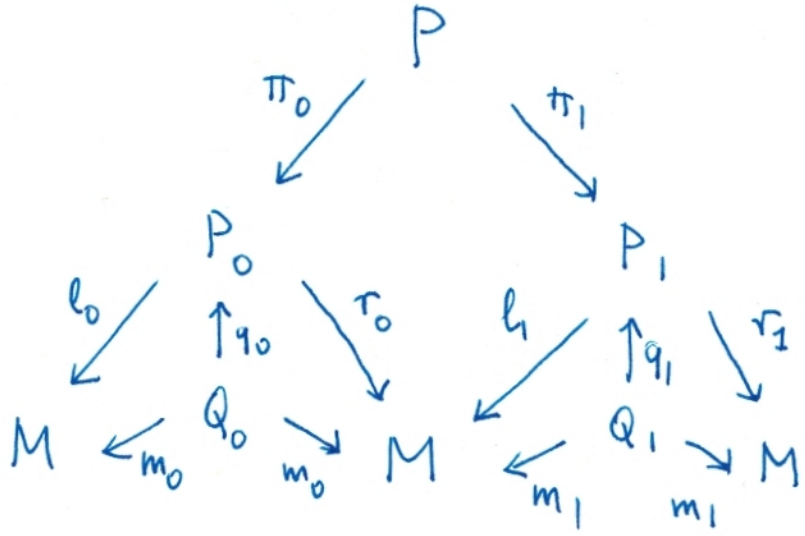
Assume the RSE's on M , \underline{P}_0 and \underline{P}_1 , with witnesses (Q_0, q_0, m_0) , (Q_1, q_1, m_1) for the reflexivity of \underline{P}_0 , \underline{P}_1 , respectively. We construct $\underline{P} = (P, \ell, \tau)$ by the pullback shown (span-composition):



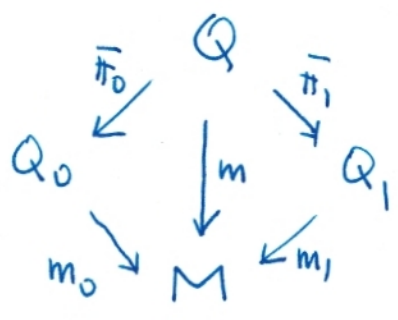
$$\ell \stackrel{\text{def}}{=} \ell_0 \pi_0$$

$$\tau \stackrel{\text{def}}{=} \tau_1 \pi_1$$

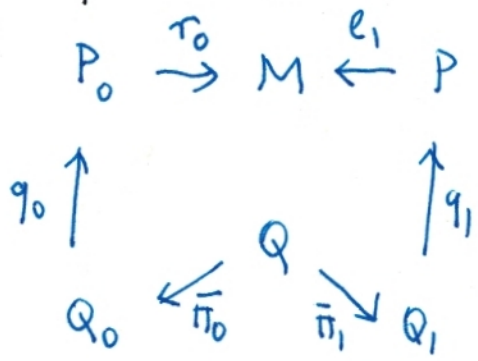
By what we said above, we do have that ℓ and τ are FS. Let's introduce the reflexivity witnesses into the commutative diagram:



Produce Q and $m: Q \rightarrow M$ as the pullback

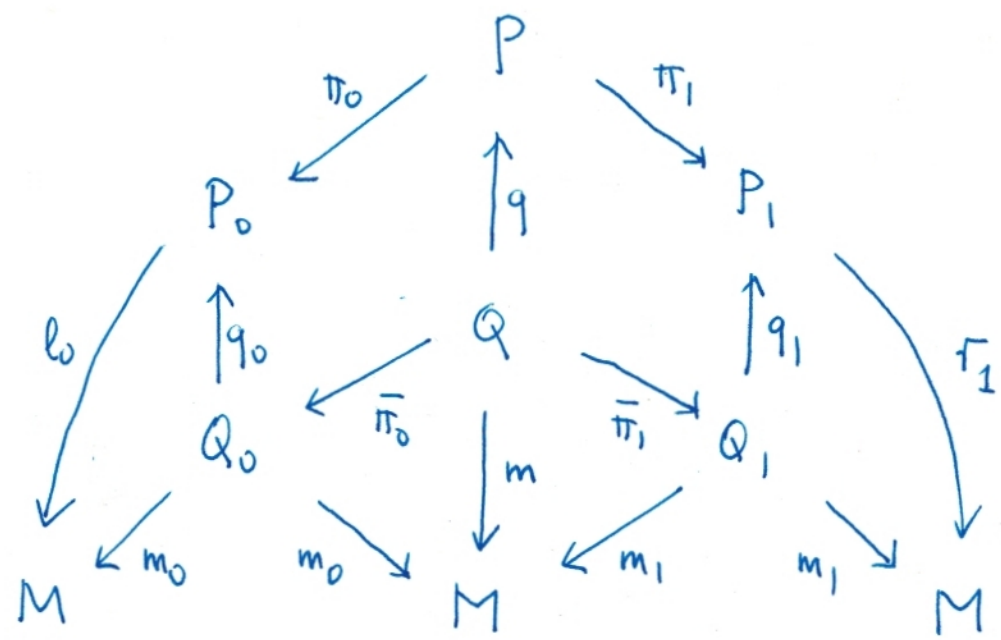


The maps $\bar{\pi}_0, \bar{\pi}_1, m$ are FS. Using that P is a pullback, and that the diagram



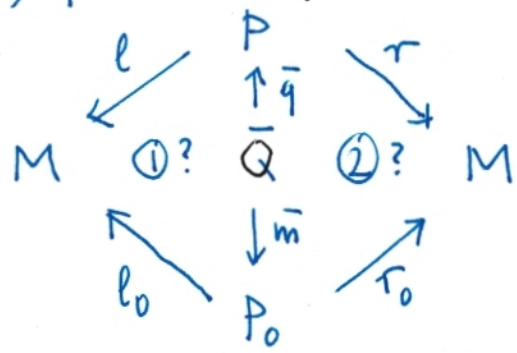
commutes (since $\tau_0 q_0 \bar{\pi}_0 = m_0 \bar{\pi}_0 = m_1 \bar{\pi}_1 = l_1 q_1 \bar{\pi}_1$)

we obtain $q: Q \rightarrow P$ such that the following commutes:



This shows that $\underline{P} = (P, l, r)$ is reflexive, with (Q, q, m) witnessing the fact.

Now, we show that $\underline{P} = \underline{P}_0 \otimes \underline{P}_1$ extends both \underline{P}_0 and \underline{P}_1 . We show that the reflexivity of \underline{P}_1 suffices to have that \underline{P} extends \underline{P}_0 ; symmetrically, we will have that the reflexivity of \underline{P}_0 implies that \underline{P} extends \underline{P}_1 . We will produce \bar{Q} , \bar{q} and \bar{m} , the last FS as in:



\bar{Q} and \bar{q} are defined by the pullback

$$\begin{array}{ccc} \bar{Q} & \xrightarrow{\bar{q}} & P \\ \pi \downarrow & & \downarrow \pi_1 \\ Q_1 & \xrightarrow{q_1} & P_1 \end{array}$$

Put together with the pullback defining P , we obtain a third pullback:

$$\begin{array}{ccccc} \bar{Q} & \xrightarrow{\bar{q}} & P & \xrightarrow{\pi_0} & P_0 & \bar{Q} & \xrightarrow{\pi_0 \bar{q}} & P_0 \\ m \downarrow \lrcorner & & \downarrow \pi_1 \lrcorner & & \downarrow \tau_0 & \pi \downarrow \lrcorner & & \downarrow \tau_0 \\ Q_1 & \xrightarrow{q_1} & P_1 & \xrightarrow{l_1} & M & = & Q_1 & \xrightarrow{l_1 q_1} & M \end{array}$$

The upper horizontal in the last square serves as the definition of $\bar{m} \stackrel{\text{def}}{=} \pi_0 \bar{q}: \bar{Q} \rightarrow P_0$. The lower horizontal is $m_1 = l_1 \pi_1$ - reflexivity of P_1 . As a pullback of an FS map, m_1 is FS. Considering

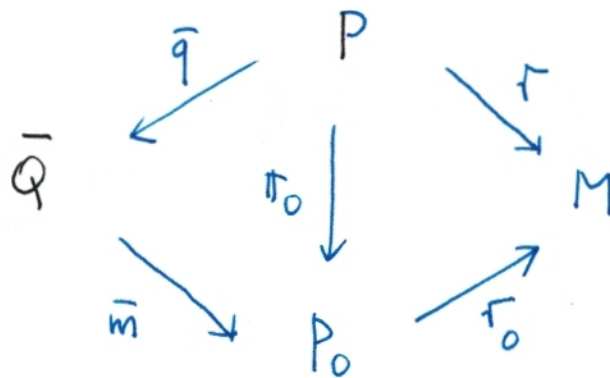
$$\begin{array}{ccc} & P & \\ \ell \swarrow & & \nwarrow \bar{q} \\ M & & \bar{Q} \\ \ell_0 \swarrow & \downarrow \pi_0 & \nwarrow \bar{m} \\ & P_0 & \end{array}$$

(definition of $\ell: P \rightarrow M$,
definition of $\bar{m}: \bar{Q} \rightarrow P_0$)

we see that the desired commutativity ①?

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holds. ②? is similar:



□

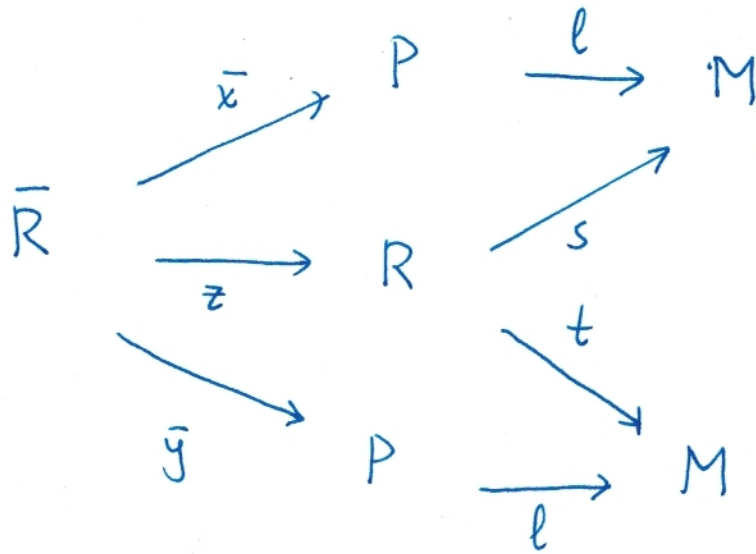
① Invariance of intrinsic equivalence under FOLDS equivalence.

Step 1 Suppose we have $\underline{P} = (P, \ell, r): M \underset{L}{\sim} N$

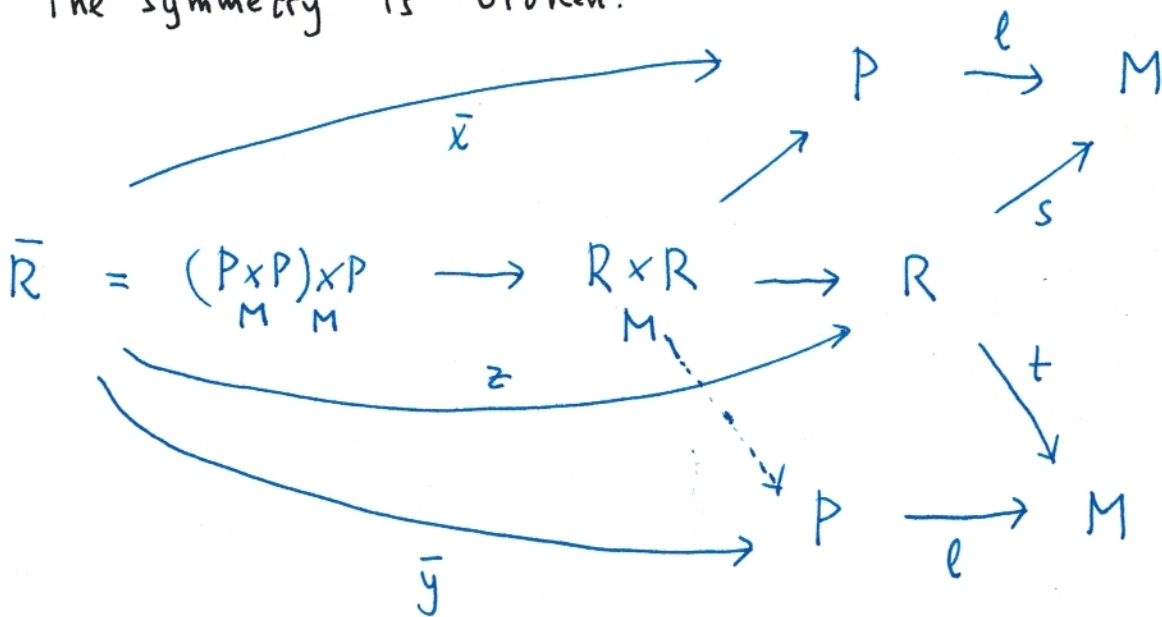
and $\underline{R} = (R, s, t): M \underset{L}{\sim} M$ a self equivalence of M . We construct a self equivalence

$$\underline{\bar{R}} = (\bar{R}, \bar{s}, \bar{t}): N \underset{L}{\sim} N \text{ of } N.$$

\bar{R} is constructed as a limit, a "double pullback", in Set^L , with projections $\bar{x}: \bar{R} \rightarrow P$, $\bar{z}: \bar{R} \rightarrow R$, $\bar{y}: \bar{R} \rightarrow P$, making the following commute:



with $(\bar{R}, \bar{x}, z, \bar{y})$ the terminal such quadruple making the diagram commute. (\bar{R}, \dots) can be construed as a repeated pullback, although in this way the symmetry is broken:

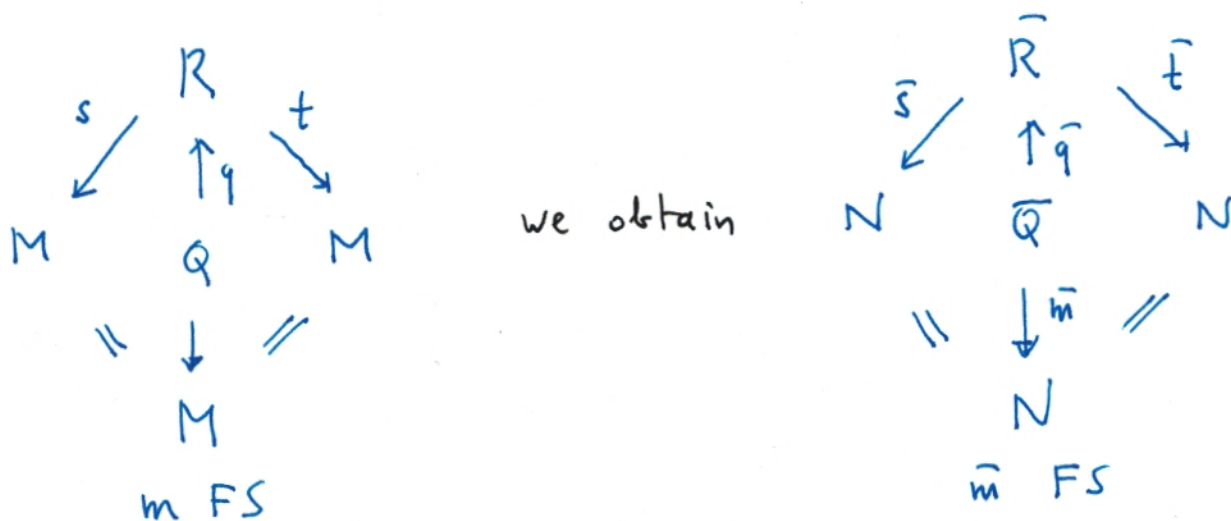


However, this way we see that the maps \bar{x}, z, \bar{y} are all FS. We define $\bar{s}: \bar{R} \rightarrow N, \bar{t}: \bar{R} \rightarrow N$

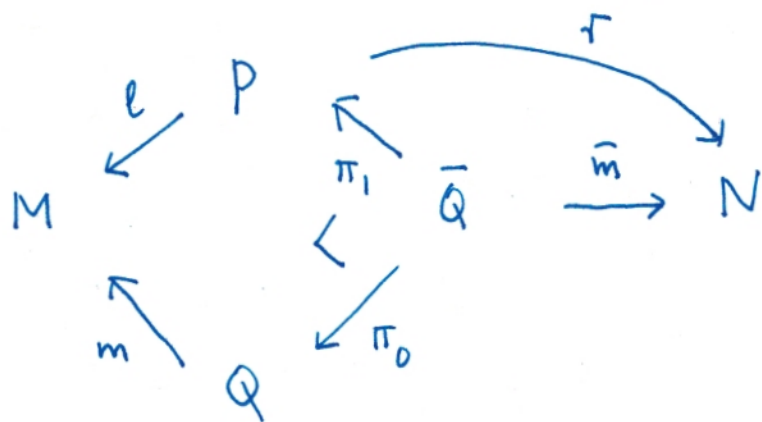
by $\bar{s} = \tau \bar{r}$, $\bar{t} = \tau \bar{y}$; \bar{s} and

\bar{t} are FS maps.

Step 2 We show that if \underline{R} is reflexive, then so is $\underline{\bar{R}}$: from



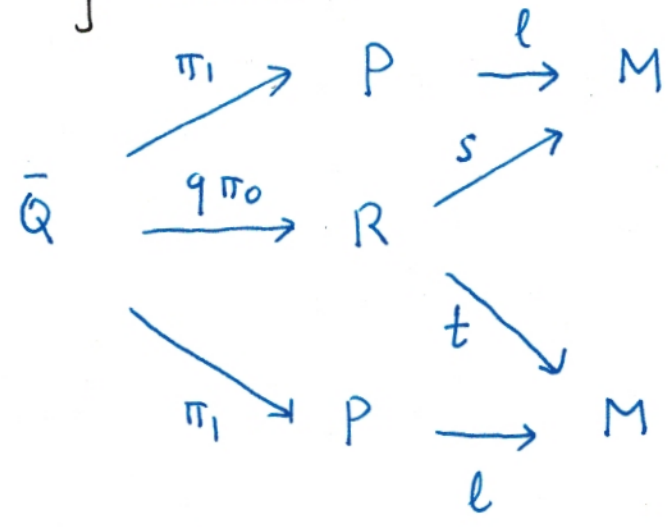
\bar{Q} is defined as the pullback in the diagram



and \bar{m} as the composite $\bar{m} = \tau \pi_1$; \bar{m} is FS.

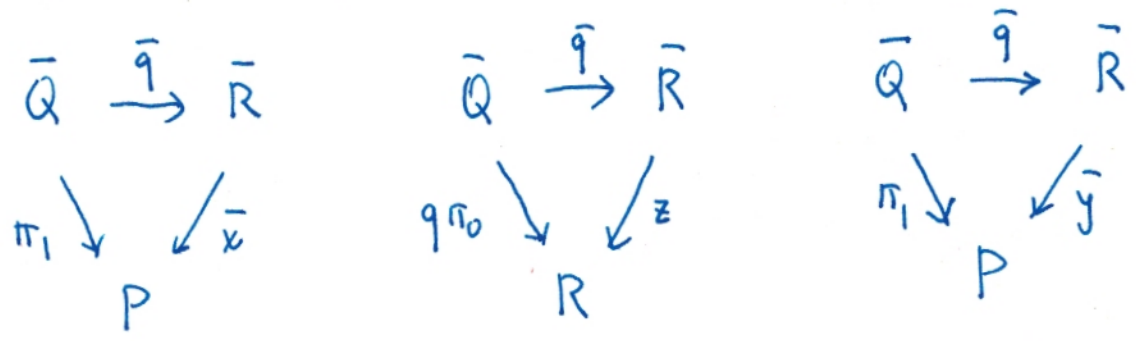
To define $\bar{q}: \bar{Q} \rightarrow \bar{R}$, we use the definition of \bar{R} as a double pull back.

The following commutes:



because: $l\pi_1 = m\pi_0 = sq\pi_0$
 $l\pi_1 = m\pi_0 = tq\pi_0$

Therefore, we have $\bar{Q} \xrightarrow{\bar{q}} \bar{R}$ such that



all commute. Therefore $\bar{s}\bar{q} = r\bar{x}\bar{q} = r\pi_1 = \bar{m}$,
 and similarly $\bar{t}\bar{q} = \bar{m}$

Step 3 Suppose $\alpha, \beta: U \rightarrow M$,

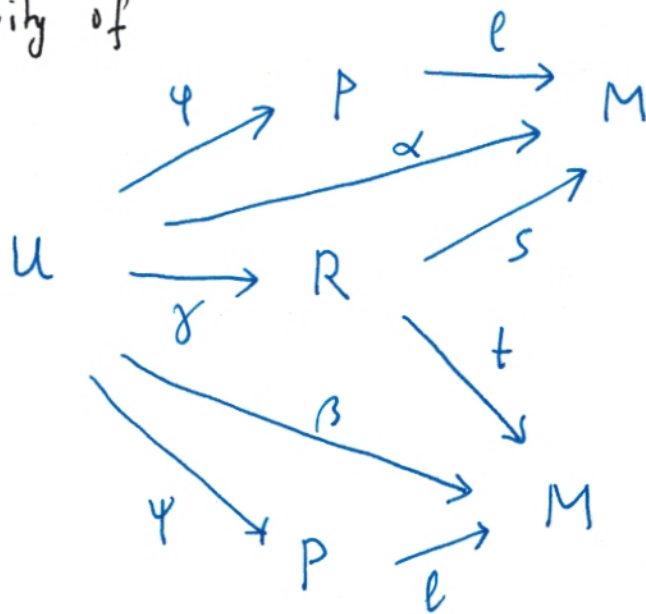
$\bar{\alpha}, \bar{\beta}: U \rightarrow N$; $\varphi: U \rightarrow P$ witnesses $\underline{P}: \alpha \sim \bar{\alpha}$,

$\psi: U \rightarrow P$ witnesses $\underline{P}: \beta \sim \bar{\beta}$.

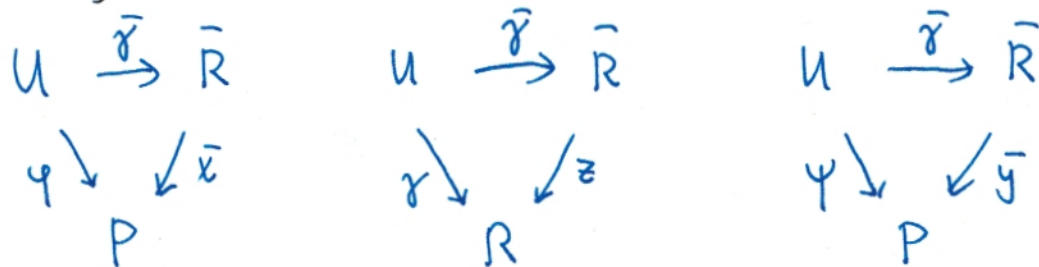
Suppose $\gamma: U \rightarrow R$ witnesses $\underline{R}: \alpha \sim \beta$.

We construct $\bar{\gamma}: U \rightarrow \bar{R}$ witnessing $\underline{\bar{R}}: \bar{\alpha} \sim \bar{\beta}$.

To use the limit definition of \bar{R} , we observe the commutativity of



Therefore, there is $U \xrightarrow{\bar{\gamma}} \bar{R}$ such that



all commute. But then

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$$\bar{s} \bar{r} = r \bar{x} \bar{r} = r \varphi = \bar{\alpha},$$

$$\bar{t} \bar{r} = r \bar{y} \bar{r} = r \psi = \bar{\beta}$$

showing that \bar{r} witnesses $\bar{R} = \bar{\alpha} \sim \bar{\beta}$.