Geometric group theory, homework 6.

Problem 1. Let X be a geodesic metric space whose all geodesic triangles are δ–slim. Find δ' for which all geodesic triangles are δ'–thin.

Hint: let xyz be a δ–slim geodesic triangle and m a point on the side xy. Let m_x be the point on the side xz at the same distance from x as m, and let m_y be the point on the side yz at the same distance from y as m. Show that $|mm_x| \leq 2\delta$ or $|mm_y| \leq 2\delta$. Apply the hint first to m, m_x, m_y in the single fiber of the tripod projection.

Definition. The *Gromov product* of points x, y with respect to a basepoint w is defined by

$$(x|y)_w = \frac{1}{2}(|xw| + |yw| - |xy|).$$

Problem 2. Let X be a δ–hyperbolic metric space. Show that for any points w, x, y, z we have

$$|xy| + |wz| \leq \max\{|xz| + |yw|, |xw| + |yz|\} + 2\delta,$$

and that this is equivalent to

$$(x|y)_w \geq \min\{(x|z)_w, (z|y)_w\} - \delta.$$

Problem 3. Suppose $\langle S|R\rangle$ is a presentation of a hyperbolic group for which Dehn's Algorithm gives correct output. Show that each element of finite order is conjugate to an element whose word length in S is shorter than the length of the longest relator in R. In particular there are only finitely many conjugacy classes of elements of finite order.

Definition. A function $e\colon \mathbf{N} \to \mathbf{N}$ is a *divergence function* for a metric space X if it satisfies the following condition. Suppose that points $x, y, y' \in X$ satisfy $|xy| = |xy'| = k + n$ and that points z, z' lying on some geodesics xy, xy' at distance k from x satisfy $|zz'| > e(0)$. Then the shortest path between y and y' outside the ball $B_{k+n}(x)$ has length at least $e(n)$.

Problem 4. Show that if X is hyperbolic, then there exists an exponential divergence function.

Hint: use the δ–thin triangle condition for the triangle xyy' and point z. Next, let m be the midpoint of the path yy' from the definition of divergence function. Use the δ–thin triangle condition for the triangle $yy'm$. Consider the midpoints of ym and my', etc.

Problem 5. Let X be a δ–hyperbolic metric space. Let $xy \subset X$ be a geodesic of length $\geq 6R + 2\delta$, whose middle segment of length $2R$ is denoted by I. Let $x'y' \subset X$ be another geodesic with midpoint m and endpoints satisfying $|xx'| \leq R, |yy'| \leq R$. Show that we have $|m, I| \leq 2\delta$.

Problem 6. Show that the *Conjugacy Problem* is *decidable* in hyperbolic groups, i.e. given a hyperbolic group $G = \langle S \rangle$ there is an algorithm deciding if words v, u over S represent conjugate elements of G.

Hint: find an algorithm replacing a word with a word representing a conjugate element such that all its cyclic translates are geodesic. Then show that for two words in such form representing conjugate elements, they are either both of length $\leq 4\delta$ or they have cyclic translates conjugated by an element of word length $\leq 2\delta$.