Geometric group theory, homework 8.

Problem 1. Let X be a hyperbolic proper metric space. Show that for any $x_0 \in X$ each point in ∂X is represented by a sequence (x_i) satisfying $|x_k, x_l| = |k - l|$.

Problem 2. Let X be a δ–hyperbolic metric space and $w \in X$. Show that for each $x, y \in \partial X$ represented by $(x_i), (y_j)$ we have

$$(x|y)_w - 2\delta \leq \liminf_{i,j\to\infty} (x_i|y_j)_w.$$

Problem 3. Construct an example showing that 2δ in Problem 2 cannot be removed.

Problem 4. Show that if X is a hyperbolic proper metric space, then ∂X is compact.

Problem 5. Let X, Y be δ–hyperbolic and let $f \colon X \to Y$ be an (L, C)–quasi-isometric embedding. Show that there is $A = A(\delta, C, L)$, such that for all $w, x, y, z \in X$ we have:

(i)

$$\frac{1}{L}(x|y)_w - A \leq (f(x)|f(y))_{f(w)} \leq L(x|y)_w + A,$$

(ii)

$$\frac{1}{L}|(x|y)_w - (y|z)_w)| - A \leq |(f(x)|f(y))_{f(w)} - (f(y)|f(z))_{f(w)}| \leq$$
$$\leq L|(x|y)_w - (y|z)_w)| + A$$

Definition. A map φ is *Hölder* if there are $\alpha > 0, c > 0$ such that

$$|\varphi(x), \varphi(y)| \leq c|x, y|^\alpha.$$

The *dilatation* of a map φ at x is

$$\limsup_{r\to 0} \frac{\sup_{|x,y|=r} |\varphi(x), \varphi(y)|}{\inf_{|x,y|=r} |\varphi(x), \varphi(y)|}.$$

A map φ is *quasiconformal* if the dilatation is uniformly bounded.

Problem 6. Show that a quasi-isometric embedding between Gromov hyperbolic spaces induces a map between their boundaries that is Hölder and quasiconformal.

Problem 7. Let X be a δ-hyperbolic metric space. Let $\gamma, \gamma' \colon [0, \infty) \to X$ be two (C, L)-quasigeodesic rays with $\gamma(0) = \gamma'(0)$ determining the same point at infinity (guaranteed by Problem 6). Show that there is $K = K(\delta, C, L)$ such that for each t there is t' with $|\gamma(t), \gamma'(t')| \leq K$.

Problem 8. (Ping-pong lemma) Let g, h be bijections of a set Ω, one of which is not an involution. Let $A, B \subset \Omega$ be disjoint, nonempty, and such that $g^m(A) \subset B, h^m(B) \subset A$ for all nontrivial g^m, h^m. Show that $\langle g, h\rangle = \langle g\rangle * \langle h\rangle$.