Proof of Proposition, p. 26

(Many readers will find this superfluous since the proof is the same as that for the corresponding fact for simplicial sets.)

For any functor $V : L \to Set$, let us write $|V|$ for the set $\{(K, v) \mid K \in L,\ v \in V(K)\}$ (the set of objects of the category $[V]$ we used earlier). Call a subset X of $|V|$ <u>closed</u> if for all arrows $p : K \to K_p$ in L,

$$(K, v) \in X \text{ implies } (K_p,\ V(p)(v)) \in X.$$

The mapping

$$\begin{aligned} Sub(V) &\longrightarrow \mathcal{P}(|V|) \\ U &\longmapsto |U| \end{aligned}$$

of subfunctors U of V to subsets of $|V|$ is a bijection onto the closed subsets of $|V|$; the mapping is a complete-lattice embedding of

$\mathrm{Sub}(V)$ into $\mathcal{P}(|V|)$; in particular, it preserves arbitrary unions.. It will be remembered that <u>directed</u> unions of subfunctors are <u>colimits</u>.

Furthermore: if X and Z are closed subsets of $|V|$, and Z is a single-element extension of X: $Z = X \dot{\cup} \{z\}$, then we have a pushout diagram: in Set^{L}:

$$\begin{tikzcd}
U \arrow[r, "\text{Incl.}"] & W \\
\mathring{K} \arrow[u] \arrow[r, "i_K"'] & \hat{K} \arrow[u, "\hat{w}"']
\end{tikzcd}$$

where U and W are the subfunctors of V corresponding to X and Z, resp. ($X = |U|$, $Z = |W|$), $z = (K, w)$ ($w \in W(K) \subseteq V(K)$) and $\hat{w}$, as usual, given by $\hat{w}_K(1_K) = w$. The reason is that the composite $\hat{w} \circ i_K : \mathring{K} \to W$ factors through $U \xrightarrow[\text{incl}]{} W$, resulting in the above diagram of functors and natural transformations, which, when evaluated at any object $K' \in L$, becomes a pushout diagram in $\mathbf{Set}$: for $K' \neq K$, we get

$$\begin{tikzcd}
U(K') \arrow[r, equal] & W(K') \\
L(K,K') \arrow[u] \arrow[r, equal] & L(K,K') \arrow[u]
\end{tikzcd}$$

and for $K' = K$:

$$\begin{tikzcd}
U(K) \arrow[r, "\text{incl.}"] & U(K) \dot\cup \{w\} & w \\
\emptyset \arrow[u] \arrow[r] & \{1_K\} \arrow[u, "\hat{w}_K"'] & 1_K \arrow[u, mapsto]
\end{tikzcd}$$

Define the binary relation $\prec$ on the set $|V|$ as follows:

$$(K', w) \prec (K, v)$$

$$\underset{\text{def}}{\Longleftrightarrow}$$

There is a proper (non-identity) arrow $p : K \to K'$ such that $w = V(p)(v)$

Note the obvious fact that a subset X of $|V|$ is closed in the sense defined above if and only if it is downward closed under $\prec$:

for $x, y \in |V|$,

$$x \in X \;\&\; y \prec x \;\Rightarrow\; y \in X.$$

The relation $\prec$ is well-founded: with any $\prec$-descending chain

$$(K, v) = (K_0, v_0) \succ (K_1, v_1) \succ \cdots$$

We have at least one corresponding sequence of proper arrows

$$K = K_0 \xrightarrow[\neq 1]{p_1} K_1 \xrightarrow[\neq 1]{p_2} \cdots$$

and, as part of the definition of L being a FOLDS signature, any such sequence has a maximal finite length that we called the dimension of K,

Now, let $f: \mathcal{U} \rightarrowtail V$ be any monomorphism in $\mathbf{Set}^L$. Let

$$X_0 = \{ (K, f_K(u)) \mid (K, u) \in |\mathcal{U}| \}$$

X_0 equals $|\bar{\mathcal{U}}|$ for the subfunctor $\bar{\mathcal{U}}$ of V that is the image of $\mathcal{U}$ under f.

It is a general fact that every well-founded relation can be extended to a (total) well-ordering. Therefore, there is a one-to-one ordinal indexing $\langle x_\beta \rangle_{\beta<\alpha}$ of the set $|V| - X_0$:

$$|V| - X_0 = \{ x_\beta \mid \beta < \alpha \}$$

and $\gamma \neq \beta \Rightarrow x_\gamma \neq x_\beta$.

<u>such that</u> $x_\gamma \prec x_\beta$ implies $\gamma < \beta$ $(\gamma, \beta < \alpha)$

Define the sets X_β for $\beta \leq \alpha$ as follows:

$$X_0 = X_0 \quad \text{as before;}$$

for limit ordinal $\beta \leq \alpha$:

$$X_\beta = \bigcup_{\gamma<\beta} X_\gamma \;;$$

for $\beta = \gamma + 1 \leq \alpha$:

$$X_{\gamma+1} = X_\gamma \,\dot\cup\, \{x_\gamma\}$$

More simply: $X_\beta = X_0 \,\dot\cup\, \{x_\gamma \mid \gamma<\beta\}$

The sets X_β are closed:

Suppose $y \prec x$ and $x \in X_\beta$: then either $x \in X_0$ or $x = x_\gamma$ for some $\gamma<\beta$;

in the first case, $y \in X_0 \subseteq X_\beta$, since X_0 is closed

in the second case: either $y \in X_0$, or $y = x_\delta$ for some $\delta < \alpha$; in the first sub-case $y \in X_\beta$;

in the second sub-case, we have

$x_\delta \prec x_\gamma$, <u>hence</u> $\delta < \gamma < \beta$,

and so $y = x_\delta \in X_\beta$ as desired.

Also : $X_\alpha = |V|$.

We have the transfinite diagram

$$\vec{\mathcal{U}} = \langle \mathcal{U}_\beta ; u_{\gamma\beta} : \mathcal{U}_\gamma \to \mathcal{U}_\beta \rangle_{\gamma \le \beta \le \alpha}$$

where $\mathcal{U}_0 = \mathcal{U}$;

for $0 < \beta \le \alpha$: $\mathcal{U}_\beta$ is the subfunctor of V corresponding to the closed set X_β: $X_\beta = |\mathcal{U}_\beta|$;

in particular, $\mathcal{U}_\alpha =: V$;

for $0 < \gamma \le \beta \le \alpha$: $u_{\gamma\beta} : \mathcal{U}_\gamma \to \mathcal{U}_\beta$ is the inclusion (corresponding to the inclusion $X_\gamma \subseteq X_\beta$)

$u_{0\beta} : \mathcal{U}_0 \to \mathcal{U}_\beta$ $(0 < \beta \le \alpha)$ is the composite

$$\mathcal{U}_0 = \mathcal{U} \xrightarrow[g]{\cong} \bar{\mathcal{U}} \longrightarrow \mathcal{U}_\beta$$

where g is the isomorphism of $\mathcal{U}$ onto its f-image $\bar{\mathcal{U}}$, and the inclusion $\bar{\mathcal{U}} \longrightarrow \mathcal{U}_\beta$ corresponds to the inclusion $X_0 \subseteq X_\beta$.

By what we saw above, $\vec{\mathcal{U}}$ is a transfinite composition in which each successor step $u_{\gamma,\gamma+1} : \mathcal{U}_\gamma \to \mathcal{U}_{\gamma+1}$ is a pushout of a sphere inclusion $\mathring{K} \overset{i_K}{\rightarrowtail} \hat{K}$; and $u_{0\alpha} : \mathcal{U}_0 \to \mathcal{U}_\alpha$ equals the given monomorphism $f : \mathcal{U} \rightarrowtail V$.

This proves that $f : \mathcal{U} \rightarrowtail V$ belongs to the saturation of the set $\{\mathring{K} \xrightarrow{i_K} \hat{K} \mid K \in L\}$.

$\square$ Proposition, p $\boxed{26}$.

Explanations to page $\boxed{51}$, for the bijection $\Sigma \restriction \overline{\Phi}$:

The fact that the bijection $\Sigma \restriction \overline{\Phi}$ is well-defined is to say that: <u>if</u> we have

$$\mathcal{U} \xrightarrow{\gamma} H[M] \quad \text{and} \quad \dot{H}(\mathcal{U}) \xrightarrow{\sigma} M$$

and the given bijection Σ relates the two: $\gamma \longmapsto \sigma$

<u>then</u> the following conditions (1) and (2) are equivalent:

(1) γ factors through $(H \restriction \Phi)[M] \xrightarrow{\text{incl}} H[M]$

(2) σ belongs to the subset $(\dot{H} \restriction \overline{\Phi})[M](\mathcal{U})$ of $\dot{H}[M](\mathcal{U})$.

Condition (1) on any $\mathcal{U} \xrightarrow{\gamma} H[M]$ is the same as:

$$\text{for all } K \in L,\ \gamma_K : \mathcal{U}(K) \to H[M](K)\ (= \mathrm{Set}^L(H(K), M))$$

factors through the subset $(H \restriction \Phi)[M](K)$ of $H[M](K)$; hence, the same as

$$\text{for all } K \in L \text{ and for all } u \in \mathcal{U}(K)$$

$$\gamma_K(u) \in (H \restriction \Phi)[M](K).$$

On the other hand, (2) is to say that, for all $(K, u) \in [\mathcal{U}]$ the composite $\sigma \circ \dot{H}(\hat{u}) : H(K) \to M$ belongs to $(H \restriction \Phi)[M](K)$

$$\sigma \circ \dot{H}(\hat{u}) \in (H \restriction \overline{\Phi})[M](K).$$

But, if $\Sigma : \gamma \mapsto \sigma$, then

$$\gamma_K(u) = \sigma \circ \dot{H}(\vec{u})$$

as shown on p. $\boxed{15}$ as the relation $\odot$ (now, $D = H$, $\delta = \gamma$, $\delta^{\#} = \sigma$).

$\square$

Explanations to page $\boxed{53}$, for the bijection $(\Sigma \upharpoonright \Phi)[M]$:

This is purely diagrammatic, based on three facts:

One: the diagram on p. $\boxed{51}$; we abbreviate it as

$$\begin{tikzcd} 5 \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "1"] & 6 \arrow[d] \\ 7 \arrow[r] & 8 \end{tikzcd}$$

Two: the pullback square in Diagram 1, p $\boxed{46}$, with the horizontal turned vertical and vice versa and with the prefix $Set^L(U, \cdots$ prefixed – it remains a pullback:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 5 \arrow[d] \\ 3 \arrow[r] & 7 \end{tikzcd}$$

(the verticals $\begin{smallmatrix} 5 \\ \downarrow \\ 7 \end{smallmatrix}$ in the two are the same).

Three: similar to Two, with Diagram 2 in place of Diagram 1:

$$\begin{tikzcd} 2 \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & 6 \arrow[d] \\ 4 \arrow[r] & 8 \end{tikzcd}$$

The square we seek, on page $\boxed{51}$, is abbreviated as:

$$\begin{tikzcd} 1 \arrow[r, dashed, "?", "\cong"'] \arrow[d] & 2 \arrow[d] \\ 3 \arrow[r, "\cong"'] & 4 \end{tikzcd}$$

The four fit together in a cube – spread out as:

$$\begin{tikzcd} 1 \arrow[r, dashed, "?"] \arrow[d] \arrow[dr, phantom, "4"] & 2 \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & 6 \arrow[d] \\ 3 \arrow[r, "\cong"'] & 4 \arrow[r] & 8 \end{tikzcd}$$

$=$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & 5 \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "1"] & 6 \arrow[d] \\ 3 \arrow[r] & 7 \arrow[r, "\cong"'] & 8 \end{tikzcd}$$

The "composite" ② □ ① $=$ $\begin{matrix} 1 & \to & 6 \\ \downarrow & & \downarrow \\ 3 & \to & 8 \end{matrix}$ is a pullback as a "composite" of two pullbacks; there is an arrow $1 \dashrightarrow 2$ that makes ④ commute by using the pullback ③; ④ □ ③ $=$ ② □ ① is a pullback; it follows that ④ is a pullback; the pullback of the isomorphism $3 \xrightarrow{\cong} 4$ is an isomorphism: $1 \overset{\cong}{\dashrightarrow} 2$ is an isomorphism.

$\square$

Explanation on invariance, bottom page $\boxed{53}$:

The LHS of the bi-implication is equivalent to saying:

for all $(K, u) \in [U]$, $x \underset{\text{def}}{=} \theta \circ \dot{H}(\check{u})$ belongs to $(H \restriction \Phi)[P](K)$

and the RHS:

for all $(K, u) \in [U]$, $z \underset{\text{def}}{=} \sigma \circ \dot{H}(\check{u})$ belongs to $(H \restriction \Phi)[M](K)$

We have:

$$\begin{tikzcd} & & P \arrow[d, "p"] \\ H(K) \arrow[r, "\dot{H}(\check{u})"] \arrow[urr, bend left, "x"] \arrow[drr, bend right, "z"'] & \dot{H}(U) \arrow[ur, "\theta"] \arrow[dr, "\sigma"'] & M \\ & & \end{tikzcd}$$

$\therefore \; z = px$

Also:

$$x \in (H \upharpoonright \underline{\overline{\Phi}})[P](K) \iff (P, x) \in \underline{\overline{\Phi}}_K$$

$$z \in (H \upharpoonright \underline{\overline{\Phi}})[M](K) \iff (M, z) \in \underline{\overline{\Phi}}_K$$

and the invariance property is that

$$(P, x) \in \underline{\overline{\Phi}}_K \text{ iff } (M, z) \in \underline{\overline{\Phi}}_K$$

provided p is FS and $z = px$

as it is the case now.

$\square$

Explanations for the Introduction

⊙ "If $\underline{P}_1$ extends $\underline{P}_0$, then $Eq[\underline{P}_0, U] \subseteq Eq[\underline{P}_1, U]$"

Suppose
$$\begin{tikzcd} & P_0 \arrow[dl, "l_0"'] \arrow[dr, "r_0"] & \\ M & & M \\ & U \arrow[ul, "\alpha"] \arrow[uu, "\gamma_0"] \arrow[ur, "\beta"'] & \end{tikzcd}$$
witnessing $\alpha \underset{\underline{P}_0}{\sim} \beta$. With the diagonal $\delta : U \to Q$ given by m being FS, we have

$$\begin{tikzcd} \emptyset \arrow[r, "!"] \arrow[d, "!"'] & Q \arrow[r, "q"] \arrow[d, "m"] & P_1 \arrow[d, "l_1"] \\ U \arrow[r, "\gamma_0"'] \arrow[ur, dotted, "\delta"] \arrow[rr, bend right, "\alpha"'] & P_0 \arrow[r, "l_0"'] & M \end{tikzcd}$$

and similarly with r's in place of l's,

$\gamma_1 \underset{def}{=} q\delta : U \to P_1$, $\alpha = l_1 \gamma_1$, $\beta = r_1 \gamma_1$

witnessing $\alpha \underset{\underline{P}_1}{\sim} \beta$

⊙ "$\underline{P}_0$ and $\underline{P}_1$ can both be extended to $\underline{P}_0 \otimes \underline{P}_1$"

First of all, note that the composite of two

FS maps is FS, and a pullback of an FS map is FS. These are cases of the general elementary facts that the class of all maps that have the right lifting property relative to a fixed map has the said closure properties.

Assume the RSE's on M, $\underline{P}_0$ and $\underline{P}_1$, with witnesses (Q_0, q_0, m_0), (Q_1, q_1, m_1) for the reflexivity of $\underline{P}_0$, $\underline{P}_1$, respectively. We construct $\underline{P} = (P, \ell, r)$ by the pullback shown (span-composition):

$$\begin{tikzcd}
& & P \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] \arrow[dd, phantom, "\lrcorner"] \arrow[ddll, bend right, "\ell"'] \arrow[ddrr, bend left, "r"] & & \\
& P_0 \arrow[dl, "\ell_0"'] \arrow[dr, "r_0"] & & P_1 \arrow[dl, "\ell_1"'] \arrow[dr, "r_1"] & \\
M & & M & & M
\end{tikzcd}$$

$$\ell \underset{\text{def}}{=} \ell_0 \pi_0$$

$$r \underset{\text{def}}{=} r_1 \pi_1$$

By what we said above, we do have that ℓ and r are FS. Let's introduce the reflexivity witnesses into the commutative diagram:

$$
\begin{tikzcd}
 & & P \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & & \\
 & P_0 \arrow[ddl, "l_0"'] \arrow[ddr, "r_0"] & & P_1 \arrow[ddl, "l_1"'] \arrow[ddr, "r_1"] & \\
 & Q_0 \arrow[u, "q_0"] \arrow[dl, "m_0"] \arrow[dr, "m_0"'] & & Q_1 \arrow[u, "q_1"] \arrow[dl, "m_1"] \arrow[dr, "m_1"'] & \\
M & & M & & M
\end{tikzcd}
$$

Produce Q and $m: Q \to M$ as the pullback

$$
\begin{tikzcd}
 & Q \arrow[dl, "\bar{\pi}_0"'] \arrow[dr, "\bar{\pi}_1"] \arrow[dd, "m"] & \\
Q_0 \arrow[dr, "m_0"'] & & Q_1 \arrow[dl, "m_1"] \\
 & M &
\end{tikzcd}
$$

The maps $\bar{\pi}_0$, $\bar{\pi}_1$, m are FS. Using that P is a pullback, and that the diagram

$$
\begin{tikzcd}
P_0 \arrow[r, "r_0"] & M & P \arrow[l, "l_1"'] \\
 & & \\
Q_0 \arrow[uu, "q_0"] & Q \arrow[l, "\bar{\pi}_0"] \arrow[r, "\bar{\pi}_1"'] & Q_1 \arrow[uu, "q_1"']
\end{tikzcd}
$$

commutes (since $r_0 q_0 \bar{\pi}_0 = m_0 \bar{\pi}_0 = m_1 \bar{\pi}_1 = l_1 q_1 \bar{\pi}_1$)

we obtain $q : Q \to P$ such that the following commutes:

$$
\begin{tikzcd}
 & & P \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & & \\
 & P_0 \arrow[dddl, bend right, "l_0"'] & & P_1 \arrow[dddr, bend left, "r_1"] & \\
 & & Q \arrow[uu, "q"] \arrow[dl, "\bar{\pi}_0"] \arrow[dr, "\bar{\pi}_1"] \arrow[dd, "m"] & & \\
 & Q_0 \arrow[uu, "q_0"] \arrow[dl, "m_0"] \arrow[dr, "m_0"] & & Q_1 \arrow[uu, "q_1"'] \arrow[dl, "m_1"] \arrow[dr, "m_1"] & \\
M & & M & & M
\end{tikzcd}
$$

This shows that $\underline{P} = (P, l, r)$ is reflexive, with (Q, q, m) witnessing the fact.

Now, we show that $\underline{P} = \underline{P}_0 \otimes \underline{P}_1$ extends both $\underline{P}_0$ and $\underline{P}_1$.
We show that the reflexivity of $\underline{P}_1$ suffices to have that $\underline{P}$ extends $\underline{P}_0$; symmetrically, we will have that the reflexivity of $\underline{P}_0$ implies that $\underline{P}$ extends $\underline{P}_1$. We will produce $\bar{Q}$, $\bar{q}$ and $\bar{m}$, the last FS as in:

$$
\begin{tikzcd}
 & P \arrow[dl, "l"'] \arrow[dr, "r"] & \\
M \arrow[r, phantom, "1?"] & \bar{Q} \arrow[u, "\bar{q}"] \arrow[d, "\bar{m}"] \arrow[r, phantom, "2?"] & M \\
 & P_0 \arrow[ul, "l_0"] \arrow[ur, "r_0"'] &
\end{tikzcd}
$$

$\bar{Q}$ and $\bar{q}$ are defined by the pullback

$$\begin{tikzcd} \bar{Q} \arrow[r, "\bar{q}"] \arrow[d, "\pi"'] & P \arrow[d, "\pi_1"] \\ Q_1 \arrow[r, "q_1"'] & P_1 \end{tikzcd}$$

Put together with the pullback defining P, we obtain a third pullback:

$$\begin{tikzcd} \bar{Q} \arrow[r, "\bar{q}"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "\pi_0"] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & P_0 \arrow[d, "r_0"] \\ Q_1 \arrow[r, "q_1"'] & P_1 \arrow[r, "\ell_1"'] & M \end{tikzcd} \quad = \quad \begin{tikzcd} \bar{Q} \arrow[r, "\pi_0 \bar{q}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_0 \arrow[d, "r_0"] \\ Q_1 \arrow[r, "\ell_1 q_1"'] & M \end{tikzcd}$$

The upper horizontal in the last square serves as the definition of $\bar{m} \underset{\text{def}}{=} \pi_0 \bar{q} : \bar{Q} \to P_0$. The lower horizontal is $m_1 = \ell_1 \pi_1$ – reflexivity of $\underline{P}_1$. As a pullback of an FS map, m_1 is FS. Considering

$$\begin{tikzcd} & P \arrow[dl, "\ell"'] \arrow[dd, "\pi_0"] & \\ M & & \bar{Q} \arrow[ul, "\bar{q}"'] \arrow[dl, "\bar{m}"] \\ & P_0 \arrow[ul, "\ell_0"] & \end{tikzcd}$$

(definition of $\ell : P \to M$,
definition of $\bar{m} : \bar{Q} \to P_0$)

we see that the desired commutativity ① ? holds. ② ? is similar:

$$\begin{tikzcd} & P \arrow[dl, "\bar{q}"'] \arrow[d, "\pi_0"] \arrow[dr, "r"] & \\ \bar{Q} \arrow[dr, "\bar{m}"'] & {} & M \\ & P_0 \arrow[ur, "r_0"'] & \end{tikzcd}$$

□

⊙ Invariance of intrinsic equivalence under FOLDS equivalence.

Step 1 Suppose we have $\underline{P} = (P, \ell, r) : M \underset{L}{\sim} N$ and $\underline{R} = (R, s, t) : M \underset{L}{\sim} M$ a self equivalence of M. We construct a self equivalence

$$\underline{\bar{R}} = (\bar{R}, \bar{s}, \bar{t}) : N \underset{L}{\sim} N \quad \text{of } N.$$

$\bar{R}$ is constructed as a limit, a "double pullback", in $\mathbf{Set}^L$, with projections $\bar{x} : \bar{R} \to P$, $z : \bar{R} \to R$, $\bar{y} : \bar{R} \to P$, making the following commute:

$$\begin{tikzcd}
 & P \arrow[r, "l"] & M \\
\bar R \arrow[ur, "\bar x"] \arrow[r, "z"'] \arrow[dr, "\bar y"'] & R \arrow[ur, "s"'] \arrow[dr, "t"] & \\
 & P \arrow[r, "l"'] & M
\end{tikzcd}$$

with $(\bar R, \bar x, z, \bar y)$ the terminal such quadruple making the diagram commute. $(\bar R, \cdots)$ can be contrued as a repeated pullback, although in this way the symmetry is broken:

$$\begin{tikzcd}
 & & P \arrow[r, "l"] & M \\
\bar R = (P\times_M P)\times_M P \arrow[r] \arrow[urr, bend left, "\bar x"'] \arrow[rr, bend right, "z"] \arrow[drr, bend right, "\bar y"'] & R\times_M R \arrow[r] \arrow[ur] \arrow[dr, dashed] & R \arrow[ur, "s"'] \arrow[dr, "t"] & \\
 & & P \arrow[r, "l"'] & M
\end{tikzcd}$$

However, this way we see that the maps $\bar x, z, \bar y$ are all FS. We define $\bar s : \bar R \to N$, $\bar t : \bar R \to N$

by $\bar{s} =_{def} r\bar{x}$, $\bar{t} =_{def} r\bar{y}$; $\bar{s}$ and $\bar{t}$ are FS maps.

<u>Step 2</u> We show that if $\underline{R}$ is reflexive, then so is $\underset{\rightarrow}{\underline{R}}$: from

$$\begin{tikzcd}
& R \arrow[dl, "s"'] \arrow[dr, "t"] & \\
M \arrow[dr, equal] & Q \arrow[u, "q"] \arrow[d] & M \arrow[dl, equal] \\
& M &
\end{tikzcd}
\qquad \text{we obtain} \qquad
\begin{tikzcd}
& \bar{R} \arrow[dl, "\bar{s}"'] \arrow[dr, "\bar{t}"] & \\
N \arrow[dr, equal] & \bar{Q} \arrow[u, "\bar{q}"] \arrow[d, "\bar{m}"] & N \arrow[dl, equal] \\
& N &
\end{tikzcd}$$

m FS $\qquad\qquad\qquad$ $\bar{m}$ FS

$\bar{Q}$ is defined as the pullback in the diagram

$$\begin{tikzcd}
& P \arrow[dl, "l"'] \arrow[rrd, bend left, "r"] & & \\
M & & \bar{Q} \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_0"] \arrow[r, "\bar{m}"] \arrow[l, phantom, "<"] & N \\
& Q \arrow[ul, "m"] & &
\end{tikzcd}$$

and $\bar{m}$ as the composite $\bar{m} = r\pi_1$; $\bar{m}$ is FS.

To define $\bar{q} : \bar{Q} \to \bar{R}$, we use the definition of $\bar{R}$ as a double pull back.

The following commutes:

$$\begin{tikzcd}
 & P \arrow[r, "\ell"] & M \\
\bar{Q} \arrow[ur, "\pi_1"] \arrow[r, "q\pi_0"] \arrow[dr, "\pi_1"'] & R \arrow[ur, "s"] \arrow[dr, "t"] & \\
 & P \arrow[r, "\ell"'] & M
\end{tikzcd}$$

because: $\ell\pi_1 = m\pi_0 = sq\pi_0$,
$\ell\pi_1 = m\pi_0 = tq\pi_0$

Therefore, we have $\bar{Q} \xrightarrow{\bar{q}} \bar{R}$ such that

$$\begin{tikzcd}
\bar{Q} \arrow[r, "\bar{q}"] \arrow[dr, "\pi_1"'] & \bar{R} \arrow[d, "\bar{x}"] \\
 & P
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{Q} \arrow[r, "\bar{q}"] \arrow[dr, "q\pi_0"'] & \bar{R} \arrow[d, "z"] \\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{Q} \arrow[r, "\bar{q}"] \arrow[dr, "\pi_1"'] & \bar{R} \arrow[d, "\bar{y}"] \\
 & P
\end{tikzcd}$$

all commute. Therefore $\bar{s}\bar{q} = r\bar{x}\bar{q} = r\pi_1 = \bar{m}$,
and similarly $\bar{t}\bar{q} = \bar{m}$

<u>Step 3</u> Suppose $\alpha, \beta : U \to M$,

$\bar{\alpha}, \bar{\beta} : U \to N$; $\varphi : U \to P$ witnesses $\underline{P} : \alpha \sim \bar{\alpha}$,

$\psi : U \to P$ witnesses $\underline{P} : \beta \sim \bar{\beta}$.

Suppose $\gamma : U \to R$ witnesses $\underline{R} : \alpha \sim \beta$.

We construct $\bar{\gamma} : U \to \bar{R}$ witnessing $\underline{\bar{R}} : \bar{\alpha} \sim \bar{\beta}$.

To use the limit definition of $\bar{R}$, we observe the commutativity of

$$\begin{tikzcd}
 & P \arrow[r, "\ell"] & M \\
U \arrow[ur, "\varphi"] \arrow[urr, "\alpha"] \arrow[r, "\gamma"'] \arrow[drr, "\beta"] \arrow[dr, "\psi"'] & R \arrow[ur, "s"'] \arrow[dr, "t"] & \\
 & P \arrow[r, "\ell"'] & M
\end{tikzcd}$$

Therefore, there is $U \xrightarrow{\bar{\gamma}} \bar{R}$ such that

$$\begin{tikzcd}
U \arrow[r, "\bar{\gamma}"] \arrow[dr, "\varphi"'] & \bar{R} \arrow[d, "\bar{x}"] \\
 & P
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\bar{\gamma}"] \arrow[dr, "\gamma"'] & \bar{R} \arrow[d, "z"] \\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\bar{\gamma}"] \arrow[dr, "\psi"'] & \bar{R} \arrow[d, "\bar{y}"] \\
 & P
\end{tikzcd}$$

all commute. But then

$$\bar{s}\,\bar{\gamma} = r\bar{x}\,\bar{\gamma} = r\varphi = \bar{\alpha},$$

$$\bar{t}\,\bar{\gamma} = r\bar{y}\,\bar{\gamma} = r\psi = \bar{\beta}$$

showing that $\bar{\gamma}$ witnesses $\underline{\bar{R}} : \bar{\alpha} \sim \bar{\beta}$.