Geometric group theory
preparation to Test II.

Problem 1. Suppose that in a presentation of G all relators have length four. Equip the presentation complex with the piecewise Euclidean metric where all the 2–cells are squares. Show that if the girth of all vertex links is $\geq 2\pi$, then G satisfies $C(4)$ and $T(4)$.

Problem 2. Find a group presentation satisfying $C(6)$ for which Dehn's Algorithm gives incorrect output.

Problem 3. Let X be a δ–hyperbolic metric space. Let d be the distance of x from a geodesic yz. Show that we have

$$d - \delta \leq (y|z)_x \leq d.$$

Problem 4. Let X be a geodesic metric space. For a geodesic triangle xyz, let $c_x \in yz, c_y \in xz, c_z \in xy$ be the points satisfying $|xc_y| = |xc_z|, |yc_x| = |yc_z|, |zc_x| = |zc_y|$. Show that X is hyperbolic if and only if there is a uniform bound on the diameter of $\{c_x, c_y, c_z\}$.

Problem 5. Let X be a geodesic metric space satisfying

$$|xy| + |wz| \leq \max\{|xz| + |yw|, |xw| + |yz|\} + 2\delta.$$

Find δ' for which X is δ'–hyperbolic.

Problem 6. Let X be a metric space of bounded diameter. Show that X is quasi-isometric to Y if and only if Y has bounded diameter.

Problem 7. Given an exact sequence

$$0 \to F_1 \to G \to H \to F_2 \to 0,$$

where F_1, F_2 are finite, show that G and H are quasi-isometric.

Problem 8. Show that the fundamental groups of all closed surfaces of negative Euler characteristic are quasi-isometric.

Problem 9. Let X be a tree whose vertex degrees are not bounded from above and are bounded from below by 3. Show that X is not quasi-isometric to the tree all of whose vertices have degree 3.

Problem 10. Let

$$0 \to N \to G \to B \to 0$$

be a short exact sequence of infinite groups, where G is hyperbolic. Show that $N \to G$ is not a quasi-isometric embedding.

Problem 11. Show that there exists $L = L(\delta, l)$ such that if the Cayley graph of a group G is δ–hyperbolic, and $w, v \in G$ are of word length $\leq l$ and conjugate, then there is $t \in G$ satisfying $t^{-1}wt = v$ of word length $\leq L$.

Problem 12. Find a hyperbolic metric space X and points $x_0 \in X, a \in \partial X$ such that a is not represented by a sequence (x_i) satisfying $|x_k x_l| = |k - l|$.

Problem 13. Let X be a hyperbolic proper metric space. Show that for each pair of points $a \neq b \in \partial X$ there is a bi-infinite geodesic representing a and b.

Problem 14. Let G be an infinite hyperbolic group that is not virtually $\mathbf{Z}$. Show that any G–orbit is dense in ∂G

Problem 15. Let X be a (possibly non-proper) hyperbolic metric space. Show that ∂X with Gromov's metric is a complete metric space.

Problem 16. Find the growth of

(i) the isometry group of $\mathbf{R}^2$ preserving the standard tiling by squares,

(ii) $\mathbf{Z}_2 * \mathbf{Z}_3$,

(iii) $\langle s_1, s_2, t_1, t_2, r \mid [s_i, r] = [t_i, r] = 1, [s_1, s_2] = [t_1, t_2] = 1, [s_i, t_j] = \delta_{ij} r \rangle$,

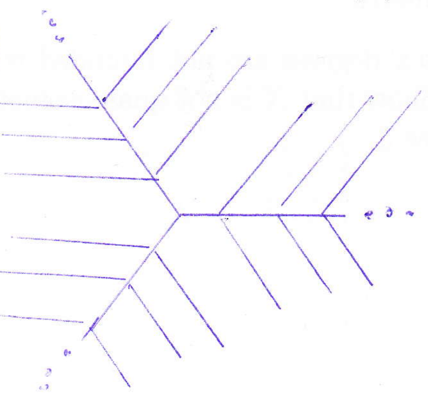
(iv) a Gromov hyperbolic group.

Problem 17. Let X be a hyperbolic proper metric space. Define a natural map $\partial X \to$ *Ends(X)* and prove that it is continous.

Problem 18. Let H be a finite subgroup of $\mathrm{Out}(F_n)$, the outer automorphism group of the free group F_n. Show that there is a metric graph with fundamental group F_n whose isometry group contains H. Hint: consider the preimage of H in $\mathrm{Aut}(F_n)$ and use Dunwoody's Theorem.

Problem 19. Describe *Ends(X)* for X below.

(i)



(ii)