Geometric group theory, homework 2.

Problem 1. Describe the Bass–Serre tree of $\mathbf{Z} * \mathbf{Z}$.

Problem 2. Let $G_0 = G_1 = G_2 = \mathbf{Z}$, and let $\varphi^1 = \varphi^2$ be the multiplication by 2. Describe the simplest possible space $X = X_1 \cup (X_0 \times [0,1]) \cup X_2$ from class. Describe $\widetilde{X}$ as the "tree of spaces".

Definition. Assume that we have two injective homomorphisms ϕ^1, ϕ^2 of a group G_0 into a group G_1. The HNN-*extension* $G_1 *_{G_0}$ is the group

$$G_1 * \mathbf{Z}/_{\langle\langle \phi^1(g)t = t\phi^2(g) \rangle\rangle},$$

where t is the generator of $\mathbf{Z}$, and we quotient by the normal closure of $t^{-1}\phi^1(g)t\phi^2(g)^{-1}$ over all $g \in G_0$.

Problem 3. Prove the following variant of van Kampen Theorem (using the standard van Kampen Theorem). Suppose that we have path connected based CW complexes X_0, X_1 with fundamental groups G_0, G_1 and cellular based maps $f^1, f^2 \colon X_0 \to X_1$ satisfying $f^i_\sharp = \phi^i$. Consider the CW complex

$$X = X_1 \cup X_0 \times [0,1]/\sim,$$

where we identify

$$X_0 \times [0,1] \ni (x,0) \sim f^1(x) \in X_1,$$
$$X_0 \times [0,1] \ni (x,1) \sim f^2(x) \in X_1.$$

Show that we have $\pi_1(X) = G_1 *_{G_0}$.

Problem 4. Find a nontrivial (i.e. without a global fixed point) isometric action of $G_1 *_{G_0}$ on the real line $\mathbf{R}$.

Problem 5. Find a tree with $G_1 *_{G_0}$ action so that vertex stabilizers are subgroups conjugate to G_1 and edge stabilizers are subgroups conjugate to G_0.

Problem 6. Show that $G_1 \to G_1 *_{G_0}$ is an embedding.

Definition. A group G is *Hopfian* if every epimorphism $G \to G$ is an isomorphism.

Problem 7. Consider $G_0 = G_1 = \mathbf{Z}$ and $\phi^1(n) = 2n$, $\phi^2(n) = 3n$. Show that $G_1 *_{G_0}$ is not Hopfian.

Hint: use an epimorphism whose restriction to G_1 is the multiplication by 2, and which maps t to t. Find an element of the kernel with a nontrivial normal form.

Problem 8. We say that a group is *residually finite* if the intersection of its finite index subgroups is trivial. Show that a finitely generated residually finite group is Hopfian.

Hint: if a group G admits an epimorphism $\varphi\colon G \to G$ with nontrivial element $g \in \mathrm{Ker}(\varphi)$, and $H \subset G$ is a nontrivial finite index subgroup that does not contain g, consider the sequence $\varphi^{-n}(H)$ over $n \geq 0$.