Geometric group theory, homework 1.

You do not need to hand in homework. At the start of the tutorial part of the course you will be asked which problems you have solved and might be chosen to show a declared solution on board.

Recall that *Gruško Theorem* states that a finite rank free group F with an epimorphism $\phi\colon F \to G_1 * G_2$ decomposes as $F_1 * F_2$, where $\phi(F_i) = G_i$.

Problem 1. Let $\mu(G)$ be the minimal cardinality of a generating set of a group G. Show

$$\mu(G_1 * G_2) = \mu(G_1) + \mu(G_2).$$

A group is *indecomposable* if it does not decompose as $A * B$ with non-trivial A and B.

Problem 2. Let G be a finitely generated group. Show that for some n we have $G = G_1 * \ldots * G_n$, where G_i are indecomposable.

Problem 3. Prove *Kuroš Theorem*: If H is a subgroup of $G_1 * G_2$, then it decomposes as the free product of a free group and conjugates of subgroups of G_i.

Problem 4. Let $G = G_1 * G_2$. Suppose that for some $g, h \in G$, the commutator $[g, h]$ is a nontrivial element of G_1. Show that g and h belong to G_1 as well. Hint: apply Kuroš Theorem and Gruško Theorem to describe $\langle g, h\rangle \subset G$

Problem 5. Show that each indecomposable subgroup of $G_1 * G_2$ is contained in some conjugate of G_1 or G_2 or is infinite cyclic (i.e. isomorphic to $\mathbf{Z}$).

Problem 6. Show that if $G = G_1 * G_2$ and the intersection $w^{-1}G_1w \cap G_i$ is nontrivial for some $w \in G,\ i = 1, 2$, then we have $i = 1$ and $w \in G_1$.

Problem 7. (unique decomposition theorem) Let G be a finitely generated group. Show that the decomposition from Problem 2 is unique in the following sense. Assume $G = G_1 * \ldots * G_n = H_1 * \ldots * H_m$, where G_i, H_j are indecomposable. Then $m = n$ and, possibly after interchanging the indices, G_i is isomorphic to H_i. Moreover, for each G_i not isomorphic to $\mathbf{Z}$, G_i is conjugate to H_i. Hint: use Problems 5 and 6.