Geometric group theory, homework 9.

Definition. Let S be a finite generating set of a group G. The *growth function* $\beta_{G,S}\colon \mathbf{N} \to \mathbf{N}$ assigns to each n the number of vertices in the Cayley graph $Cay(G,S)$ at distance $\leq n$ from the identity.

Definition. We write $\beta \prec \alpha$ if there are c_1, c_2 satisfying $\beta(n) \leq c_1\alpha(c_2 n)$.We say that the functions α and β are *equivalent* if $\alpha \prec \beta$ and $\beta \prec \alpha$.

Problem 1. Suppose that $Cay(G,S)$ and $Cay(G',S')$ are quasi-isometric. Show that then $\beta_{G,S}$ and $\beta_{G',S'}$ are equivalent. In particular the equivalence class of the growth of a group G does not depend on the generating set. This class is called the *growth* of G.

Problem 2. Find the growth of

(i) $\mathbf{Z}^k$,

(ii) the free group F_k,

(iii) the Heisenberg group $\langle s,t,r | [s,t] = [s,r] = 1,\ [t,r] = s\rangle$,

(iv) the solvable group $\langle \mathbf{Z}^2, t \mid t^{-1}vt = Av \text{ for all } v \in \mathbf{Z}^2\rangle$, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hint: if v is not an eigenvector of A, then for all finite $I \subset \mathbf{N}$ the values $\sum_{i\in I} A^i v$ are distinct.

Problem 3. Show that if the growth of a group G is linear, then G is virtually $\mathbf{Z}$. Hints:

(i) Show that there exists n with $\beta_{G,S}(2n) \leq 2\beta_{G,S}(n) + n$.

(ii) Show that *Cay(G,S)* contains a bi-infinite geodesic γ.

(iii) Show that each vertex v is at distance $< 2n$ from γ. To do that, for simplicity say that in (i) $\beta_{G,S}(n)$ denotes the number of vertices at distance $< n$ from the identity. Consider the closest vertex $w \in \gamma$ to v and balls od radius n centred at the vertices of γ at distance n from w.

Problem 4. Describe $Ends(X)$ for

(i) $\mathbf{R} \times [0,1]$,

(ii) the Cayley graph of $\mathbf{Z}^2$ with the two standard generators,

(iii) the Cayley graph of $\mathbf{Z}_3 * \mathbf{Z}_3$ with the two standard generators,

(iv) $X = \{(x,y) \in \mathbf{R}^2 \mid x \in \mathbf{Z} \text{ or } y = 0\}$ with the metric defined as the infimum of lengths of paths.

Problem 5. Show that the following conditions are equivalent.

(i) G has a finite normal subgroup H such that G/H is $\mathbf{Z}$ or $\mathbf{Z}_2 * \mathbf{Z}_2$.

(ii) $G = H*_H$ where H is finite or $G = A *_H B$ where H is finite of index 2 in A and B.

Theorem (Stallings). *A finitely generated group with ≥ 2 ends can be expressed as a free product with amalgamation or an* HNN-*extension over a finite group.*

Problem 6. Show that the conditions in Problem 5 are equivalent to G being virtually $\mathbf{Z}$.

Problem 7. Let G be a finitely generated torsion-free virtually free group. Show that G is free.