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Convergence to Sharp Traveling Waves of Solutions for Burgers-Fisher-KPP Equations with Degenerate Diffusion

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Abstract

This paper is concerned with the convergence to sharp traveling waves of solutions with semi-compactly supported initial data for Burgers-Fisher-KPP equations with degenerate diffusion. We characterize the motion of the free boundary in the long-time asymptotic of the solution to Cauchy problem and the convergence to sharp traveling wave with almost exponential decay rates. Here a key difficulty lies in the intrinsic presence of nonlinear advection effect. After providing the analysis of the nonlinear advection effect on the asymptotic propagation speed of the free boundary, we construct sub- and super-solutions with semi-compact supports to estimate the motion of the free boundary. The new method overcomes the difficulties of the non-integrability of the generalized derivatives of sharp traveling waves at the free boundary.

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1 Introduction and Main Results

We consider the Cauchy problem for the following Burgers-Fisher-KPP equations with degenerate diffusion

$$u_t + f(u)_x = (u^m)_{xx} + g(u), \tag{1.1}$$

with initial data

$$u(x, 0) = u_0(x) \in \mathscr{A}. \tag{1.2}$$

Here, $\mathscr{A}$ is the function class with semi-compact supports such that

$$\mathscr{A} := \{u_0 \in L^\infty(\mathbb{R});\ u_0(x) \geq 0,\ u_0 \text{ is piecewise continuous},\\ \operatorname{supp} u_0 \subset (-\infty, x_0] \text{ for some } x_0 \in \mathbb{R}, \liminf_{x\to-\infty} u_0(x) > 0\}.$$

The degenerate diffusion in (1.1) is of porous medium type with $m > 1$, which arises from a density-dependent dispersal in biological dynamics (Aronson 1980; Shiguesada et al. 1979; Murray 2002) or a temperature-dependent thermal conductivity (Mendez and Fort (2001)), etc. The advection in (1.1) is Burgers type and the reaction in (1.1) is Fisher-KPP type such that

$$f \in C^2, \quad g \in C^1,\ g(0) = g(1) = 0,\ g(u) > 0,\ \forall u \in (0, 1),\\ g(u) < 0,\ \forall u > 1, \quad g'(0) \geq 0,\ g'(1) < 0. \tag{1.3}$$

We further assume that $g(s)$ satisfies the following growth condition

$$\sup_{s\in(0,1)} \frac{g'(s)s}{g(s)} < m. \tag{1.4}$$

The growth condition (1.4) is compatible with the degenerate diffusion and is fulfilled by the classical Fisher-KPP type source such that $g''(s) < 0$.

The direction of the advection has influence on the propagation properties and certainly on the minimal admissible wave speed of traveling waves. For the sake of simplicity, we focus on the case that the traveling wave $\phi_c(\xi)$ with $\xi = x - ct$ connects 1 at $-\infty$ and 0 at $+\infty$, i.e., $\phi_c(-\infty) = 1$ and $\phi_c(+\infty) = 0$. Other cases can be treated via changes of variables in a similar way.

Compared with the extensive studies on the attractiveness of smooth traveling waves for degenerate diffusion equations (see, for example, Liu and Yu (1997); Fife and McLeod (1980); Mellet et al. (2009); Sattinger (1976); Kienzler (2016); Gnann et al. (2019)), the convergence results to sharp traveling waves are rarely studied. These attractiveness property of sharp traveling waves are proved for reaction diffusion equations without convection for initial data with compact (or semi-compact) supports by Biró (1997, 2002) and for initial data with exponentially decaying in Díaz and Kamin (2012); Kamin and Rosenau (2004, ?). Even without advection effect, developing the convergence theory for traveling waves of degenerate diffusion equations has been proved challenging: It was Du et al. (2020) who studied the large time behavior of solutions for Fisher-KPP equations with degenerate diffusion in higher dimension $u_t = \Delta u^m + u(1-u)$, and show the logarithmic shift phenomenon (see also Medvedev et al. (2003)) for one-dimensional case.

Related to the attractiveness property of sharp fronts, the stability results (spectral or nonlinear) of traveling waves for degenerate diffusion equations with reaction under small perturbations are very scarce. As far as we know, it was Hosono (1986) who first formulated the rigorous stability analysis of a sharp traveling front for a degenerate diffusion equation of Nagumo or bistable reaction in 1986. The spectral stability of non-critical smooth traveling waves for (1.1) without the advection term $f(u)_x$ has only been shown recently by Leyva and Plaza (2020), which is a key further step in the study of nonlinear stability of traveling waves for degenerate diffusion equations. Then Leyva et al. (2022) extended the previous spectral theory to the bistable case, contributing to the stability analysis for more general density-dependent degenerate diffusions. Recently, Dalibard et al. (2023) established the nonlinear stability of degenerate diffusion sharp traveling fronts in the porous medium case.

The attractiveness property and the corresponding convergence rate of sharp traveling waves for the Burgers-Fisher-KPP equations with degenerate diffusion remain open due to some technical issues. The effect of advection to the degenerate diffusion equations (1.1) is essential, which causes the study on the convergence results of sharp traveling waves to be difficult and challenging. The main purpose of the present paper is to give a rigorous proof of convergence result of sharp traveling waves corresponding to the degenerate Burgers-Fisher-KPP equations (1.1).

Regarding the degenerate diffusion equations, the existence of sharp traveling waves has been well-studied in Aronson (1980, 1985); De Pablo and Vázquez (1991); Sánchez-Garduño and Maini (1994a, b, 1995, 1997); Malaguti and Marcelli (2003) for the porous medium equation with Fisher-KPP source or other reactions, and further investigated for the Burgers-Fisher-KPP equations by Ma and Ou (2021) and Mendoza and Muriel (2021) in the linear diffusion case $m = 1$, and by Gilding and Kersner (2005) in the degenerate diffusion case $m > 1$. We summarize these results as follows.

Proposition 1.1 *There exists a constant $c^* = c^*(m, f, g)$ such that (1.1) has traveling wave solution $\phi_c(x-ct)$ connecting 1 and 0 ($\phi_c(-\infty) = 1$ and $\phi_c(+\infty) = 0$), if and only if $c \geq c^*$, such that*

(i) if $c > c^*$, then $0 < \phi_c(\xi) < 1$ and $\phi_c'(\xi) < 0$ for all $\xi \in \mathbb{R}$;
(ii) if $c = c^*$, then there exists $\xi_0 \in \mathbb{R}$, such that $\phi_{c^*}(\xi) = 0$ for all $\xi \geq \xi_0$, and $0 < \phi_{c^*}(\xi) < 1$ with $\phi_{c^*}'(\xi) < 0$ for all $\xi < \xi_0$.

Moreover, $c^* > f'(0)$, $\phi_{c^*} \in C^\alpha(\mathbb{R})$ with $\alpha = \min\{1, \frac{1}{m-1}\}$,

$$\phi_{c^*}'(\xi) \sim -\frac{c^* - f'(0)}{m}\phi_{c^*}^{2-m}(\xi),$$

as $\phi_{c^*}(\xi) \to 0^+$ or equivalently $\xi \to \xi_0^-$.

Without loss of generality, we always assume that $\xi_0 = 0$.

Main results. The main results of this paper are the following convergence theorems.

Theorem 1.1 *For any solution $u(x,t)$ of the Cauchy problem (1.1)–(1.2) with initial data $u_0 \in \mathscr{A}$, there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges in form and in speed to $\phi_{c^*}(x - c^*t - x_0)$:*

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi_{c^*}(x - c^*t - x_0)| = 0, \tag{1.5}$$

where $\phi_{c^*}(x - c^*t)$ is the sharp traveling wave with critical wave speed c^* in Proposition 1.1 (with $\xi_0 = 0$).

Theorem 1.1 implies the motion of the free boundary of the solution to Cauchy problem and presents a sharp traveling wave type asymptotic profile of the solution.

Corollary 1.1 *Let $u(x,t)$ be the solution of the Cauchy problem (1.1)–(1.2) with initial data $u_0 \in \mathscr{A}$, and let $\zeta(t)$ be the free boundary of the semi-compact support of the solution $u(x,t)$, i.e., $\zeta(t) := \sup\{x \in \mathbb{R}; u(x,t) > 0\}$. Then there exists a $x_0 \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} (\zeta(t) - c^*t) = x_0,$$

and $u(x,t) = 0$ for all $x \geq \zeta(t)$,

$$\lim_{t\to+\infty} \sup_{y\in\mathbb{R}} |u(\zeta(t) + y, t) - \phi_{c^*}(y)| = 0.$$

Then our theorem on almost exponential convergence rate to sharp traveling waves is given as follows.

Theorem 1.2 *Let $u(x,t)$ and x_0 be as stated in Theorem 1.1. For any $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon) > 0$ such that*

$$\sup_{|x-c^*t-x_0|\geq\varepsilon} |u(x,t) - \phi_{c^*}(x - c^*t - x_0)| \leq Ce^{-\sigma t}, \ \forall t > 0, \tag{1.6}$$

for some positive constant $C > 0$.

Background of studies. For the regular Fisher-KPP equations without advection effect and degeneracy of diffusion,

$$u_t = u_{xx} + g(u),$$

where $g(u)$ satisfies the mono-stable condition (1.3) (the Fisher-KPP type), the stability of the traveling waves has been a quite hot and attractive topic (see Bramson

(1983); Chen (1997); Chern et al. (2015); Fife and McLeod (1980); Gallay (1994); Kirchgassner (1992); Lau (1985); Lin et al. (2014); Mei et al. (2009a, b); Moet (1979); Mei et al. (2012); Sattinger (1976); Uchiyama (1978), the monograph Volpert et al. (1994), and the references therein). The first study addressing the stability of non-critical traveling waves was given by Sattinger (1976) by the spectral analysis method in 1976 for the 1-D Fisher-KPP equation. Two years later, Uchiyama (1978) showed the local stability for the traveling waves including the critical waves by the maximum principle method, but no convergence rate for the critical waves case was related. Then Bramson (1983) derived the sufficient and necessary condition for the stability of non-critical and critical waves without convergence rates. The same results were also obtained by Lau (1985) later in a different way. In Moet (1979), Moet showed that the critical waves are algebraically stable in the form of $O(t^{-1/2})$ by the Green function method. Kirchgassner (1992) also obtained the stability for the critical waves in the form $O(t^{-1/4})$ by the spectral method, which was further improved to be $O(t^{-3/2})$ by Gallay (1994) by using the renormalization group method, with a much faster decay for the initinal perturbations. For the n-D Fisher-KPP equations, the stability of planar faster traveling waves with $c > c^*$ was obtained by Mallordy and Roquejoffre in Mallordy and Roquejoffre (1995). For the mono-stable reaction-diffusion equations with or without time-delay, including the regular Fisher-KPP equations, by using the weighted energy method, Mei and his collaborators Chern et al. (2015); Lin et al. (2014); Mei et al. (2009a, b, 2012) showed the time-exponential stability for the non-critical traveling waves, and the time-algebraic stability for the critical traveling waves, including the oscillating waves caused by the large time-delay.

For the Fisher-KPP equations with degenerate diffusion (i.e., $m > 1$),

$$u_t = (u^m)_{xx} + g(u),$$

the traveling waves usually lose the regularity due to the degeneracy of diffusion, in particular for the sharp traveling waves in the critical case $c = c^*$. The study on the wave stability or attractiveness is quite limited as we know. For the sharp traveling waves with critical wave speed $c = c^*$, Biró (2002) first proved their attractiveness by technically constructing a special pair of sub- and super-solution. For the non-critical traveling waves with $c > c^*$, even if the degeneracy of diffusion is strong, the waves are still C^2-smooth. In this case, by using the weighted energy method and the viscosity vanishing technique, Huang et al. (2018) and Liu et al. (2022) obtained the stability of these smooth non-critical traveling waves to the local/nonlocal degenerate diffusion equations with time-delay, respectively.

On the other hand, without the reaction term $g(u)$, the equation (1.1) becomes the viscous Burgers equation with degenerate diffusion

$$u_t + f(u)_x = (u^m)_{xx}.$$

When $m = 1$, the stability of traveling waves (the so-called viscous shocks) has been intensively studied. In 1960, Il'in and Il'in and Oleinik (1960) first proved the stability of these viscous shocks by means of maximum principle, once the flux function $f(u)$ is convex. The same stability results were then obtained by Sattinger (1976) in 1976 by the spectral analysis method, and by Matsumura and Nishihara (1985) in 1985 and Goodman (1986) in 1986 independently by the energy method, respectively. The time-algebraic convergence rates were further derived by Kawashima and Matsumura (1985) and Jones et al. (1993). When the flux function is non-convex, the viscous shocks may be degenerate if the entropy condition is degenerate to the wave speed at the constant states. The stability of these degenerate viscous shocks and the optimal convergence rates were showed by Matsumura and Nishihara (1994) and Mei (1995). The L^1-stability was then obtained by Freistühler and Serre (1998).

For the Burgers equations with degenerate diffusion, Kurganov and Rosenau (1997) proposed and studied a new variant of the Burgers equation with saturating dissipation

$$u_t + f(u)_x = \nu Q(u_x)_x, \quad \nu > 0,$$

where the flux function satisfies $|Q(s)| \leq 1$, $Q'(s) > 0$ for all s and $Q'(s) \to 0$ as $|s| \to \infty$. A typical flux limited function is $Q(s) = \frac{s}{\sqrt{1+s^2}}$. As shown in Kurganov et al. (1998); Kurganov and Rosenau (1997), compared with the Burgers equation, traveling waves solutions develop discontinuities within finite time above the critical threshold, while small solutions remain smooth. The existence of sharp traveling waves (sharp viscous shocks) for the degenerate Burgers equations with $m > 1$ was also presented in the text book (Gilding and Kersner 2004).

However, it seems that the stability or attractiveness of sharp traveling waves is not trivial due to the difficulty arising from the flux term and the degeneracy of diffusion. Regarding the Burgers-Fisher-KPP equations with degenerate diffusion (1.1), we note that it possesses some good and bad features from both Burgers equations and Fisher-KPP equations. The stability or attractiveness of sharp traveling waves is also not treated so far. We observe that, even though there is the bad affect from the advection, the advantage from the reaction term is enough to allow us to prove the convergence to sharp traveling waves by the monotonicity techniques. With this motivation, we shall give a rigorous proof of sharp wave convergence results for (1.1).

Features, difficulties and strategies. Due to the degeneracy and semi-compacted properties of sharp waves, the technical steps we need to take are different and much more complex compared to the standard parabolic equations with linear diffusion and smooth traveling waves. For Burgers-Fisher-KPP equation with degenerate diffusion, there arise the following distinguishing features.

Both the sharp traveling wave and the solution are generally at most Hölder continuous near the free boundaries, and any smoothing approximations may destroy the free boundaries, therefore the energy methods which require regularities are not applicable. In the present work, we resolve these mathematical difficulties and provide a delicate analysis for relationship between the nonlinear advection and the asymptotic propagation speed of the free boundary.

The nonlinear advection affects the behavior of solutions for Cauchy problem. For Burgers-Fisher-KPP equation with degenerate diffusion, the initial disturbances propagates in a special way: if $u_0(x)$ is semi-compact (i.e., supp $u_0 = (-\infty, \zeta_0]$), then $u(x, t)$ is always semi-compact such that

$$\mathrm{supp}\, u(\cdot, t) = (-\infty, \zeta(t)], \quad \zeta(0) = \zeta_0,$$

where $\zeta(t)$ is a continuous function describing the edge of the support, also called the free boundary, of the solution $u(x, t)$. Note that the nonlinear advection could lead to expanding or shrinking of the free boundaries. The free boundary of Cauchy problem is nonmonotone. However, for the reaction diffusion equations, the solutions of Cauchy equations always expand. For the Burgers equation with degenerate diffusion, the asymptotic behaviors of the solution is also unknown. We will prove that the solutions of these equations converge to sharp traveling waves. Our convergence result is also the first result on the large time behavior of Cauchy problem for Burgers-Fisher-KPP equation with degenerate diffusion.

The key step to show the L^∞ convergence to sharp traveling wave is to estimate the motion of the free boundary. Both the sharp traveling wave and the solution are semi-compact, such that

$$\phi_{c^*}(x - c^*t - x_0) > 0 \text{ for } x < x_0 + c^*t, \quad \phi_{c^*}(x - c^*t - x_0) = 0 \text{ for } x \geq x_0 + c^*t,$$
$$u(x, t) > 0 \text{ for } x < \zeta(t), \quad u(x, t) = 0 \text{ for } x \geq \zeta(t),$$

the difference of the free boundaries has large impact on the difference of the two solutions. A natural way is to construct sub- and super-solutions with semi-compact supports, such that the amplitude decreases or increases to the same as sharp traveling wave, and the free boundaries of sub- and super-solutions converges to some scope. The new method overcomes the difficulties of the non-integrability of the generalized derivatives of sharp traveling wave solutions at the free boundary.

2 Proof of the main results

The aim of this paper is to study the convergence results of sharp traveling waves for Burgers-Fisher-KPP equations with degenerate diffusion. We recall the existence of critical wave speed and sharp traveling waves for porous medium diffusion equations in Proposition 1.1. For the degenerate diffusion equation with Fisher-KPP type reaction (i.e., without convection), the above statements are proved in many articles, using phase plane analysis method (see Huang et al. (2018); Audrito and Vázquez (2017); Xu et al. (2020a, b)). For the case with advection, we refer to Sánchez-Garduño and Pérez-Velázquez (2016); Gilding and Kersner (2005).

Noticing that the sharp traveling wave $\phi_{c^*}(x - c^*t)$ is the unique (up to translation) traveling wave with semi-finite support, it is natural to imagine that the solution $u(x, t)$ time asymptotically converges to a shift of the unique sharp traveling wave, $\phi_{c^*}(x - c^*t - x_0)$ with some $x_0 \in \mathbb{R}$. The solutions to Cauchy problem with compact (or semi-compact) initial data always remain compact (or semi-compact), and therefore the propagation property of compactly supported initial data behaves similarly not only in speed but also in form as the traveling wave with the critical wave speed.

The first step to show the L^∞ convergence to sharp traveling wave is to estimate the evolution of the free boundary. A natural way is to construct sub- and super-solutions with semi-compact supports, such that the amplitude decreases or increases to the same as sharp traveling wave, and the free boundaries of sub- and super-solutions converges to some scope. This is achieved by Biró (2002) for the case of $f(u) = 0$ and $g(u) = u^p - u^q$ in a delicate manner. We extend these precise estimates to a more general case in this paper.

Lemma 2.1 *Let $F(t)$ and $G(t)$ be the solutions to the following ordinary differential system*

$$\begin{cases} F'(t) = \varepsilon_1 A(F(t), G(t), t) F(t)(1 - F(t)), \\ G'(t) = c^* F^{m-1}(t) - \varepsilon_2 B(F(t), G(t), t)(1 - F(t)), \\ F(0) = F_0 > 0, \quad G(0) = G_0, \end{cases} \tag{2.1}$$

where ε_1 and ε_2 are positive constants, $A(F, G, t)$ and $B(F, G, t)$ are bounded functions with positive infimum. Then

(i) $\lim_{t\to+\infty} F(t) = 1$, *and the convergence rate is exponential;*
(ii) *if $F_0 < 1$, then $F(t)$ is strictly decreasing; while if $F_0 > 1$, then $F(t)$ is strictly increasing. Therefore* $F(t) \in [\min\{F_0, 1\}, \max\{F_0, 1\}]$*;*
(iii) $\lim_{t\to+\infty} G'(t) = c^*$*;*
(iv) *there exists $x_0 \in \mathbb{R}$ such that* $\lim_{t\to+\infty}(G(t) - c^* t) = x_0$*, and the convergence rate is exponential.*

The conditions on $A(F, G, t)$ and $B(F, G, t)$ can be relaxed to be locally satisfied with respect to F:

$$0 < \inf_{F\in[F_1,F_2],G\in\mathbb{R},t>0} A(F, G, t) \le \sup_{F\in[F_1,F_2],G\in\mathbb{R},t>0} A(F, G, t) < +\infty,$$

and

$$\sup_{F\in[F_1,F_2],G\in\mathbb{R},t>0} B(F, G, t) < +\infty,$$

for any $0 < F_1 < F_2 < +\infty$.

Proof The positivity of $\varepsilon_1 A(F, G, t)$ shows the monotonicity of $F(t)$ as (ii). Therefore, $F(t) \in [\min\{F_0, 1\}, \max\{F_0, 1\}]$, and $A(F(t), G(t), t)$ has positive infimum. This yields (i) and further (iii). According to (2.1), we have

$$\frac{G'(t) - c^*}{F'(t)} = \frac{c^*(F^{m-1}(t) - 1) - \varepsilon_2 B(F, G, t)(1 - F(t))}{\varepsilon_1 A(F, G, t) F(t)(1 - F(t))},$$

where the right hand side is uniformly bounded by some positive constant L (using L'Hospital's rule as F tends to 1). Therefore, $|G'(t) - c^*| \le L|F'(t)|$. Noticing that $F(t)$ is monotone and exponentially converges to 1, we find that $|F'(t)|$ and then $|G'(t) - c^*|$ are integrable. This implies (iv). □

Lemma 2.2 *Under the growth condition (1.4), there holds*

$$g(\lambda u) < \lambda^m g(u), \quad \forall \lambda > 1,\ \forall u \in (0, 1). \tag{2.2}$$

Furthermore, define the monotone increasing part of $g(u)$ *as*

$$g^*(u) := \max_{s \le u} g(s), \quad \forall u \ge 0, \tag{2.3}$$

and define the following function for $\lambda \in (0, +\infty)$ *and* $u \in (0, 1)$

$$H(\lambda, u) := \begin{cases} \dfrac{\lambda^m g(u) - g(\lambda u)}{(\lambda - 1) g^*(u)}, & \lambda \ne 1, \\ \dfrac{m g(u) - g'(u) u}{g^*(u)}, & \lambda = 1. \end{cases}$$

Then $H(\lambda, u)$ *is positive and continuous on the open set* $(\lambda, u) \in (0, +\infty) \times (0, 1)$.

Proof The continuity of $H(\lambda, u)$ with respect to λ at $\lambda = 1$ is easily obtained by the L'Hospital's rule and the positivity of $H(\lambda, u)$ at $\lambda = 1$ is directly shown by the growth condition (1.4). According to (1.4), we have

$$(\ln g(u))' < (\ln u^m)'. \tag{2.4}$$

For any $\lambda > 1$, integrating the above inequality (2.4) over $(u, \lambda u)$ yields

$$\frac{g(\lambda u)}{g(u)} < \lambda^m, \quad \forall \lambda > 1, \quad \forall u \in (0, 1). \tag{2.5}$$

This implies the positivity of $H(\lambda, u)$ for $\lambda > 1$. Taking $\mu := \lambda^{-1}$ and $\hat{u} := \lambda u$ in (2.5), further, we have

$$\frac{g(\hat{u})}{g(\mu \hat{u})} < \mu^{-m}, \quad \forall \mu \in (0, 1), \quad \forall \hat{u} \in (0, 1).$$

That is,

$$\frac{g(\lambda u)}{g(u)} > \lambda^m, \quad \forall \lambda \in (0, 1), \quad \forall u \in (0, 1),$$

which shows the positivity of $H(\lambda, u)$ for $\lambda \in (0, 1)$. □

Lemma 2.3 *For any positive constants* $\delta_0, \delta_2 > 0$*, there exists a* $\delta_1^* > 0$ *such that for all* $0 < \delta_1 \le \delta_1^*$*, the following inequality is true*

$$-\delta_1 \phi_{c^*}(\xi) - \delta_2 \phi'_{c^*}(\xi) + \delta_0 g^*(\phi_{c^*}(\xi)) \ge 0, \quad \forall \xi \in \mathbb{R},$$

where $g^*(u)$ *is defined in Lemma 2.2.*

Proof Note that $\phi_{c^*}(\xi) \equiv 0$, for all $\xi \geq 0$, we only need to consider the case $\xi < 0$. According to Proposition 1.1, $\phi_{c^*}(-\infty) = 1$, $\phi_{c^*}(\xi)$ decreases to 0 as $\xi \to 0^-$, and more precisely, $\phi'_{c^*}(\xi) \sim -\frac{c^*-f'(0)}{m}\phi_{c^*}^{2-m}(\xi)$, with $c^* > f'(0)$. Therefore, the function

$$J(\xi) := -\delta_2\phi'_{c^*}(\xi) + \delta_0 g^*(\phi_{c^*}(\xi))$$

behaves similarly as $\delta_2\frac{c^*-f'(0)}{m}\phi_{c^*}^{2-m}(\xi) + \delta_0 g^*(\phi_{c^*}(\xi))$ when $\xi \to 0^-$. It follows that

$$\liminf_{\xi\to 0^-}\frac{J(\xi)}{\phi_{c^*}(\xi)} \geq \liminf_{\xi\to 0^-}\frac{\delta_2\frac{c^*-f'(0)}{m}\phi_{c^*}^{2-m}(\xi)}{\phi_{c^*}(\xi)} \geq \liminf_{\xi\to 0^-}\delta_2\frac{c^*-f'(0)}{m}\phi_{c^*}^{1-m}(\xi) = +\infty,$$

and

$$\liminf_{\xi\to -\infty}\frac{J(\xi)}{\phi_{c^*}(\xi)} = \liminf_{\xi\to -\infty}\frac{\delta_0 g^*(\phi_{c^*}(\xi))}{\phi_{c^*}(\xi)} = \delta_0\max_{u\in[0,1]} g(u) > 0.$$

The above asymptotic analysis, together with the continuity and positivity of the function $\frac{J(\xi)}{\phi_{c^*}(\xi)}$ over $(-\infty, 0)$, imply the existence of a positive constant $\delta_1^* > 0$ such that $J(\xi) \geq \delta_1^*\phi_{c^*}(\xi)$ for all $\xi < 0$. □

The following constructions of sub- and super-solutions with semi-compact supports for degenerate diffusion equations with reaction and convection are useful for the convergence results.

Lemma 2.4 *Define*

$$W(x,t) := F(t)\phi_{c^*}(x - G(t)), \tag{2.6}$$

where $F(t)$ and $G(t)$ are the solutions in Lemma 2.1 with $A(F,G,t) \equiv 1 \equiv B(F,G,t)$ and initial data (F_0, G_0). Further assume that ε_1 is sufficiently small and ε_2 is sufficiently large. Then $W(x,t)$ is a sub-solution to (1.1) if $F_0 < 1$; $W(x,t)$ is a super-solution to (1.1) if $F_0 > 1$. Moreover there exists $x_0 \in \mathbb{R}$ such that

$$\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}}|W(\xi + c^*t, t) - \phi_{c^*}(\xi - x_0)| = 0,$$

and the convergence rate is exponential.

Proof Since $\phi_{c^*}(x - c^*t)$, with $\xi = x - c^*t$, is the sharp traveling wave, we have (we write $\phi = \phi_{c^*}$ in this proof for simplicity)

$$-c^*\phi' + f'(\phi)\phi' = (\phi^m)'' + g(\phi), \quad \xi < 0. \tag{2.7}$$

Note that $\phi(\xi) \equiv 0$ for all $\xi \geq 0$, and $\phi(\xi) > 0$ for all $\xi < 0$. It suffices to prove that for any $t > 0$ and $x < G(t)$,

$$\begin{aligned}\mathscr{L}(F(t)\phi(\eta)) :=& F'(t)\phi(\eta) - F(t)\phi'(\eta)G'(t) + f'(F(t)\phi(\eta))F(t)\phi'(\eta) \\ & - F^m(t)(\phi^m)''(\eta) - g(F(t)\phi(\eta)) \leq 0,\end{aligned} \tag{2.8}$$

where $\eta := x - G(t) < 0$. Substituting (2.7) into (2.8), we obtain

$$\begin{aligned}\mathscr{L}(F(t)\phi(\eta)) =& F'(t)\phi(\eta) - (F(t)G'(t) - c^* F^m(t))\phi'(\eta) \\ & + (f'(F(t)\phi(\eta))F(t) - f'(\phi)F^m(t))\phi'(\eta) + F^m(t)g(\phi) \\ & - g(F(t)\phi(\eta)).\end{aligned}$$

According to the differential system (2.1), we further have

$$\begin{aligned}\mathscr{L}(F(t)\phi(\eta)) =& \varepsilon_1 F(t)(1 - F(t))\phi(\eta) + \varepsilon_2 F(t)(1 - F(t))\phi'(\eta) \\ & + (f'(F(t)\phi(\eta))F(t) - f'(\phi)F^m(t))\phi'(\eta) + F^m(t)g(\phi) \\ & - g(F(t)\phi(\eta)).\end{aligned}$$

Define

$$\Phi(F, \phi) := f'(F\phi)F - f'(\phi)F^m, \quad \Psi(F, \phi) := F^m g(\phi) - g(F\phi).$$

Using L'Hospital's rule, we find

$$\begin{aligned}\lim_{F \to 1} \frac{\Phi(F, \phi)}{F - 1} &= \lim_{F \to 1} \frac{f'(F\phi)F - f'(\phi)F^m}{F - 1} \\ &= \lim_{F \to 1} (f'(F\phi) - f''(F\phi)\phi F - mf'(\phi)F^{m-1}) \\ &= f'(\phi) - f''(\phi)\phi - mf'(\phi),\end{aligned}$$

and

$$\begin{aligned}\lim_{F \to 1} \frac{\Psi(F, \phi)}{F - 1} &= \lim_{F \to 1} \frac{F^m g(\phi) - g(F\phi)}{F - 1} \\ &= \lim_{F \to 1} (mF^{m-1}g(\phi) - g'(F\phi)\phi) \\ &= mg(\phi) - g'(\phi)\phi.\end{aligned}$$

According to Lemma 2.2, the function

$$H(F, \phi) = \frac{\Psi(F, \phi)}{(F - 1)g^*(\phi)}$$

is positive and continuous on $[\min\{F_0, 1\}, \max\{F_0, 1\}] \times (0, 1)$. The analysis of the limits as $\phi \to 0^+$ and $\phi \to 1^-$ shows that $H(F, \phi)$ has positive infimum $\delta_0 > 0$.

Similar analysis shows that

$$\left|\frac{\Phi(F,\phi)}{F-1}\right| \leq M_0, \quad \forall (F,\phi) \in [\min\{F_0,1\}, \max\{F_0,1\}] \times (0,1),$$

for some $M_0 > 0$. Therefore, we can estimate

$$\begin{aligned}\mathscr{L}(F(t)\phi(\eta)) = (F(t)-1)\cdot\Big(&-\varepsilon_1 F(t)\phi(\eta) - \varepsilon_2 F(t)\phi'(\eta)\\ &-\frac{\Phi(F(t),\phi(\eta))}{F(t)-1}\phi'(\eta) + H(F(t),\phi(\eta))g^*(\phi)\Big)\\ =:&\ (F(t)-1)\cdot K(F(t),\phi(\eta),t),\end{aligned}$$

where (note that $\phi'(\eta) < 0$ at $\eta < 0$)

$$\begin{aligned}K(F(t),\phi(\eta),t) &\geq -\varepsilon_1 F(t)\phi(\eta) - \varepsilon_2 F(t)\phi'(\eta) + M_0\phi'(\eta) + \delta_0 g^*(\phi)\\ &\geq -\varepsilon_1 \max\{F_0,1\}\phi(\eta) - \varepsilon_2 \min\{F_0,1\}\phi'(\eta) + M_0\phi'(\eta) + \delta_0 g^*(\phi)\\ &\geq 0,\end{aligned}$$

provided that $\varepsilon_1 \max\{F_0,1\}$ is sufficiently small, $\varepsilon_2 \min\{F_0,1\} > M_0$, according to Lemma 2.3. Therefore, $\mathscr{L}(W(x,t)) = \mathscr{L}(F(t)\phi(\eta))$ has the same sign as $F(t)-1$, and $W(x,t)$ is a sub- or super-solution to (1.1) for $F_0 > 1$ or $0 < F_0 < 1$ respectively.

The convergence follows from the properties of $F(t)$ and $G(t)$ as proved in Lemma 2.1. In fact,

$$\begin{aligned}&\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}} |W(\xi + c^*t, t) - \phi_{c^*}(\xi - x_0)|\\ =&\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}} |F(t)\phi_{c^*}(\xi + c^*t - G(t)) - \phi_{c^*}(\xi - x_0)|\\ \leq&\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}} \big(|(F(t)-1)\phi_{c^*}(\xi + c^*t - G(t))| + |\phi_{c^*}(\xi + c^*t - G(t)) - \phi_{c^*}(\xi - x_0)|\big)\\ \leq&\lim_{t\to+\infty}|F(t)-1| + \lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}}|\phi_{c^*}(\xi + c^*t - G(t)) - \phi_{c^*}(\xi - x_0)|\\ =&0,\end{aligned}$$

since $\lim_{t\to+\infty} F(t) = 1$ and $\lim_{t\to+\infty}(G(t) - c^*t) = x_0$. Moreover, the convergence is exponential. □

Lemma 2.5 *Let $u(x,t)$ be the solution of the Cauchy problem (1.1) and (1.2) with initial data $u_0 \in \mathscr{A}$. Assume that u_0 satisfies the following profile condition: there exist $0 < \delta_1 < 1$, $\delta_2 > 1$, $X_1 > 0$, $X_2 > 0$, and $x_0 > 0$ such that*

$$\delta_1\phi_{c^*}(x - x_0 + X_1) < u_0(x) < \delta_2\phi_{c^*}(x - x_0 - X_2). \tag{2.9}$$

Then there exist $\omega_1(\delta_1) \geq 0$ and $\omega_2(\delta_2) \geq 0$ such that for any $t > 0$, there holds

$$\delta_1\phi_{c^*}(\xi - x_0 + X_1 + \omega_1(\delta_1)) < u(\xi + c^*t, t) < \delta_2\phi_{c^*}(\xi - x_0 - X_2 - \omega_2(\delta_2)),$$

where $\omega_1(\delta_1) > 0$, $\omega_2(\delta_2) > 0$, *and* $\lim_{\delta_1\to 1^-}\omega_1(\delta_1) = 0$, $\lim_{\delta_2\to 1^+}\omega_2(\delta_2) = 0$.

Proof The Lyapunov-type stability results follows from the idea of Biró (2002) for the case without convection. According to Lemma 2.4, the following two functions are sub- and super-solutions

$$W_1(x,t) := F_1(t)\phi_{c^*}(x - x_0 - G_1(t)), \quad W_2(x,t) := F_2(t)\phi_{c^*}(x - x_0 - G_2(t)),$$

with $F_1(0) = \delta_1$, $G_1(0) = -X_1$, and $F_2(0) = \delta_2$, $G_2(0) = X_2$. The profile condition (2.9) implies the comparison of the initial data. Thus by the comparison principle of parabolic equations, we have

$$F_1(t)\phi_{c^*}(x - x_0 - G_1(t)) \le u(x,t) \le F_2(t)\phi_{c^*}(x - x_0 - G_2(t)).$$

That is,

$$F_1(t)\phi_{c^*}(\xi + c^*t - x_0 - G_1(t)) \le u(\xi + c^*t, t) \le F_2(t)\phi_{c^*}(\xi + c^*t - x_0 - G_2(t)). \tag{2.10}$$

Lemma 2.1 shows that $G_2(t) - c^*t$ and $G_1(t) - c^*t$ both converges exponentially. In fact,

$$\begin{aligned} G_2(t) - c^*t =& X_2 + \int_0^t (G_2'(\tau) - c^*)\mathrm{d}\tau \\ =& X_2 + \int_0^t \left(c^*(F_2^{m-1}(\tau) - 1) - \varepsilon_2(1 - F_2(\tau))\right)\mathrm{d}\tau \\ =& X_2 + \int_0^t \left(c^*(F_2^{m-1}(\tau) - 1) - \frac{\varepsilon_2}{\varepsilon_1}\frac{F_2'(\tau)}{F_2(\tau)}\right)\mathrm{d}\tau. \end{aligned} \tag{2.11}$$

Note that $F_2(t)$ decreases to 1, $F_2(t) \in (1, \delta_2]$,

$$-\frac{\varepsilon_2}{\varepsilon_1}\int_0^t \frac{F_2'(\tau)}{F_2(\tau)}\mathrm{d}\tau = -\frac{\varepsilon_2}{\varepsilon_1}\ln(F_2(t))\Big|_0^t = -\frac{\varepsilon_2}{\varepsilon_1}\ln\frac{F_2(t)}{\delta_2} \le \frac{\varepsilon_2}{\varepsilon_1}\ln(\delta_2).$$

Further,

$$\begin{aligned} \left|\int_0^t c^*(F_2^{m-1}(\tau) - 1)\mathrm{d}\tau\right| =& \int_0^t |c^*|\frac{F_2^{m-1}(\tau) - 1}{F_2(t) - 1}(F_2(t) - 1)\mathrm{d}\tau \\ \le& |c^*| \max_{s\in[1,\delta_2]}\frac{s^{m-1} - 1}{s - 1}\cdot\int_0^t \frac{-F_2'(\tau)}{\varepsilon_1 F_2(\tau)}\mathrm{d}\tau \\ \le& \max_{s\in[1,\delta_2]}\frac{s^{m-1} - 1}{s - 1}\cdot\frac{|c^*|}{\varepsilon_1}\ln(\delta_2). \end{aligned}$$

Substitute the above two estimates into (2.11),

$$G_2(t) - c^*t \le X_2 + \frac{\varepsilon_2}{\varepsilon_1}\ln(\delta_2) + \max_{s\in[1,\delta_2]}\frac{s^{m-1}-1}{s-1}\cdot\frac{|c^*|}{\varepsilon_1}\ln(\delta_2) =: X_2 + \omega_2(\delta_2).$$

Similarly, note that $F_1(t)$ increases to 1 and $F_1(t) \in [\delta_1, 1)$,

$$\begin{aligned} c^*t - G_1(t) &= X_1 + \int_0^t (c^* - G_1'(\tau))\mathrm{d}\tau \\ &= X_1 + \int_0^t \Big(c^*(1 - F_1^{m-1}(\tau)) + \varepsilon_2(1 - F_1(\tau))\Big)\mathrm{d}\tau \\ &\le X_1 + \int_0^t \left(|c^*|\frac{1 - F_1^{m-1}(\tau)}{1 - F_1(\tau)} + \varepsilon_2\right)\cdot(1 - F_1(\tau))\mathrm{d}\tau \\ &\le X_1 + \left(|c^*|\max_{s\in[\delta_1,1]}\frac{1 - s^{m-1}}{1-s} + \varepsilon_2\right)\int_0^t \frac{F_1'(\tau)}{\varepsilon_1 F_1(\tau)}\mathrm{d}\tau \\ &= X_1 + \left(|c^*|\max_{s\in[\delta_1,1]}\frac{1 - s^{m-1}}{1-s} + \varepsilon_2\right)\cdot\frac{1}{\varepsilon_1}\ln\frac{F_1(t)}{\delta_1} \\ &\le X_1 + \left(|c^*|\max_{s\in[\delta_1,1]}\frac{1 - s^{m-1}}{1-s} + \varepsilon_2\right)\cdot\frac{1}{\varepsilon_1}\ln\frac{1}{\delta_1} \\ &=: X_1 + \omega_1(\delta_1). \end{aligned} \tag{2.12}$$

The proof is completed according to (2.10), the monotonicity of $\phi_{c^*}(\xi)$, and the estimates (2.11), (2.12). □

Now we are ready to show the convergence results.

Lemma 2.6 *Let $u(x,t)$ be the solution of the Cauchy problem (1.1) and (1.2) with initial data $u_0 \in \mathscr{A}$. Then there exists a $x_0 \in \mathbb{R}$ such that $u(x,t)$ converges in form and in speed to $\phi_{c^*}(x - c^*t - x_0)$: for any $0 < \varepsilon < 1$, there exists $T > 0$ such that*

$$\begin{aligned} (1-\varepsilon)\phi_{c^*}(\xi - x_0 + \omega_1(1-\varepsilon) + \varepsilon) &\le u(\xi + c^*t, t) \\ &\le (1+\varepsilon)\phi_{c^*}(\xi - x_0 - \omega_2(1+\varepsilon) - \varepsilon), \end{aligned} \tag{2.13}$$

for all $t \ge T$ and $\xi \in \mathbb{R}$, where ω_1 and ω_2 are defined in Lemma 2.5.

Proof The case of $f(u) \equiv 0$ and $g(u) = u^p - u^q$ with $1 \le p < \min\{m, q\}$ was proved by Z. Biró in Biró (2002). We show that the attractiveness property holds true for general convection $f(u)$ and reaction $g(u)$ once the sub- and super-solutions proved in Lemma 2.4 are constructed. For any $u_0 \in \mathscr{A}$, there exist sub- and super-solutions in the form (2.6):

$$W_1(x,t) := F_1(t)\phi_{c^*}(x - x_1 - G_1(t)), \quad W_2(x,t) := F_2(t)\phi_{c^*}(x - x_2 - G_2(t)),$$

such that $W_1(x,0) < u_0(x) < W_2(x,0)$. In fact, this is achieved by choosing $F_2(0) > \sup_{x\in\mathbb{R}} u_0(x)$ and $0 < F_1(0) < \liminf_{x\to-\infty} u_0(x)$, and then shifting x_1 and x_2. Similar to Lemma 2.5, there exist ξ_1 and ξ_2 such that

$$F_1(0)\phi_{c^*}(\xi-\xi_1) \le u(\xi+c^*t,t) \le F_2(0)\phi_{c^*}(\xi-\xi_2), \quad t>0,\ \xi\in\mathbb{R}, \tag{2.14}$$

which presents a preliminary outline of the evolution edge of the support.

Denote $z(\xi,t) := u(\xi+c^*t,t)$ in the moving coordinates. For any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = +\infty$, denote $z_n(\xi) := u(\xi+c^*t_n,t_n)$. The compact analysis shows the existence of a function $z(\xi) \in C^\alpha(\mathbb{R})$ and a convergent subsequence of $\{z_n\}$, denoted by $\{z_n\}$ itself for simplicity, such that

$$\lim_{n\to\infty}\sup_{\xi\in\mathbb{R}} |z_n(\xi)-z(\xi)| = 0. \tag{2.15}$$

Testing the equation $z_t - c^*z_\xi + f(z)_\xi = (z^m)_{\xi\xi} + g(z)$ by any smooth function $\varphi(\xi,t)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\Big(z(\xi,t_b)\varphi(\xi,t_b)-z(\xi,t_a)\varphi(\xi,t_a)\Big)\mathrm{d}\xi-\int_{t_a}^{t_b}\int_{\mathbb{R}} z(\xi,t)\varphi_t(\xi,t)\mathrm{d}\xi\mathrm{d}t\\ =&\int_{t_a}^{t_b}\int_{\mathbb{R}}\Big(z^m(\xi,t)\varphi_{\xi\xi}(\xi,t)-\big(c^*z(\xi,t)-f(z(\xi,t))\big)\varphi_\xi(\xi,t)+g(z(\xi,t))\varphi(\xi,t)\Big)\mathrm{d}\xi\mathrm{d}t,\end{aligned}$$

for any $0 < t_a < t_b < +\infty$. Let $t_a = t_n$, $t_b = t_{n+1}$, and choose $\varphi(\xi,t)$ independent of t. Mean Value Theorem shows the existence of $\theta_n \in (t_n, t_{n+1})$, such that

$$\begin{aligned}&\int_{\mathbb{R}}\Big(z(\xi,t_{n+1})-z(\xi,t_n)\Big)\varphi(\xi)\mathrm{d}\xi\\ =&(t_{n+1}-t_n)\int_{\mathbb{R}}\Big(z^m(\xi,\theta_n)\varphi''(\xi)-\big(c^*z(\xi,\theta_n)-f(z(\xi,\theta_n))\big)\varphi'(\xi)+g(z(\xi,\theta_n))\varphi(\xi)\Big)\mathrm{d}\xi.\end{aligned}$$

The compactness of $\{z(\xi,\theta_n)\}$ and its free boundary $\{\zeta_n\}$, with $\zeta_n := \sup_\xi\{z(\xi,\theta_n) > 0\} \in [\xi_1,\xi_2]$ according to (2.14), together with the uniform convergence (2.15), imply the following convergence (passing to subsequences if necessary)

$$\lim_{n\to\infty}\zeta_n = x_0, \quad \lim_{n\to\infty} z(\xi,\theta_n) = w(\xi), \tag{2.16}$$

for some $x_0 \in [\xi_1,\xi_2]$ and $w(\xi) \in C^\alpha(\mathbb{R})$ such that

$$\int_{\mathbb{R}}\Big(w^m(\xi)\varphi''(\xi)-\big(c^*w(\xi)-f(w(\xi))\big)\varphi'(\xi)+g(w(\xi))\varphi(\xi)\Big)\mathrm{d}\xi = 0.$$

Therefore, $w(\xi)$ is the shift of the unique sharp traveling wave $\phi_{c^*}(\xi)$, i.e.,

$$w(\xi) = \phi_{c^*}(\xi - x_0).$$

According to Lemma 4 of Biró (2002), for any $0 < \varepsilon < 1$, there exists $\delta > 0$ such that $\mu(\delta) < \varepsilon$, let n be sufficiently large in (2.16) such that

$$|\zeta_n - x_0| < \delta, \quad |z(\xi, \theta_n) - \phi_{c^*}(\xi - x_0)| < \delta.$$

Take $T = \theta_n$, using Lemma 4 of Biró (2002) again, we see that

$$(1-\varepsilon)\phi_{c^*}(\xi - x_0 + \varepsilon) < z(\xi, T) = u(\xi + c^*T, T) < (1+\varepsilon)\phi_{c^*}(\xi - x_0 - \varepsilon).$$

Applying Lemma 2.5 (starting from time T), we obtain

$$\begin{aligned}(1-\varepsilon)\phi_{c^*}(\xi - x_0 + \varepsilon + \omega_1(1-\varepsilon)) &< u(\xi + c^*t, t)\\ &< (1+\varepsilon)\phi_{c^*}(\xi - x_0 - \varepsilon - \omega_2(1+\varepsilon)),\end{aligned}$$

for all $t \geq T$. The proof is completed. □

Proof of Theorem 1.1. This is proved in Lemma 2.6. □

We prove the almost exponential convergence rates of the attractiveness of sharp traveling wave.

Proof of Theorem 1.2. Lemma 2.5 shows that $\lim_{\delta_1 \to 1^-} \omega_1(\delta_1) = 0$, $\lim_{\delta_2 \to 1^+} \omega_2(\delta_2) = 0$. For any $\varepsilon > 0$, we choose $0 < \varepsilon_0 < \varepsilon$, where ε_0 will be determined latter. There exists $0 < \hat{\varepsilon} < \varepsilon_0$ such that $\hat{\varepsilon} + \omega_1(1-\hat{\varepsilon}) < \varepsilon_0$ and $\hat{\varepsilon} + \omega_2(1+\hat{\varepsilon}) < \varepsilon_0$. For this $\hat{\varepsilon}$, Lemma 2.6 implies the existence of $T > 0$ such that

$$\begin{aligned}(1-\hat{\varepsilon})\phi_{c^*}(\xi - x_0 + \omega_1(1-\hat{\varepsilon}) + \hat{\varepsilon}) &\leq u(\xi + c^*t, t)\\ &\leq (1+\hat{\varepsilon})\phi_{c^*}(\xi - x_0 - \omega_2(1+\hat{\varepsilon}) - \hat{\varepsilon}),\end{aligned}$$

for all $t \geq T$ and $\xi \in \mathbb{R}$. Therefore,

$$(1-\varepsilon_0)\phi_{c^*}(\xi - x_0 + \varepsilon_0) \leq u(\xi + c^*t, t) \leq (1+\varepsilon_0)\phi_{c^*}(\xi - x_0 - \varepsilon_0), \quad \forall t \geq T.$$

Starting from time T, we construct sub- and super-solutions

$$\begin{aligned}W_1(x,t) &:= F_1(t)\phi_{c^*}(x - x_0 - c^*T - G_1(t)),\\ W_2(x,t) &:= F_2(t)\phi_{c^*}(x - x_0 - c^*T - G_2(t)),\end{aligned}$$

with $F_1(T) = 1 - \varepsilon_0$, $F_2(T) = 1 + \varepsilon_0$, and $G_1(T) = -\varepsilon_0$, $G_2(T) = \varepsilon_0$. Comparison principle implies for all $t > T$,

$$F_1(t)\phi_{c^*}(x - x_0 - c^*T - G_1(t)) \leq u(x,t) \leq F_2(t)\phi_{c^*}(x - x_0 - c^*T - G_2(t)).$$

Lemma 2.1 tells us that both $F_1(t)$ and $F_2(t)$ exponentially converge to 1, both $G_1(t) - c^*(t-T)$ and $G_2(t) - c^*(t-T)$ exponentially converge.

According to the proof of Lemma 2.5 (see the estimates of (2.11) and (2.12)), both $|G_1(t)-c^*(t-T)|$ and $|G_2(t)-c^*(t-T)|$ are $O(\varepsilon_0)$ as $\varepsilon_0 \to 0^+$. We may take $\varepsilon_0 > 0$ smaller such that $|G_1(t) - c^*(t - T)| < \varepsilon/2$ and $|G_2(t) - c^*(t - T)| < \varepsilon/2$. Thus, both free boundaries of sub- and super-solutions remains in the $\varepsilon/2$ neighborhood of $x_0 + c^*t$, and exponentially converge to some point within this neighborhood. Together with the exponential convergence of the altitudes $F_1(t)$ and $F_2(t)$, the proof is completed. □

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