Geometric group theory, homework 5.

Problem 1. Realise the group presented by

$$\langle p, r, s, t \mid prst = pr^{-1}s^{-1}t^{-1} = 1\rangle$$

as a subgroup of isometries of $\mathbf{H}^2$. Show that for this presentation Dehn's Algorithm (with 5 replaced by 3) gives incorrect output.

Problem 2. Realise the group presented by

$$\langle r, s, t \mid r^2s^2t^2 = 1\rangle$$

as a subgroup of isometries of $\mathbf{H}^2$. Show that for this presentation Dehn's Algorithm (with 5 replaced by 4) gives correct output.

Problem 3. Show that the groups in Problem 1 and 2 are isomorphic.

Problem 4. What strongest possible small cancelation conditions are satisfied by groups

(i) $\langle a_1, a_2, b_1, b_2, c_1, c_2 \mid [a_1, a_2][b_1, b_2][c_1, c_2] = 1\rangle$,

(ii) in Problem 1,

(iii) in Problem 2,

(iv) $\langle a, t \mid t^{-1}at = a^2\rangle$,

(v) $\langle r, s, t \mid rst = tsr\rangle$,

(vi) $\langle s, t \mid s^2t^2 = ts\rangle$?

Problem 5. Let X be the presentation complex of a group $G = \langle S|R\rangle$ satisfying $C'(\frac{1}{6})$ such that R does not contain proper powers. Show that X does not admit a homotopically nontrivial map from the 2–sphere.

Definition. Let X be a piecewise Euclidean 2–complex. The *vertex link* at a vertex $v \in X$ is the metric graph with vertices corresponding to edges issuing from v, and edges corresponding to corners of 2–cells at v with the length equal to the angle at the corner. The *girth* of a graph is the length of the shortest loop that is not homotopically trivial.

Problem 6. Suppose that in a presentation of G all relators have length six. Equip the presentation complex with the piecewise Euclidean metric where all the 2–cells are regular hexagons. Show that if the girth of all vertex links is $\geq 2\pi$, then G satisfies $C(6)$.

Problem 7. Let $|S| = 2$ and let $G = \langle S|r\rangle$, where r is a single cyclically reduced word of length L, chosen at random. Show that when $L \to \infty$, the probability that G satisfies $C'(\frac{1}{6})$ tends to 1.