Geometric group theory
preparation to Midterm I.

Problem 1. Show that each torsion free group generated by 3 elements that splits (nontrivially) into free product, admits an epimorphism onto $\mathbf{Z}$.

Problem 2. Show that each free group of finite rank is Hopfian.

Problem 3. Show that the group $G = \langle g_0, h_0, g_1, h_1, \ldots | g_{i-1} = [g_i, h_i] \text{ for all } i\rangle$ cannot be expressed as a free product of indecomposable subgroups. Hints:

(i) Show that $G = \mathbf{Z} * G$.

(ii) Use the same proof as for the unique decomposition theorem (Problem 7 from homework 1) to show that G is not a free product of a finite number of factors.

(iii) If G splits as a free product of infinite number of factors, then show that we have $g_0 \in A$ for some nontrivial decomposition $G = A * B$. Use Problem 4 from homework 1 to reach a contradiction.

Problem 4 (Higman and Neumann embedding theorem)**.** Show that every countable group G can be embedded in a 2-generator group. Hints:

(i) Show that the group $G_1 = G * \mathbf{Z}$ is generated by elements of infinite order (call them $y_0, y_1, \ldots$).

(ii) Let $G_2 = \langle G_1, t_0, t_1, \ldots | t_0^{-1} y_0 t_0 = y_1, t_1^{-1} y_1 t_1 = y_2, \ldots\rangle$, obtained from G_1 by a sequence of HNN-extensions. Show that $\{t_i\}$ generate a free subgroup of G_2.

(iii) Find an embedding of the infinite rank free group F_∞ into the free group F_2 of rank 2, so that one generator is common. Prove that the corresponding $G_3 = G_2 *_{F_\infty} F_2$ is generated by 3 elements, among which there are two pairs generating free subgroups.

Problem 5. Show that if H is a finitely generated subgroup of $A *_C B$ or $A *_C$, such that it does not split as a free product with amalgamation and does not admit an epimorphism onto $\mathbf{Z}$, then H is contained in the conjugate of A or B.

Problem 6. Show that in Problem 5 the condition that H is finitely generated cannot be removed, by finding appropriate subgroup H of the HNN-extension $\langle a, t | t^{-1}at = a^2 \rangle$.

Problem 7. Let A, B be finitely generated and nontrivial. Show that finitely generated normal sugroups of $A * B$ are trivial or of finite index.

Problem 8. Let $\mathcal{G}$ be a graph of groups. Let T be the Bass-Serre tree of $\mathcal{G}$ with $\pi_1(\mathcal{G})$ action. Show that the quotient graph of groups $T/\pi_1(\mathcal{G})$ equals $\mathcal{G}$.

Problem 9. Draw Cayley graphs of the following groups and determine, which of them have property (FA).

(i) $\langle s, t, r \mid s^2 = t^2 = r^2 = (st)^2 = (tr)^3 = (rs)^5 = 1 \rangle$,

(ii) $\langle s, t, r \mid s^2 = t^2 = r^2 = (st)^2 = (tr)^4 = (rs)^4 = 1 \rangle$,

(iii) $\langle s, t, r \mid s^2 = t^2 = r^2 = (st)^2 = (tr)^3 = (rs)^6 = 1 \rangle$,

(iv) $\langle s, t, r \mid s^3 = t^2 = r^2 = (tr)^3 = sts^{-1}r = 1 \rangle$,

(v) $\langle s, t \mid t^{-1}st = s^2 \rangle$,

(vi) $\langle \mathbf{Z}^2, t \mid t^{-1}vt = Av$ for all $v \in \mathbf{Z}^2 \rangle$, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Problem 10. Let H be a normal subgroup of G. Show that if both H and G/H have property (FA), then G has property (FA) as well.

Problem 11. Show that each group G that is not finitely generated admits a nontrivial action on a tree. Hint: construct appropriate graph of groups with fundamental group G and consider the action of G on the associated Bass–Serre tree.

Problem 12. Let T be a tree, and let g be a hyperbolic, and h an elliptic isometry of T. Show that if the fixed point set of h is disjoint from the axis of g, then the composition $g \circ h$ is hyperbolic.

Problem 13. Show that each orientation preserving isometry of $\mathbf{H}^2$ can be expressed as a composition of two reflexions.

Problem 14. Show that there does not exists a hyperbolic metric on a torus. Hint: investigate when two isometries of $\mathbf{H}^2$ commute.

Problem 15. Let g be an isometry of $\mathbf{H}^2$. Describe the sets of points $x \in \mathbf{H}^2$ with constant hyperbolic translation length $|x, g(x)|$.

Problem 16. (i) Realise the group presented by

$$G = \langle s, t, r \mid s^2 = r^2 = t^2 = (sr)^4 = (rt)^4 = (st)^4 = 1 \rangle$$

as a subgroup of isometries of $\mathbf{H}^2$.

(ii) Compute the area of the quotient $D = \mathbf{H}^2 / G$.

(iii) Let $P \subset \mathbf{H}^2$ be a polygon homeomorphic to a disc, tiled by D, in the tiling of $\mathbf{H}^2$ by D. Estimate the number of copies of D in P by the number of edges on the boundary of P.

Problem 17. Realise the group presented by

$$\langle a_1, a_2, b \mid [a_1, a_2] b^2 = 1 \rangle$$

as a subgroup of isometries of $\mathbf{H}^2$. Show that for this presentation Dehn's Algorithm gives correct output.

Problem 18. Consider $\mathbf{H}^2$ with the octagonal tiling coming from the closed surface of genus 2.

(i) Find and prove a formula relating the area and the sum of the exterior angles of an annulus obtained as the quotient by $\mathbf{Z}$ of a $\mathbf{Z}$–invariant polygonal strip in $\mathbf{H}^2$.

(ii) Let w, v be boundary words of such an annulus. Assuming that Dehn's Algorithm cannot simplify neither w nor v, what is the possible form of the annulus?

(iii) Show that the Conjugacy Problem is decidable for the fundamental group of the closed genus 2 surface: there is an algorithm which given two words w, v as input decides if they represent conjugate elements.