Geometric group theory, homework 7.

Problem 1. Show that the composition of quasi-isometric embeddings is a quasi-isometric embedding and the composition of quasi-isometries is a quasi-isometry.

Problem 2. A quasi-isometry $h\colon X \to X$ is *bounded* if for some constant C we have $|x, h(x)| \leq C$ for every $x \in X$. Show that for each quasi-isometry $f\colon X \to Y$ there is a quasi-isometry $g\colon Y \to X$ such that the compositions $f \circ g$ and $g \circ f$ are bounded.

Problem 3. Let $f\colon X \to Y$ be a quasi-isometry and $g\colon Y \to Z$ a Lipschitz map. Show that g is a quasi-isometric embedding if and only if $g \circ f$ is a quasi-isometric embedding.

Note that for Z a geodesic metric space, applying Problem 3 to $X = \mathbf{Z} \subset \mathbf{R} = Y$, we obtain that each quasi-isometric embedding $\mathbf{Z} \to Z$ extends to a quasi-geodesic $\mathbf{R} \to Z$.

Problem 4. Let $n \geq 3$ and suppose that T, T' are trees with all vertices of degree ≥ 3 and $\leq n$. Show that T and T' are quasi-isometric.

Problem 5. Using the lemma on the stability of quasi-geodesics, show that if X is a geodesic metric space, Y is hyperbolic, and $f\colon X \to Y$ is a quasi-isometry, then X is also hyperbolic.

Problem 6. Find distorted cyclic subgroups of

(i) $B(1,2) = \langle a, t \mid t^{-1}at = a^2 \rangle$,

(ii) the Heisenberg group $\langle s, t, r \mid [s,t] = [s,r] = 1,\ [t,r] = s \rangle$.

Problem 7. Show that in a δ–hyperbolic metric space a 10δ–local geodesic α is a quasi-geodesic.

Hint: use the lemma that a 4δ–local geodesic α is 3δ–close to a geodesic γ with the same endpoints to show that $x, y \in \alpha$ at distance $\leq 3\delta$ from $x', y' \in \gamma$ with $|x'y'| \leq \frac{\delta}{2}$ bound a geodesic subpath xy of α.

Problem 8. Let g be an element of infinite order in a Gromov hyperbolic group. Show that the quotient of the centraliser $C(\langle g \rangle)$ by $\langle g \rangle$ is a finite group. In particular, a Gromov hyperbolic group does not contain $\mathbf{Z} \oplus \mathbf{Z}$.

Hint: consider the geodesic square in the Cayley graph with vertices $\mathrm{Id}, f, g^m f, g^m$ where $|g^m|_S > 2|f|_S + 2\delta$.

Problem 9. Let g be an element of infinite order in a Gromov hyperbolic group. Show that the centraliser $C(\langle g \rangle)$ is of index at most 2 in the normaliser $N(\langle g \rangle)$.