Geometric group theory, homework 4.

Problem 1. Show that a Möbius transformation of the unit disc preserving the center is a Euclidean rotation or reflection. Hint: use the fact that inversions carry circles and lines to circles and lines and that they preserve angles.

Problem 2. Show that a Möbius transformation g of the unit disc preserves the hyperbolic metric. Hint: use Problem 1 to show that it suffices to consider g an inversion and to compare the scaling factor at the center of the disc with the scaling factor at its image under g.

Definition. The *halfspace model* for the hyperbolic plane $\mathbf{H}^2$ consists of the subset of the complex plane $\{z \colon \mathrm{Im} z > 0\}$ with scaling factor $\frac{1}{\mathrm{Im} z}$ at each point z. Note that homotheties centred at the real axis are obviously isometries, and from Problem 3 it will follow that any Möbius transformation of the halfspace model is an isometry.

Problem 3. Show that the Poincaré disc model and the halfspace model for $\mathbf{H}^2$ are isometric. Hint: consider an inversion in the circle of radius $\sqrt{2}$ centred at $-i$. By Problem 2 it suffices to compare the scaling factors in the two models at the fixed point $(\sqrt{2}-1)i$.

Problem 4. Compute the area of an ideal hyperbolic triangle.

Problem 5. Show that in the halfspace model the group of orientation preserving isometries can be identified with $\mathrm{PSL}(2, \mathbf{R})$ acting by homographies.

Problem 6. Show that an isometry of $\mathbf{H}^2$ distinct from the identity is elliptic, parabolic or hyperbolic, depending on whether the absolute value of its trace in $\mathrm{PSL}(2, \mathbf{R})$ is < 2, equal 2, or > 2.

Problem 7. Realise the group presented by

$$\langle p, r, s, t, u \mid p^2 = r^2 = s^2 = t^2 = u^2 = [p, r] = [r, s] = [s, t] = [t, u] = [u, p] = 1\rangle$$

as a subgroup of isometries of $\mathbf{H}^2$.

Problem 8. Compute the area of a right-angled pentagon in $\mathbf{H}^2$.

Problem 9. Let P be a polygon homeomorphic to a disc in the tessellation of $\mathbf{H}^2$ by right-angled pentagons. Estimate the number of pentagons in P by the number of edges on the perimeter of P.