TOPICS IN GEOMETRY: THE GEOMETRY OF THE EUCLIDEAN PLANE

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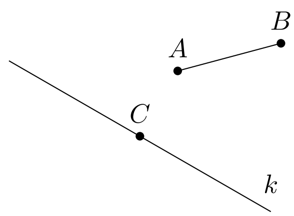
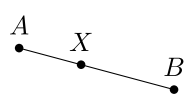
1. Introduction

The Euclidean plane is a collection of **points**. **Lines** are subsets of the Euclidean plane. We will typically refer to points by capital letters A, B, C etc and lines by lower case letters l, k, m etc.



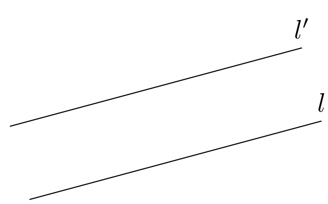
In the figure above, the point C lies on the line k, and we have drawn the **line segment** AB between the points A and B. If a point X lies on the segment AB, then we say X is **between** A and B.



We take the point of view that lines, circles, and any other **geometric figures** are sets of points. Consequently, we will often use the notation $X \in k$ (read: X is in/on k) when X is a point and k is a geometric figure.

Another important concept is the notion of a transformation of the plane. A transformation of the plane should be thought of as a "movement" of points. The first chapter is devoted to exploring **reflections**, and the second chapter explores **rotations**, both of which are transformations of the plane.

Definition 1.1. Parallel lines Two lines l and l' are **parallel** if they do not intersect



Before we begin proving theorems, let us note a two intuitive facts:

Fact 1.2. For any two distinct points A and B there exists one and only one line $k = AB$ which contains these two points.

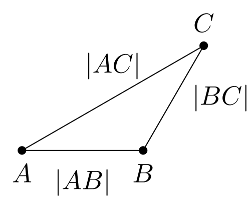
Fact 1.3. For any line k there is at least one point P not contained in that line.

Given a pair of points A, B we denote the **distance** between them as $|AB|$. The next fact about Euclidean geometry is less obvious then the previous facts, so take some time to convince yourself that it is reasonable.

Fact 1.4 (Triangle Inequality)**.** If A, B, C are three points in the plane, then their distances satisfy

$$|AC| \leq |AB| + |BC|,$$

and equality holds if and only if B lies on the line segment AC.



Definition 1.5 (angles)**.** If A, B, C are three distinct points in the plane, then we may construct the **angle** $\angle ABC$, which has a **degree-measure** in between $0°$ and $180°$.

$\angle ABC = 0°$ a right angle $90°$ $\angle ABC = 180°$

For now we are only considering unoriented angles (when we consider oriented angles, we will need to allow angles with degree-measure in between $180°$ and $360°$).

Given three points A, B, C, we say they form a **triangle** if they do not lie on a common line L. If three points A, B, C lie on a straight line then we say they form a **degenerate triangle** or are **collinear**.

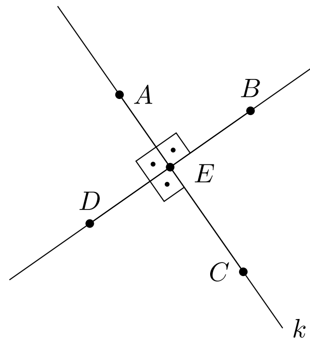
Exercise 1.6. Let ABC be a triangle. Using the triangle inequality, prove that

$$|AB| - |BC| < |AC| < |AB| + |BC|;$$

why is this inequality strict? *Hint*: non-degeneracy of the triangle ABC implies that the above inequalities are strict.

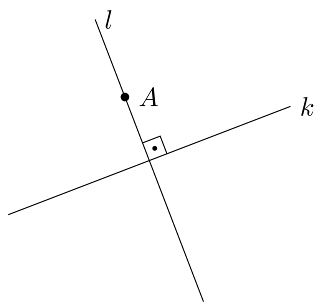
Given two lines k, l and points A, C on k and B, D on l such that line segments AC and BD intersect in a point E, we can form the four angles $\angle AEB$, $\angle BEC$, $\angle CED$, and $\angle DEA$.

If all four of these angles are equal, then the lines l, k are **perpendicular**.

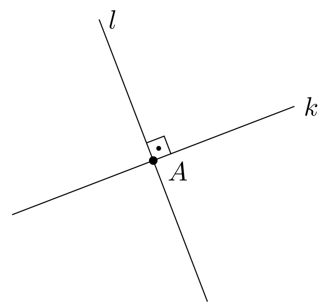


Remark. When any two lines AC and BD intesect in a point E as shown above, four angles are formed. Even if the lines are not perpendicular, we still have two equalities $\angle AEB = \angle CED$ and $\angle AED = \angle CEB$. What distinguishes a perpendicular pair of lines is that all four angles are equal; one might say that a perpendicular pair of lines has more **symmetry** than an arbitrary pair of lines.

Fact 1.7. Given a line k, and any point A, there is a unique perpendicular l to k which contains A. *Note*: this fact is true whether or not A lies on k (see the figure below).



dropping the perpendicular
from A to k



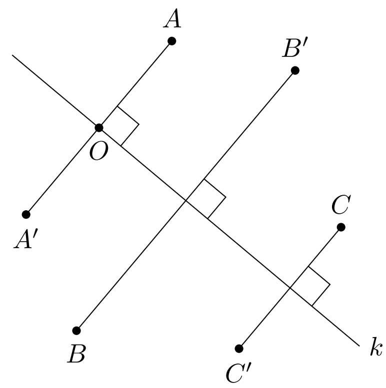
raising the perpendicular
to k from A

1.1. **Reflections.** Using Fact 1.7, we can define **reflections** through lines; reflections will be an indispensable tool when we start proving theorems.

Definition 1.8. Let k be a line. The **reflection through** k is a transformation of the Euclidean plane which sends a point A to its reflection A'. This reflection satisfies two properties.

(i) If A lies on k, then $A' = A$ (in other words, the reflection fixes the line k).

(ii) If A does not lie on k, then A' is the point such that the line AA' is perpendicular to k in a point O such that $|AO| = |OA'|$ (see figure below).

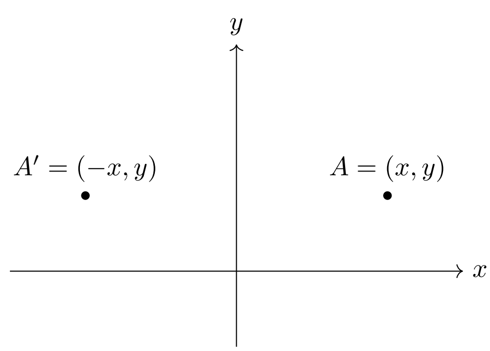


There is an important fact related to this definition.

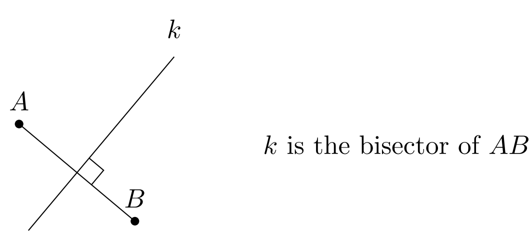
Fact 1.9. The reflection through any line k preserves distances and angles. In other words, if A, B, C are any two points whose reflections through k are A', B' and C', then $|AB| = |A'B'|$, and $\angle ABC = \angle A'B'C'$.

Exercise 1.10. Prove that the reflection of a line is a line (*hint*: use the equality case in the triangle inequality). Prove that if l is parallel to k, then its reflection l' through k is also parallel to k. Prove that l is perpendicular to k if and only if l is its own reflection (through k).

Example. If we give our plane x and y coordinates (something we will almost never do), then the reflection of $A = (x, y)$ through the y axis is easily seen to be $A' = (-x, y)$.



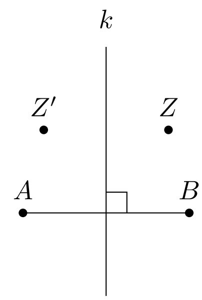
Definition 1.11. The **bisector** (or **perpendicular bisector**) of a segment AB is the line k perpendicular to AB at its centre.



Exercise 1.12. Let k be the bisector of AB. Prove that the reflection of A through k is B.

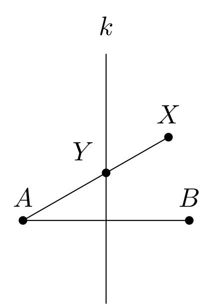
Theorem 1.13 (characterization of points on the bisector)**.** A point X lies on the bisector of a segment AB if and only if $|AX| = |XB|$.

Proof. Throughout the proof, let k denote the bisector of AB, and let Z' denote the reflection of Z through the bisector of AB:



This theorem is an "if and only if" statement, so we must prove two implications. First we prove the $\implies$ direction, namely, we will prove that if X lies on the bisector of AB then $|AX| = |XB|$. Since X lies on the bisector, $X = X'$. By the result of Exercise 1.12, $B = A'$, and since reflections preserve distances, $|AX| = |A'X'| = |BX|$, as desired.

Now we prove the $\impliedby$ direction of the proof. We assume that $|AX| = |XB|$. We prove by contradiction, and suppose that X does not lie on k. Then either AX or BX intersects k, and so without loss of generality, we can assume that AX intersects k at a point Y, as in the following figure.



Since Y lies on the bisector through AB, we know from the $\implies$ direction of this theorem (which we already proved) that $|AY| = |YB|$. Applying the triangle inequality twice yields

$$|BX| < |BY| + |YX| \qquad |AX| = |AY| + |YX|\,,$$

where we know the first inequality is strict since Y cannot lie on BX, since X and B lie on the same half-plane determined by k, while Y lies on k. We also know the second inequality is actually an equality because Y lies on AX. But since $|AY| = |BY|$, we conclude that $|BX| < |AX|$, which contradicts our assumption. Since we began our argument by assuming that X did not lie on k, the contradiction $|BX| < |AX|$ forces us to conclude that X must lie on k, and this completes the proof. □

For the next proof we require some preliminary results.

Fact 1.14. If l is a line and X is a point not on l, then there is a unique line k containing X which is parallel to l.

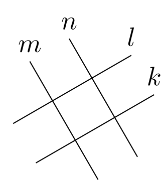
Exercise 1.15. Prove the following:

(a) If l, k are parallel lines, and m intersects k, then m also intersects l. *Hint*: use the preceeding fact.

(b) If k is a line containing distinct points A and B, then the perpendiculars to k raised from A and B are parallel lines. *Hint*: Let X lie on the perpendiculars to k raised from A and from B, then use uniqueness of perpendiculars to k dropped from X.

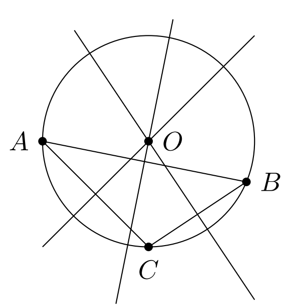
(c) If l, k are parallel, and m is perpendicular to l at A, then there is a point B on k such that m is perpendicular to k at B. *Hint*: by (a), m intersects k at some point B, and by (b) the perpendicular to m raised from B is parallel to l. Then use the fact stated before this exercise.

(d) If l, k are parallel lines, and $m \neq n$ are such that m is perpendicular to l and n is perpendicular to k, then m and n are parallel. *Hint*: use (c) to conclude that m is perpendicular to k, and so n and m are both perpendicular to k. Then use (b).



Theorem 1.16. In any triangle ABC, the bisectors of AB, AC and BC all intersect in a single point.

Proof. Since AB and BC are not parallel, their perpendicular bisectors are also not parallel (by Exercise 1.15.d), and so they intersect at some point O. By Theorem 1.13, $|OA| = |OB|$ and $|OB| = |OC|$. But then $|OC| = |OA|$, so O lies on the bisector of AC (using Theorem 1.13 again). Thus O lies on all three bisectors, which is what we wanted to show.



□

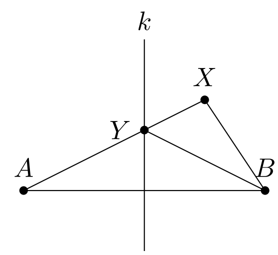
Remark. From the proof, $|AO| = |BO| = |CO|$, and so there is circle passing through ABC centred at O. This circle is unique, since any other circle with centre O' passing through ABC would have $|AO'| = |BO'| = |CO'|$, and so O' would also lie on all three bisectors of ABC. Since three lines intersect in (at most) a unique point, we conclude that $O' = O$, so the circle is unique. This circle is called the **circumscribed circle** of the triangle ABC.

Theorem 1.17. Let X be a point outside the line AB. Then X lies on the bisector of AB if and only if $\angle XAB = \angle XBA$.

Remark. The assumption that X lies outside the line AB is crucial, for it is easy to show that if X is anywhere inside the segment AB, then $\angle XAB = \angle XBA$ (even if it is not on the bisector).

Proof. Let k be the bisector of AB. We begin with the $\Longrightarrow$ direction of the theorem, so we assume that $X \in k$. If we reflect in k, then $X' = X$ and $A' = B$, and since reflections preserve angles, $\angle XAB = \angle X'A'B' = \angle XBA$, as desired.

Now we prove the $\Longleftarrow$ direction of the theorem. As in the proof of Theorem 1.13, we prove by contradiction, and so we suppose that $\angle XAB = \angle XBA$, but X does not lie on k. Without loss of generality, suppose that X and B lie on the same half plane determined by k. Let Y be the intersection point of AX and k, as shown below.



Since the lines AX and AY are equal, the angles $\angle XAB$ and $\angle YAB$ are equal. By the $\Longrightarrow$ part of this theorem, we know that $\angle YAB = \angle YBA$, and thus $\angle XAB = \angle YBA$. However, we assume that $\angle XAB = \angle XBA$, so we conclude that $\angle XBA = \angle YBA$ (if we look at the above figure, this equality is obviously not true, but we need to use logical arguments to finish the proof). Since $\angle XBA = \angle YBA$, we conclude that the lines BX and BY are equal (since X and Y lie on the same half-plane determined by line AB), and so the line YX is equal to the line BX. However, we defined Y to lie on the line AX, so the line AX is equal to the line YX. But then we have equality of lines $BX = AX$, and so X lies on the line AB, which we assume is not true. This is a contradiction, and so we must have that X lies on k, and so we have completed the proof. □

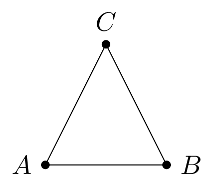
Corollary 1.18. In a triangle ABC, the following three conditions are equivalent:

(i) $|AC| = |CB|$.
(ii) $\angle CAB = \angle CBA$.

(iii) C lies on the bisector of AB.

Proof. Since ABC is a triangle, we know that C does not lie on the line AB. The statement (i) $\iff$ (iii) is Theorem 1.13, (ii) $\iff$ (iii) is Theorem 1.17, and so these three statements are equivalent. $\square$

Definition. If ABC is a triangle which satisfies any of the equivalent properties in the above corollary, then ABC is called an **isosceles** triangle. (Note: by our definition, equilateral triangles are also technically isosceles triangles).



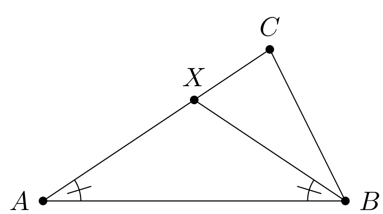
Exercise 1.19. Let $T = ABC$ be a triangle. Prove that T is isosceles if and only if there exists a line k such that T is its own reflection when we reflect through k. (we say that the reflection through k is a **symmetry** of the triangle T).

Prove that T is equilateral if and only if there exist two distinct lines k and l such that T is its own reflection through k and through l.

Remark: we could summarize this exercise by saying that non-isosceles triangles have no symmetry, isoceles triangles have some symmetry, and equilateral triangles have the most symmetry.

Theorem 1.20. Let ABC be a triangle with $\angle CAB < \angle CBA$. Then $|BC| < |AC|$. (this theorem says that, opposite a smaller angle, you have a smaller side).

Proof. Since $\angle CAB < \angle CBA$, we know that there exists a point X in the segment AC such that $\angle XBA = \angle CAB$. This is shown in the figure below.



Then we apply the triangle equality to conclude

$$(*) \qquad |AX| + |XC| = |AC|\,,$$

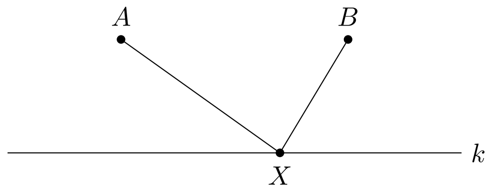
and the triangle inequality to conclude that

$$(**) \qquad |BX| + |XC| > |BC|\,.$$

We can conclude a strict inequality above because non-degeneracy of ABC implies that BXC cannot lie on a common line. By Theorem 1.17 we know that $|AX| = |BX|$, and so we combine $(*)$ and $(**)$ to conclude that $|BC| < |AC|$, as desired. □

1.2. **Some applications of reflections.**

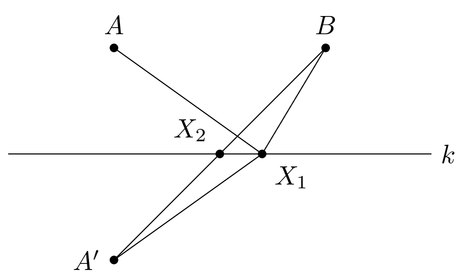
Problem 1.21. Given a line k and points $A, B \in k$ on the same side of k, find $X \in k$ such that $|AX| + |XB|$ is minimized.



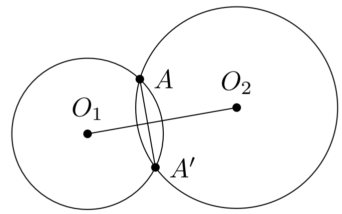
To solve this problem, we will consider the reflection through k. The idea for the solution is summarized nicely in the following figure. Let X be any point on k, and let A' denote the reflection of A through k. Then $|AX| = |A'X|$, so the problem of minimizing the distance $|AX| + |XB|$ is the same as the problem of minimizing the distance $|A'X| + |XB|$. Suppose that $X_1 \in k$ does not lie on the segment AB. Then the triangle inequality tells us that $|A'X_1| + |X_1B| > |A'B|$. Now let $X_2 \in k$ be the intersection of segment $A'B$ with k (X_2 exists because A and B lie on the same side of k, by assumption, therefore A' and B lie on opposite sides of k). Since X_2 lies on the segment AB, $|A'X_2| + |X_2B| = |A'B|$, and so we deduce that for all $X_1 \neq X_2$ on k, we have

$$|AX_2| + |X_2B| < |AX_1| + |X_1B| .$$

This inequality tells us that the point X_2 is the unique minimizer of $|AX| + |XB|$, and this completes the solution.



Problem 1.22. Show that two distinct circles can intersect in at most two points.



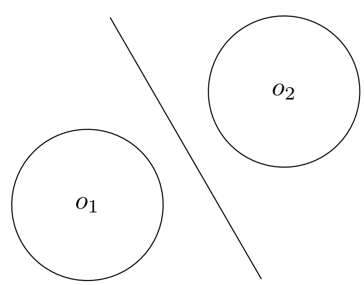
Before we solve this problem, let us recall the definition of a circle. If O is a point and r a positive real number, then the **circle centred at O with radius r** is the collection of all points X such that $|OX| = r$.

Now suppose that we have two distinct circles o_1 and o_2 with centres O_1 and O_2, respectively. If $O_1 = O_2$, then the radii of o_1 and o_2 must be different (or else they would be the same circle!), in which case it is easy to show that o_1 and o_2 do not intersect.

Supposing now that $O_1 \neq O_2$, suppose that $A \in o_1 \cap o_2$ (read: A is in the intersection of o_1 and o_2). If A' is any other point on $o_1 \cap o_2$, then $|AO_1| = |A'O_1|$ and $|AO_2| = |A'O_2|$, by the definition of a circle. But by Theorem 1.13, this implies that the line O_1O_2 bisects the segment AA'. Thus the reflection through O_1O_2 sends A to A'. Therefore, A' is uniquely determined by A. This completes the solution.

Exercise 1.23. Show that if two circles o_1 and o_2 intersect in a single point A, then A must lie on the line O_1O_2 (therefore, in the notation of the previous example, $A' = A$).

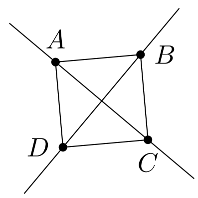
Exercise 1.24 (Congruence of circles)**.** If o_1 and o_2 are two circles with the same radius, construct a line k so that o_1 and o_2 are reflections of each other through k.



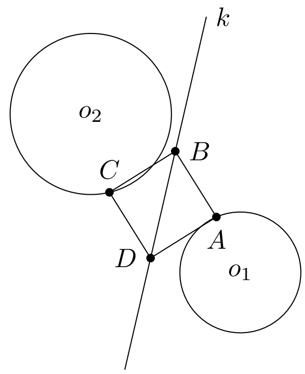
Exercise 1.25. Let k be a line and o a circle. Prove that k and o intersect in at most two points. *Hint*: Follow Example 1.2.2; if A is one intersection point, any other intersection point is uniquely determined by A.

Exercise 1.26 (Preparation for Example 1.2.3)**.** Prove that $s = ABCD$ is a square if and only if A is the reflection of C through BD, B is the reflection of D through AC, and

$|AC| = |BD|$.

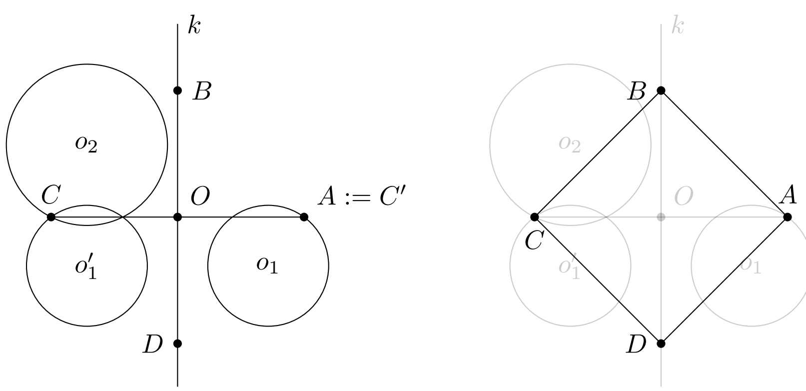


Example 1.27. Given circles o_1 and o_2 and a line k, find necessary conditions under which a square $ABCD$ such that $A \in o_1$, $C \in o_2$ and $B, D \in k$ exists.



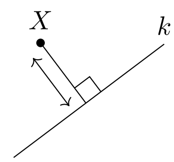
Let us begin our solution by supposing that such a square $ABCD$ exists. Thanks to Exercise 1.26, we know that the reflection through $BD = k$ sends A to C. Therefore o_1' intersects o_2, where o_1' denotes the reflection of o_1 through k, (since C is on o_1' and o_2). Thus a *necessary* condition for the existence of such a square is that o_1' and o_2 must intersect.

Now we prove that this condition on o_1, o_2 and k is *sufficient*; that is, if o_1' and o_2 intersect, then a square $ABCD$ satisfying our problem exists. Let o_1' and o_2 intersect in a point C, and let A be the reflection of C through the line k. Let O be the midpoint of AC; clearly O lies on k. Now let $B \neq D$ be the (only) two points on k such that $|OB| = |OD| = |OA| = |OC|$. This construction is summarized in the figure below.



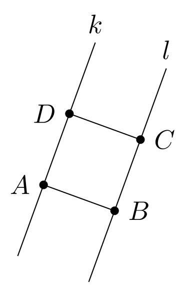
Then since BD is the bisector of AC and AC is the bisector of BD (prove this!), and $|BD| = |AC|$ (by our construction), we can apply Exercise 1.26 to conclude that $ABCD$ is a square. This completes the solution.

Definition 1.28. Here we introduce two concepts which will be used in the next example. Let X be a point and k a line. The **distance** from X to k (written $|X, k|$) is the length of the perpendicular line segment from X to k.

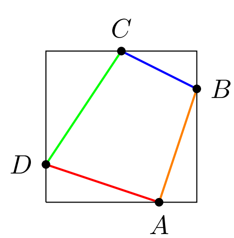


Exercise 1.29. Prove that the distance $|X, k|$ is the minimum distance between X and a point on k.

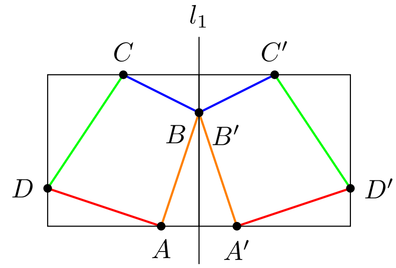
Exercise 1.30. If $s = ABCD$ is a square of length 1 and l and k are lines such that $AD = l$ and $BC = k$, then any point X on the segment AB satisfies $|X, l| + |X, k| = 1$.



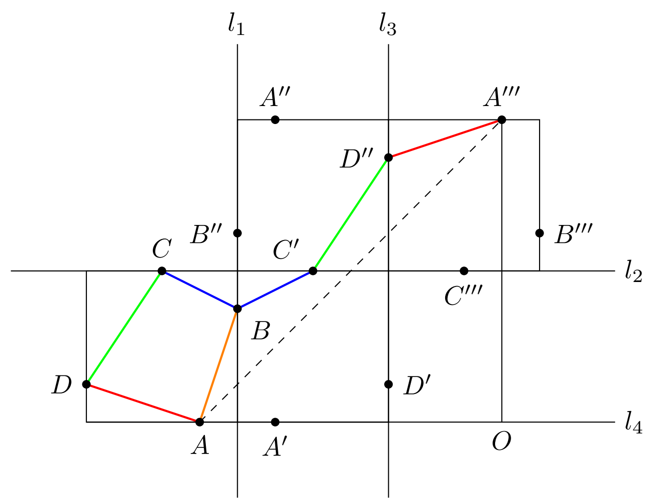
Example 1.31. Suppose the vertices of a quadrilateral A, B, C, D lie on distinct sides of a square s of side length 1. Prove that the perimeter of $ABCD$ is at least than $2\sqrt{2}$.



The solution is nicely summarized in the following figure. Let l_1 be the line coinciding with the side of s which contains the point B. Reflecting s in l_1 defines a new square s' containing the reflections $A'B'C'D'$ of our original quadrilateral $ABCD$.



We repeat this process with the line l_2 coinciding with the side of s' containing the point C'; this defines the points A'', B'', C'' and D''. Repeating the process one last time, we reflect in the line l_3 coinciding with the side of s'' containing the point D''. This construction is summarized in the figure below. Now note that $|AB| + |BC| + |CD| + |DA|$ is equal to $|AB| + |BC'| + |C'D''| + |D''A'''|$, and by the triangle inequality, this sum of distances is bounded below by $|AA'''|$.

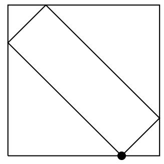


Luckily, we can calculate $|AA'''|$. Let O be the reflection of A''' through l_2. Since l_3 bisects $A''A'''$ and the reflection through l_2 fixes l_3 (since $l_3 \perp l_2$) we deduce that l_3 bisects $A'O$ (since the l_2 reflection sends A'' to A' and A''' to O). Therefore the l_3 reflection sends A' to O and so the distance from A' to O is twice the distance from A' to l_3. Since the l_1 reflection sends A to A', the distance from A to A' is twice the distance from A' to l_1. We obtain

$$|AO| = |AA'| + |A'O| = 2\,|l_1A'| + 2\,|A'l_3| = 2(|l_1A'| + |A'l_3|) = 2,$$

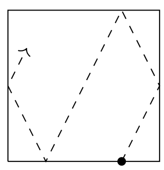
since l_1 and l_3 coincide with parallel sides of the square s' (see Exercise 1.30). An easier argument shows that $|OA'''| = 2$, and thus $|AA'''|$ is the hypotenuse of a right triangle with both non-hypotenuse side lengths equal to 2. We apply Pythagoras' theorem (which we will prove later on!) to deduce $|AA'''|^2 = 2^2 + 2^2$, so $|AA'''| = 2\sqrt{2}$.

Exercise 1.32. Let s be a unit square and let A be a point on one side of s (suppose A is not a corner). Prove that there is exactly one set of points B, C, D on the other sides of s such that the perimeter of $ABCD$ satisfies the lower bound of $2\sqrt{2}$.



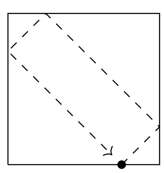
Hint: referring to the figure in Example 1.2.4, draw a straight line from A to A''' and consider "folding" this straight line back into a quadrilateral.

Exercise 1.33 (continuation of previous exercise)**.** Consider a point A and a unit square s as in the previous exercise. Suppose we "play billiards" in s, where we can hit a billiard ball initially located at A and let it bounce around the square. The rule of the game is that when the ball bounces off of an edge of the square the angle of incidence must the same as the angle of reflection.



(This rule is a very good approximation to how a billiard ball actually moves, as most billiard players know.)

Prove that if you hit the ball towards the adjacent wall in such a way that it comes back to A after hitting each of the other three walls exactly once, then the trajectory of the ball will be the unique quadrilateral from Exercise 1.32.



2. Rotations

In this section we introduce rotations as a new tool we can use to prove theorems.

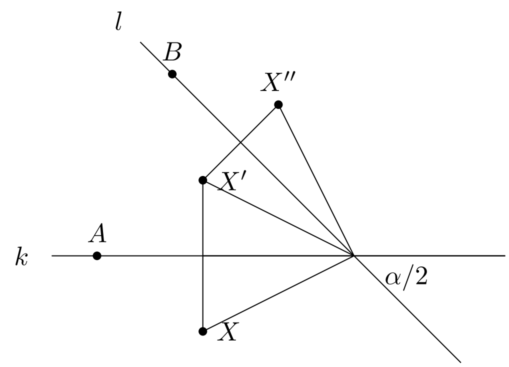
Definition 2.1. The **rotation** through an (oriented) angle α at a point O is the unique transformation of the plane satisfying two properties:

(i) The point O is the only point fixed by the rotation (this is the centre of the rotation).

(ii) If $A \neq O$, then A gets rotated to the unique point A' such that $\angle AOA' = \alpha$ and $|OA'| = |OA|$.

Theorem 2.2. Pick points A, B distinct from O such that $\angle AOB = \alpha/2$; then the rotation is the reflection through $k := OA$ followed by the reflection through $l := OB$.

Proof. The reason this theorem is true is best seen in the following figure.



Since O is clearly fixed under both reflections, it is clear that this composition of two reflections is a transformation of the plane which satisfies (i). It remains to prove (ii), namely if $X \neq O$ then $\angle XOX'' = \alpha$, and $|OX| = |OX''|$.

The angle $\angle XOX'$ is twice the angle $\angle AOX'$ since reflecting in k proves the equality $\angle AOX' = \angle XOA$ and clearly $\angle XOX' = \angle AOX' + \angle XOA$. Similarly, the angle $\angle X'OX''$ is twice the angle $\angle X'OB$. Thus

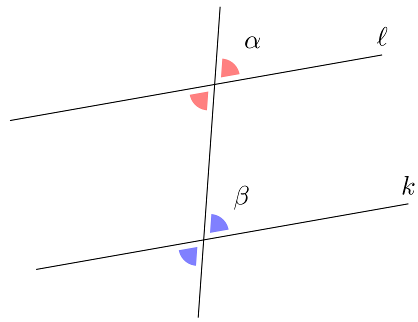
$$\angle XOX'' = \angle XOX' + \angle X'OX'' = 2\angle AOX' + 2\angle X'OB = 2\angle AOB = \alpha.$$

The second part is easy since reflections preserve distances so we know $|OX| = |OX'| = |OX''|$. We have proved this composition of two reflections is a transformation which satisfies (i) and (ii) in Definition 2.1, so it must be the reflection through α at O. $\square$

Corollary 2.3. Rotations preserve distances and angles.

Proof. This follows from Fact 1.9 (which states that reflections preserve distances and angles), since any rotation is simply a composition of two reflections. $\square$

Fact 2.4. Let k, l be lines in the plane intersecting another line m. Then ℓ and k are parallel if and only if $\alpha = \beta$ (in the figure, α is the red angle and β is the blue angle).



Exercise 2.5. We can actually prove this Fact, using the previous Fact 1.14.

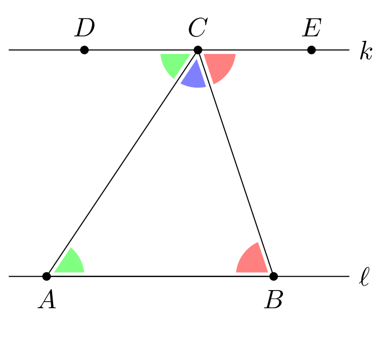
(a) Let O be a point and ℓ a line not containing O. Prove that the $180°$ rotation around O sends ℓ to a line ℓ' parallel to ℓ. *Hint*: the $180°$ rotation is a composition of reflection through *any* pair of perpendicular lines intersecting at O; choose a special pair.

(b) Suppose that $\ell \parallel k$. In the context of Fact 2.4, Let m intersect ℓ at a point A and k at a point B, and let O be the midpoint of the segment AB. Then the $180°$ rotation through O sends A to B, and by part (a) sends ℓ to a line ℓ' parallel to ℓ which contains B. Use Fact 1.14 to prove that $\ell' = k$.

(c) Use this rotation to conclude $\alpha = \beta$. *Hint:* pick point C on ℓ such that $\angle OAC = \alpha$, and argue that $\angle OBC' = \beta$, where C' is the rotated image of C. Then use the fact that rotations preserve angles.

(d) For the converse, suppose the angles are equal. As in part (b), consider the rotation around O, and picking $C \in \ell$ such that $\angle OAC = \alpha = \beta$, show that $\angle OBC' = \beta$ implies C' must lie on k, so that the rotation of $180°$ around O sends ℓ to k. Conclude that $\ell \parallel k$.

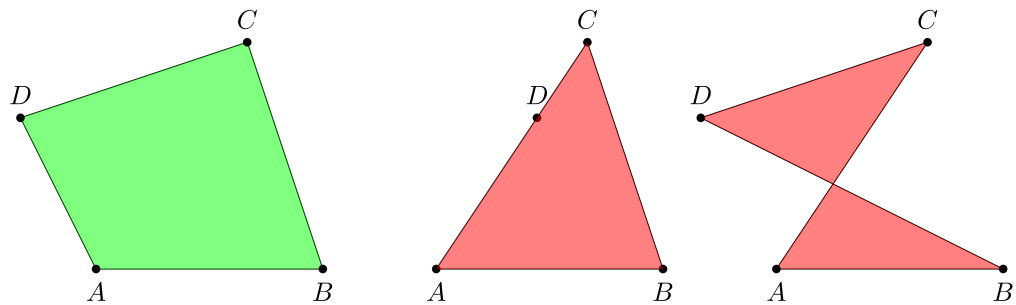
Corollary 2.6. The sum of interior angles in a triangle is $180°$.



Proof. Let k be the unique parallel to $\ell = AB$ passing through C. As in the figure, pick points D and E on k such that C is between D and E. Without loss of generality, we may assume that the three angles $\angle ACD$, $\angle BCA$ and $\angle ECB$ do not intersect (by choosing D and B on opposite sides of AC). Then $\angle ACD + \angle BCA + \angle ECB = \angle BCD = 180°$. Since k and ℓ are parallel, we may apply Fact 2.4 to conclude $\angle CAB = \angle ACD$ and $\angle ABC = \angle ECB$. Then we conclude $\angle CAB + \angle BCA + \angle ABC = 180°$, as desired. □

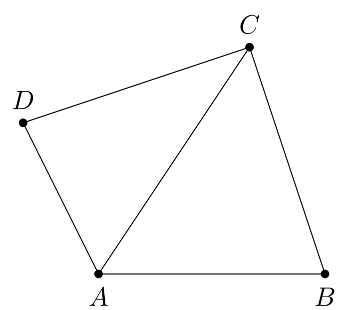
Definition 2.7. A **quadrilateral** $ABCD$ is a collection of four distinct segments called **edges** AB, BC, CD and DA; the four points A, B, C, D are called **vertices**. We require that no line contains three (or more) of the four points (i.e. $ABCD$ is "non-degenerate"). If two edges intersect at a vertex (e.g. segments AB and AD intersect at A) we say that two edges are **adjacent**. We require that non-adjacent edges do not intersect.

In the figure below, the green figure is a genuine quadrilateral, and the red figures are not quadrilaterals.



Corollary 2.8. For any quadrilateral $ABCD$, the sum of the interior angles is 360°.

Proof. As in the figure below, split the quadrilateral $ABCD$ into two triangles ABC and ADC.

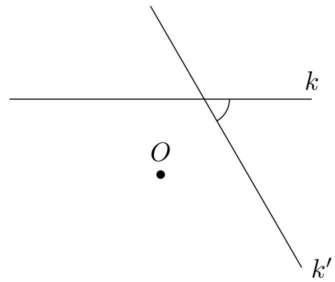


Then, using Corollary 2.6, we know that $\angle DAC + \angle ACD + \angle CDA = 180°$ and $\angle CAB + \angle ABC + \angle BCA = 180°$. Since $\angle DAB = \angle DAC + \angle CAB$ and $\angle BCD = \angle BCA + \angle ACD$, we conclude

$$\angle DAB + \angle ABC + \angle BCD + \angle CDA = 360°,$$

as desired. □

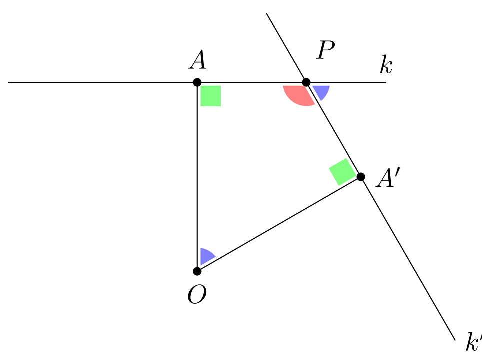
Theorem 2.9. Let k' be the rotation of k around a point O by angle $0° < \alpha < 180°$. Then the angle between k and k' is α. Here the angle between k and m is computed by starting at any half-line of k and going clockwise until you hit one of the two half-lines of l.



Proof. Let A' be the rotation of A around O by the angle α.

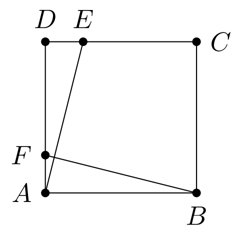
If $O \in k$, then the theorem follows immediately from the definition of a rotation. If not, since $0° < \alpha < 180°$, we claim that the two lines are not parallel and intersect at a point P. To prove this claim, suppose not (so that k and k' are parallel); then A, O and A' all lie on a common line, so α must have been $180°$.

If $O \notin k$, then pick $A \in k$ such that OA is perpendicular to k, as shown below:

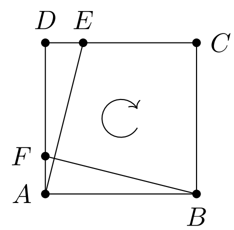


Referring to the colours in the figure above, we note that since OA is perpendicular to k, the green angle between OA and k is $90°$. Since OA' is the rotation of OA and k' is the rotation of k, we conclude (since rotations preserve angles) that the second green angle is also $90°$. Since the angles inside a quadrilateral add up to $360°$, we know that the red angle plus the blue angle at O is $180°$. But we also know that the red angle plus the blue angle at the intersection of k and k' is equal to $180°$ (since they join to make a straight line) and so we conclude that the two blue angles are in fact equal. Since the blue angle at O is $\angle AOA'$, we deduce that it is the angle of rotation, namely α. This completes the proof. $\square$

Problem 2.10. Let $ABCD$ be a square and suppose $E \in DC$ and $F \in AD$ are such that $|DE| = |AF|$. Prove that AE and BF are perpendicular.



Solution. The trick is to rotate by 90° through the centre of the square.



Then $A \mapsto D \mapsto C \mapsto B \mapsto A$, and since F is between AD, F' is between DC. Since E and F' are both between DC and located at a distance of $|DE|$ from D, we conclude they are equal $E = F'$. Hence the line FB is rotated to the line AE. We can thus apply Theorem 2.9 to conclude the angle between lines FB and AE is 90°; in other words, they are perpendicular.

Definition 2.11 (Congruence)**.** We say that two figures are **congruent** if one can be obtained from the other by a sequence of of reflections.

Exercise 2.12.

(1) Prove that all points are congruent.
(2) Prove that all lines are congruent.
(3) Prove that all circles of the same radius are congruent. Can two circles of differing radii be congruent?
(4) Consider a figure $\mathcal{F}$ defined by the union of two intersecting perpendicular lines, as shown below.

Prove that if $\mathcal{F}'$ is another figure made out of two intersecting perpendicular lines, then $\mathcal{F}$ is congruent to $\mathcal{F}'$.

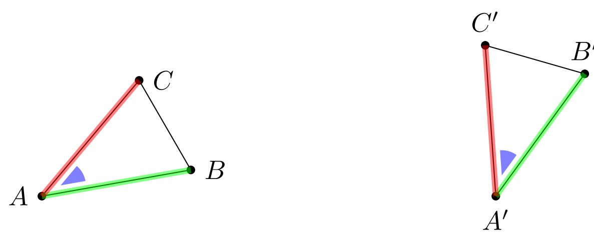
Exercise 2.13 (if you know what "equivalence relations" are)**.** Prove that "congruence" is an equivalence relation on the set of all figures.

Definition 2.14 (Congruence of Triangles)**.** For triangles ABC we modify the definition of congruence a little bit. We say ABC and $A'B'C'$ are congruent if there is a sequence of reflections taking A to A', B to B' and C to C'. In words, this modification of the definition of congruence takes the ordering of the vertices into account.

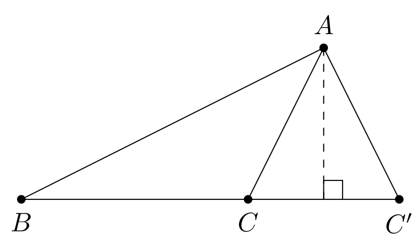
Theorem 2.15 (Triangle Congruence)**.** ABC and $A'B'C'$ are congruent if and only if one of the following condition holds.

(i) (SSS) the three sides are equal; $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|BC| = |B'C'|$.
(ii) (SAS) one of the pairs of corresponding angles are equal, and the two pairs of sides adjacent to this angle have equal length. For example: $\angle CAB = \angle C'A'B'$ and $|AC| = |A'C'|$ and $|AB| = |A'B'|$.
(iii) (ASA): two of the pairs of corresponding angles are equal, and the pair of segments which are adjacent to these angles is equal. For example: $\angle CAB = \angle C'A'B'$ and $\angle CBA = \angle C'B'A'$ and $|AB| = |A'B'|$. Note that equality in two angle pairs automatically implies the third angle pair is equal, so one does not care which ones they are.

In the figure below, we have shown an instance when the (SAS) criterion would apply:

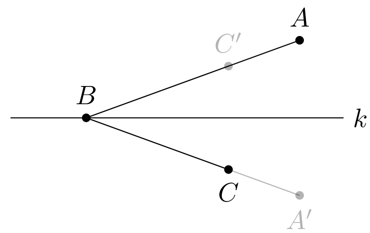


Remark. The following figure shows non-congruent triangles ABC and ABC' with side-side-angle equality $|AB| = |AB|$, $|AC| = |AC'|$ and $\angle ABC = \angle ABC'$.



While this remark demonstrates that "side-side-angle" equality is, in general, not sufficient to guarantee congruence of two triangles ABC and $A'B'C'$, there is one case where the "side-side-angle" equalities do guarantee congruence (see the Lemma below).

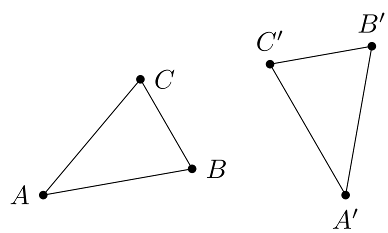
Definition 2.16. let ABC be an angle. The **bisector** of $\angle ABC$ is a line k passing through B such that the reflection through k interchanges the two half-lines defining the angle $\angle ABC$.



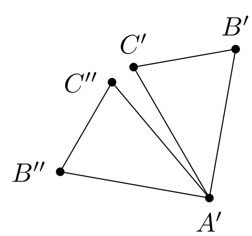
Exercise 2.17. Prove that the bisector to an angle ABC always exists. *Hint:* pick $\widetilde{A}$ on the ray BA so that $\angle\widetilde{A}BC = \angle ABC$, and such that $\left|\widetilde{A}B\right| = |BC|$. Let k be the bisector of the segment $\widetilde{A}C$. Prove that k bisects $\angle ABC$.

Proof (Proof of Theorem 2.15)**.** Obviously, if ABC and $A'B'C'$ are congruent then (i), (ii) and (iii) all hold. The point of the theorem is that (i), (ii) and (iii) contain (a priori) less information than full-blown congruence.

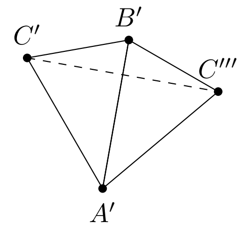
In class we proved that the (SAS) criterion implies congruence. Here we will prove that the (SSS) criterion implies congruence. Suppose we have our two triangles ABC and $A'B'C'$, as shown below, and suppose that $|AB| = |A'B'|$, $|BC| = |B'C'|$ and $|AC| = |A'C'|$.



If we reflect through the bisector of AA', then A gets sent to A', and B and C get sent to B'' and C'', respectively:

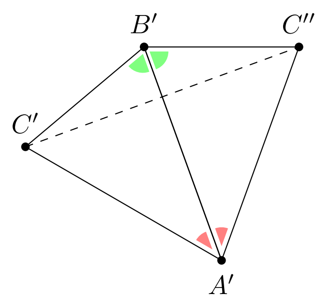


Assuming $|AB| = |A'B'|$, then $|A'B''| = |A'B'|$, and so the bisector through $B'B''$ contains A'. It follows that reflecting through the bisector of $B'B''$ sends B'' to B', A' to A' (and C'' to C'''). If $C''' = C'$ then we got lucky, and the triangles are congruent. If $C''' \neq C'$. then $|A'C'''| = |AC| = |A'C'|$ and $|B'C'''| = |BC| = |B'C'|$, and so $A'B'$ is the bisector to the segment $C'C'''$ (by Theorem 1.13). It follows that reflecting in $A'B'$ sends C' to C''' (and doesn't move A' or B'). This proves that ABC and $A'B'C'$ are congruent.

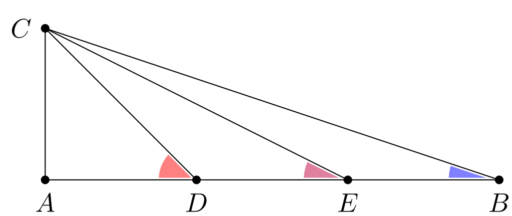


We remark that we needed, at most, three reflections to move ABC to $A'B'C'$. □

Exercise 2.18. Prove that the (ASA) criterion for two triangles ABC and $A'B'C'$ implies congruence of ABC and $A'B'C'$. *Hints*: suppose $|AB| = |A'B'|$ and, following the above proof, use two reflections to move the pair (A, B) to the pair (A', B'). Refering to the notation of the proof of Theorem 2.15, assume that C' and C'' are on opposite sides of $A'B'$. Consider $A'C'C''$ (see below). Prove that $\angle A'C'C'' = \angle A'C''C'$ (by reflecting one triangle through $A'B'$, and showing that the line $B'C''$ must be reflected to $B'C'$ and $A'C''$ must be reflected to $A'C'$, so that C'' must be reflected to C') and conclude (by Theorem 1.17) that A' lies on the bisector through $C'C''$. Similarly, conclude that B' lies on the bisector through $C'C''$, and thus $A'B'$ is the bisector through $C'C''$. It follows that the reflection through $A'B'$ sends C' to C'' (as in the proof of Theorem 2.15, this is exactly what we want!).



Problem 2.19. Let ABC be a right-angled triangle with $|AB| = 3\,|AC|$ and let D, E lie on side AB be chosen so that $|AD| = |DE| = |EB|$. Prove that $\angle CDA + \angle CEB + \angle CBA = 90°$.



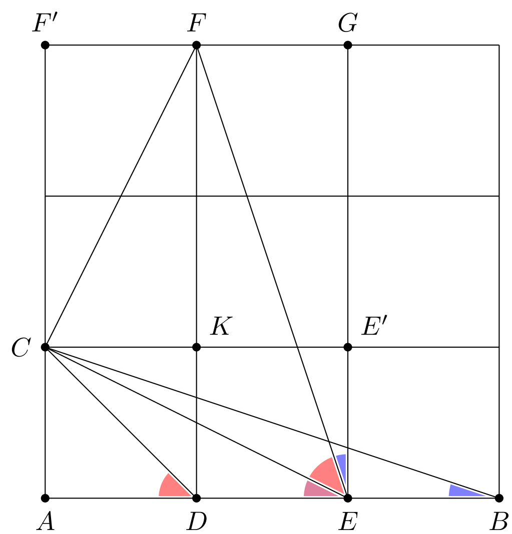
Solution. It helps to build a 3×3 grid to house our triangle as in the figure below. With the labels from the figure below, we remark that the counterclockwise rotation of $90°$ around C sends the rectangle $ACE'E$ to $KCF'F$. From this rotation we deduce two things: (1) the diagonal CE has the same length as the corresponding diagonal CF and (2) $\angle ECF = 90°$. Therefore the triangle ECF is a right angled isosceles triangle, and so we conclude

$$\angle FEC = \angle CFE = 45° = \angle CDA.$$

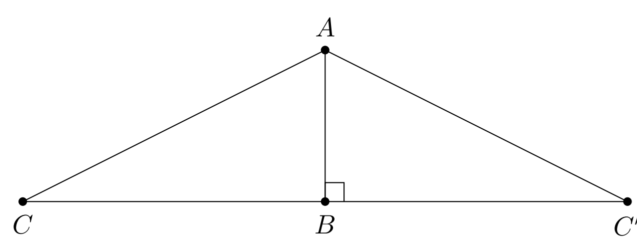
It is also clear from the figure that triangle EGF is congruent to triangle BAC, since are both right angled triangles with equal side lengths $|AB| = |EG|$ and $|FG| = |AC|$ (here we are using "side-angle-side" criterion to deduce congruence). By congruence, $\angle GEF = \angle CBA$. Combining everything we have concluded so far, we obtain

$$\angle CBA + \angle CEA + \angle CDA = \angle GEF + \angle FEC + \angle CEA = \angle GEA = 90°,$$

and this completes the solution.

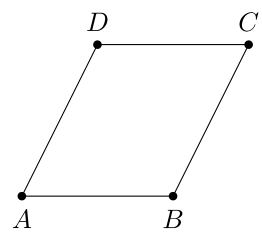


Proposition 2.20. If $|CA| = |C'A'|$, $|AB| = |A'B'|$ and $\angle ABC = \angle A'B'C' = 90°$, then ABC and $A'B'C'$ are congruent.



Proof. Since $|AB| = |A'B'|$, we can follow the proof of Theorem 2.15 and move $A'B'C'$ so that $A'B'$ coincides with AB. Using another reflection if necessary, we may assume that C and C' are on opposite sides of the line $AB = A'B'$. Since $\angle CBA = 90°$ and $\angle ABC' = 90°$, we conclude that $\angle CBC' = 180°$, and thus B lies between CC'. Since $|AC| = |AC'|$, we know that ACC' forms an isosceles triangle. Then by Corollary 1.18, we know that $\angle ACC' = \angle AC'C$ and consequently $\angle ACB = \angle AC'B$. Thus we may now use the "angle-side-angle" criterion for triangle congruence to deduce ABC is congruent with ABC'. □

Definition 2.21. A **parallelogram** is a quadrilateral $ABCD$ such that $AB \parallel CD$ and $BC \parallel AD$ and $|AB| = |DC|$, $|AD| = |BC|$.

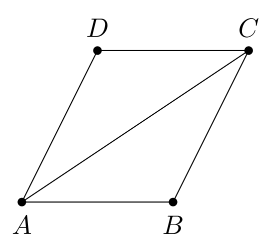


Proposition 2.22. Let $ABCD$ be a quadrilateral such that any of the following three conditions holds.

(a) $|AB| = |DC|$ and $|BC| = |AD|$.
(b) $|AB| = |DC|$ and $AB \parallel DC$.
(c) $AB \parallel DC$ and $BC \parallel AD$.

Then $ABCD$ is a parallelogram.

Proof. Without loss of generality, assume that the diagonal segment AC is contained inside the quadrilateral.



First we note that if we can show that ACB is congruent to CAD, then we will be done. Congruence easily gives us equality of all sides, and it also gives us parallel edges, as the following argument shows. If $\angle CAB = \angle ACD$, then the line AC cuts the two lines AB and CD in equal angles. Previously we have shown that this implies $AB \parallel CD$. A similar argument shows that $BC \parallel AD$. Therefore if ACB is congruent to CAD then $ABCD$ is a parallelogram.

Case (a): If $|AB| = |CD|$ and $|BC| = |AD|$, then the "side-side-side" criterion for congruence holds between triangles ACB and CAD.

Case (b): If $|AB| = |CD|$ and $AB \parallel CD$, then AC cuts AB and CD in equal angles so that $\angle ACD = \angle CAB$. Then we may use "side-angle-side" to conclude that triangle BAC is congruent to DCA.

Case (c): If $AB \parallel CD$ and $BC \parallel AD$, then AC cuts CD and AB in equal angles, and it cuts AD and BC in equal angles. Since triangles ACD and CAB share a side AC, we can use the "angle-side-angle" criterion to conclude ACD and CAB are congruent. □

Exercise 2.23. Find a quadrilateral $ABCD$ such that $|AD| = |CB|$ and $AB \parallel CD$ yet $ABCD$ is not a parallelogram.

3. Circular arcs

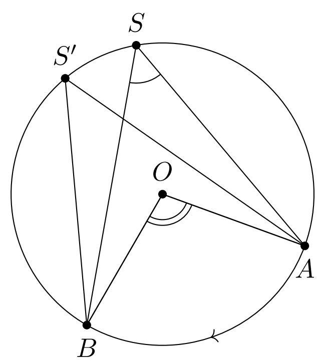
Definition 3.1. Let A, B be points on a circle o with centre O. The **arc** AB is the arc of o going clockwise from A to B.

The **central angle** corresponding to the arc AB is the oriented angle AOB.

An **inscribed angle** is the oriented angle ASB with $S \in o$ outside the arc AB.

We say that the arc AB **subtends** the central angle AOB and the inscribed angle ASB (i.e. arcs subtend angles).

In the figure below we have shown an arc AB, the central angle $\angle AOB$ it subtends and two incribed angles it subtends. We emphasize that, as shown, there is more than one inscribed angle corresponding to the arc AB.



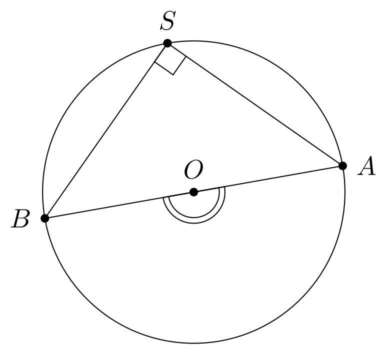
Theorem 3.2. Let A, B lie on a circle o. The measure of any inscribed angle subtended by the arc AB is half the measure of the central angle subtended by arc AB.

Corollary 3.3. Any two inscribed angles subtended by arcs of the same length have the same measure.

Remark. The length of an arc is proportional to the measure of the central angle it subtends, with proportionality constant equal to $2\pi R/360$, where R is the radius of the circle. One can easily prove this whenever the measure θ of the central angle α is equal to $m360°/n$, where m and n are natural numbers (by subdividing the circle into n congruent arcs and using a rotation to show that the arc subtending α has the same length as m of these n arcs). The case for general angle measure follows by an approximation by rational angle measures.

As a corollary to the proportionality between the subtended central angle's measure and the arc's length, we know that two arcs have equal length if and only if they subtend central angles with the same measure (assuming that the arcs lie on the same circle).

Corollary 3.4. Any inscribed angle subtended by arc AB, where the segment AB is a diameter of o, is $90°$.



Exercise 3.5. Prove Corollary 3.3 and Corollary 3.4 using Theorem 3.2.

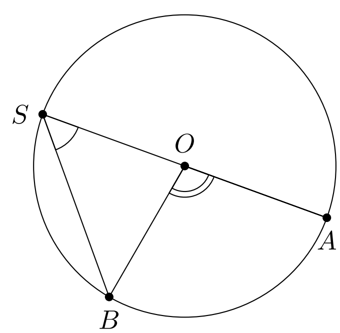
Exercise 3.6. Let ASB be a right-angled triangle, with the right angle at S. If O is the midpoint of AB, prove that $|AO| = |OB| = |OS|$.

Proof of Theorem 3.2. We will consider three cases.

Case 1: assume that $\angle AOB < 180°$, and that SA forms a diameter of the circle (see figure). This gives us $\angle BOS + \angle AOB = 180°$, and since OSB is an isosceles triangle, we have

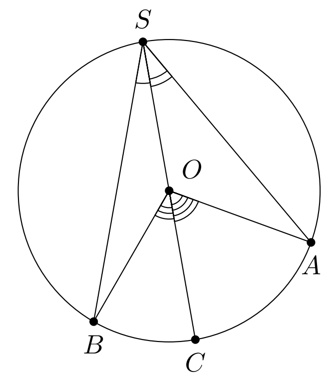
$$\angle OSB = \frac{1}{2}(180° - \angle BOS) = \frac{1}{2}\angle AOB,$$

and since $\angle ASB = \angle OSB$, we have shown $2\angle ASB = \angle AOB$, as desired.

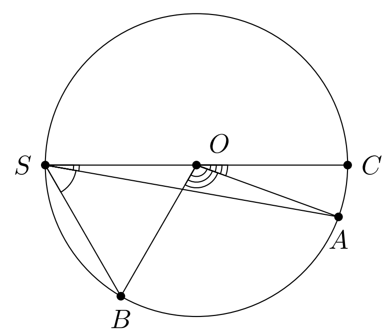


Case 2: The angle $\angle ASB$ contains O. This case follows relatively easily from the figure. From Case 1, we see that the cyan (cyan = light blue) angle is half the dark blue angle, and the orange angle is half the red angle. In other words, if we choose C so that SC form a diameter of the circle, then $2\angle ASC = \angle AOC$ and $2\angle CSB = \angle COB$. Since $\angle AOB = \angle AOC + \angle COB$, and $\angle ASB = \angle ASC + \angle CSB$, adding our previous equalities gives us

$2\angle ASB = \angle AOB$, as desired.



Case 3: The angle $\angle ASB$ does not contain O. This case is very similar to Case 2, although we replace the addition of angles used in Case 2 by subtraction of angles. The details are left to the reader.

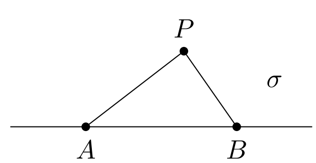


□

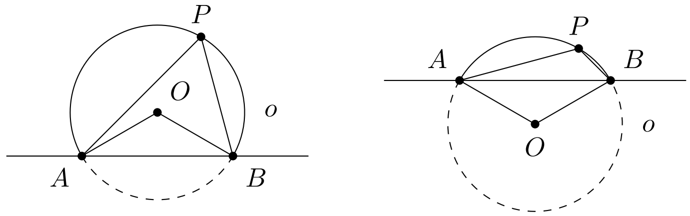
Corollary 3.7. Given a circle, any two inscribed angles with the same measure are subtended by arcs of the same length.

Proof. This follows immediately from Theorem 3.2, since it tells us that the central angle's measure is twice the inscribed angle's measure. We mentioned above that the arc's length is proportional to the measure of the central angle it subtends. □

Corollary 3.8. Given points $A \neq B$, let σ be the half plane bounded by AB containing all points P such that $\angle BPA < 180°$ (here we mean the clockwise oriented angle).



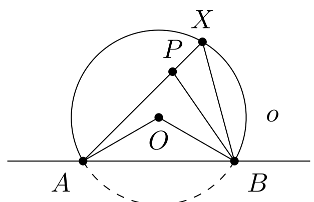
Then for any $\alpha > 180°$, the set of points P such that $\angle BPA = \alpha$ is the arc AB of the circle o with centre O on the bisector of AB satisfying $\angle BOA = 2\alpha$. **Remark**: note that O may lie on either side of AB, depending on the size of α.



Proof. By Theorem 3.2, it is clear that if P lies on the circle o, then $\angle BPA = \alpha$, since $\angle BPA$ is the inscribed angle of the arc AB subtending a central angle of measure 2α.

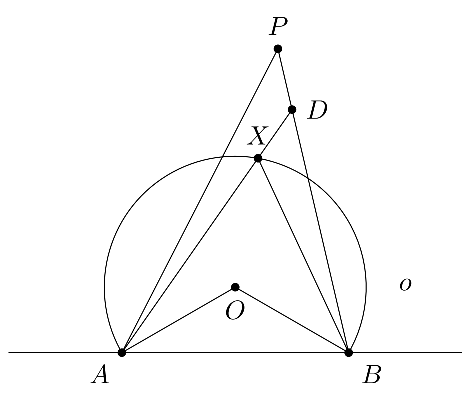
Conversely, we need to show that $\angle BPA = \alpha$ implies P lies on the circle o. In search of a contradiction, suppose that P lies off of the circle. We consider two cases.

Case 1: P lies inside the circle o. Then extend the line AP until it intersects o in a point X, as shown below.



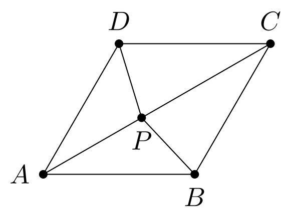
Now the idea is to look at the angles in the triangle BPX. We know that $\angle XPB + \angle BPA = \angle XPA = 180°$ (XPA lie on a straight line). But $\angle BPA = \alpha$, and so $\angle XPB = 180° - \alpha$. However, $\angle BXP = \alpha$, since X lies on the circle o (we proved this in the first part of this proof). Since $\angle BXP + \angle XPB + \angle PBX = 180°$, we conclude that $\angle PBX = 0$, which is a contradiction (since it implies the lines BX and BP are equal, and since BX intersects AP in a single point, we deduce that $X = P$, but we assumed $P \notin o$).

Case 2: P lies outside the circle o. Then pick $X \in o$ inside the angle $\angle BPA$, and let D be the intersection between AX and BP, as shown in the figure below.

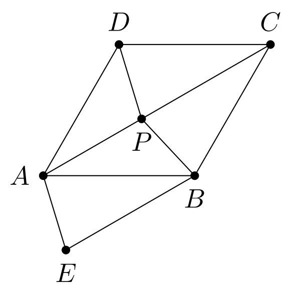


Note that $\angle BDX + \angle DXB < 180°$ (since these are two angles in a triangle). However $\angle DXB + \angle BXA = 180°$, and $\angle BXA = \alpha$ by the first part of this proof. Comparing the inequality with the equality yields $\angle BDX < \alpha$. A similar argument shows that $\angle BPA < \angle BDX < \alpha$. This contradicts our assumption that $\angle BPA = \alpha$. This completes the proof. □

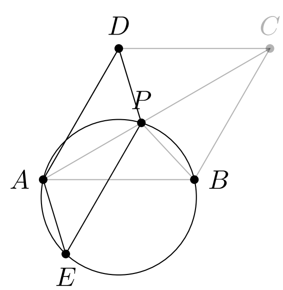
Problem 3.9. Suppose that point P lies inside a parallelogram $ABCD$ with $\angle ABP = \angle PDA$. Prove that $\angle DAP = \angle PCD$.



Solution. The idea is to translate the triangle DCP along the parallel edges AD and BC so that edge DC coincides with the parallel edge AB, and let P be translated to a point E, as shown below.

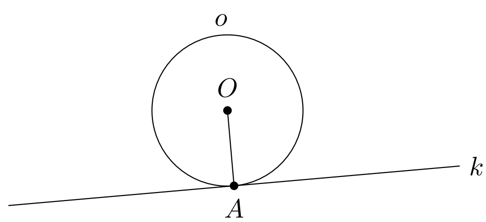


It is clear that AEB and DPC are congruent triangles, and so, in particular $|AE| = |DP|$. Then since $\angle CDP = \angle BAE$, we know that AE is parallel to DP. By the characterization of parallelograms in Proposition 2.22 we conclude that $DPAE$ is a parallelogram. In particular we conclude that the opposite angles $\angle PDA$ and $\angle AEP$ are equal. By our assumption, $\angle PDA = \angle ABP$, and so $\angle ABP = \angle AEP$. By Corollary 3.8, we conclude that $AEPD$ lie on a circle, as shown below.

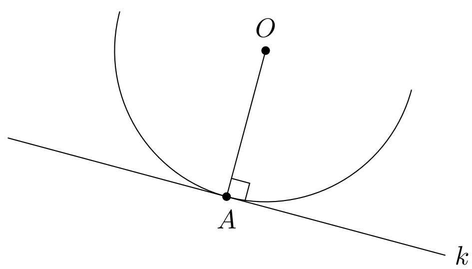


Then we apply Corollary 3.8 again to conclude $\angle EBA = \angle EPA$. Since the line AP cuts the parallel lines AD and PE in equal angles we deduce $\angle EPA = \angle DAP$. Combining our results so far, we have $\angle EBA = \angle DAP$. Since triangle DCP was congruent to ABE, we know $\angle EBA = \angle PCD$, so $\angle PCD = \angle DAP$, which was what we wanted to show.

Definition 3.10. A line k is **tangent** to a circle o if k intersects o in a single point.

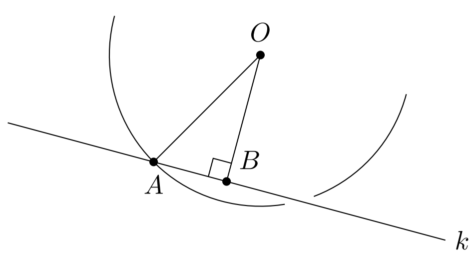


Proposition 3.11. If A lies on the circle o centred at O, then a line k is tangent to o at A if and only if k is perpendicular to the radius OA.



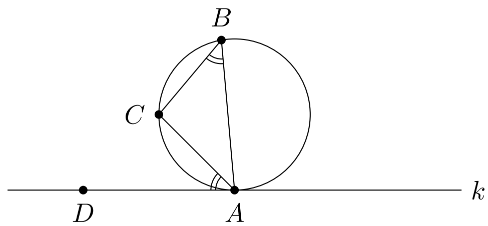
Proof. First suppose k contains A and is perpendicular to OA. Then the reflection through line OA fixes k and o (when we say the reflection "fixes" k we mean k is its own reflection). It follows that the reflection sends intersection points of k and o to intersection points of k and o. If there is another point $B \neq A$ which lies on k and o, then B cannot lie on OA, so B gets reflected to some other point $B \neq B' \neq A$. But then A, B, B' all lie on the intersection of o and k, which is impossible since we proved a line intersects a circle in at most 2 points (Exercise 1.25). Therefore the perpendicular to OA passing through A is tangent to o. It remains to prove that this is the only tangent through A.

Suppose that k is not perpendicular to OA, then let B be the projection of O onto k, as shown below.

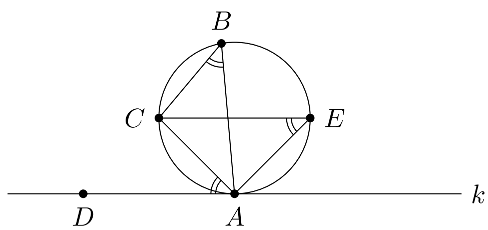


Consider the reflection in line OB; as in the first part of this proof, this reflection fixes k (since k is perpendicular OB) and fixes o (since any reflection through a line passing through the centre of a circle fixes that circle). Since A is the *only* intersection point of o and k, the reflection through OB must send A to itself, so OB and k intersect in two points A and B. It follows that $k = OB$, which is a contradiction, since it is obvious that any line passing through the centre of a circle o intersects o in two points. This completes the proof. $\square$

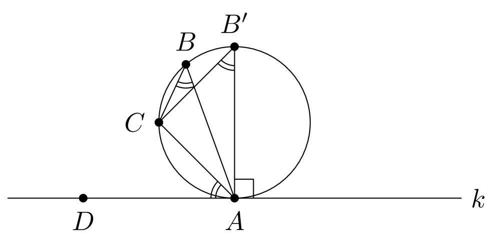
Problem 3.12. Let k be a line tangent to a circle o at A. Let B, C be distinct points on o distinct from A. Let $D \in k$ so that the angle DAB contains C. Then $\angle ABC = \angle DAC$.



Remark. If we let E be a point lying on the same side of AC as B, as shown below, then $\angle DAC$ can be thought of $\angle AEC$ in the limit as $E \to A$. This motivational remark is not meant to be a proof.



Solution. Assume that $\angle DAC < 90°$ (see the remark below). Pick B' such that AB' is a diameter and so that the arc AC subtends the inscribed angle $AB'C$ (so that $\angle ABC = \angle AB'C$).

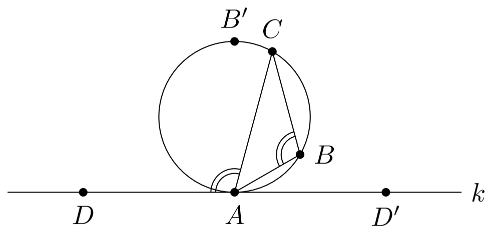


Then, since the angle $B'CA$ is subtended by a diameter, $\angle B'CA = 90°$. Therefore $\angle CAB' + \angle AB'C = 90°$. Since AD is tangent to the circle, and AB' is a diameter (so it contains the radius through A), we conclude by Proposition 3.11 that AD and AB' are orthogonal, so that $\angle DAB' = 90°$. The angle $\angle DAB'$ contains C; this requires the assumption that $\angle DAC < 90°$, in addition to the assumption that angle DAB contains C. Then we may write

$$\angle DAC + \angle CAB' = \angle DAB' = 90° = \angle CAB' + \angle AB'C,$$

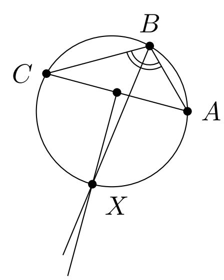
so $\angle DAC = \angle AB'C$, as desired.

Remark. The proof we give below is the case when that $\angle DAC \leq 90°$. The problem still has a solution when $\angle DAC \geq 90°$, but it will require a modified argument, since we will not be able to conclude that $\angle ABC = \angle AB'C$ (see the figure below).

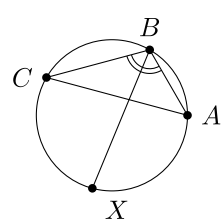


Exercise 3.13. Adapt the solution of Problem 3.12 to prove the case $\angle DAC \geq 90°$. *Hint*: prove the case $\angle DAC = 90°$ separately, and in the case $\angle DAC > 90°$, consider a point D' on line k as in the figure shown in the above Remark, and then apply the solution of Problem 3.12 with B and C interchanged.

Problem 3.14. Let ABC be a triangle with $|AB| \neq |BC|$. Show that the bisector of angle ABC intersects the bisector of segment AC in a point X lying on the circumscribed cricle.

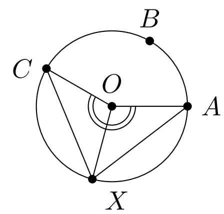


Solution. A good place to start when solving a problem which asks you to prove three figures intersect in a single point X is to guess what the point is, and then (if you guessed right) show that the three figures all contain it. In our problem, the three figures are the bisector of angle ABC, the bisector of AC and the circle. After staring at our figure for a while, we guess that X should be the point on the circle which splits the arc AC (not containing B) into two equal length arcs.

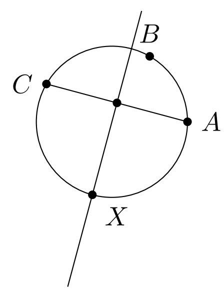


With this choice of X, we immediately realize that $\angle XBC = \angle ABX$ since the angles XBC and ABX are subtended by arcs of the same length (recall Corollary 3.7).

Furthermore, it is intuitively obvious that $|AX| = |CX|$. This can be rigourously proved by noting that AOX and COX are congruent by "side-angle-side" criterion, since the central angles subtended by arcs AX and XC are equal since the arc lengths are equal (as shown below), and the sides OA, OX and OC all have equal length (since they are all radii of the circle). Thus the third side pair is also equal $|AX| = |CX|$.

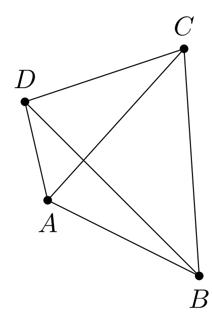


But then since $|AX| = |XC|$ we know that X lies on the bisector of AC.



We have shown that X lies on both the bisector through angle ABC and through segment AC, and this completes the solution.

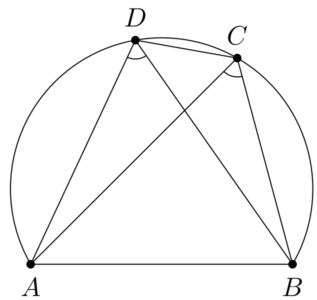
Definition 3.15. Let $ABCD$ be a quadrilateral. We say that $ABCD$ is **convex** if both diagonals AC and BD lie in the interior of $ABCD$.



Theorem 3.16. Let $ABCD$ be a convex quadrilateral. Then the following are equivalent.

(i) $ABCD$ lie on a common circle.
(ii) $\angle BDA = \angle BCA$.
(iii) $\angle DAB + \angle BCD = 180°$.

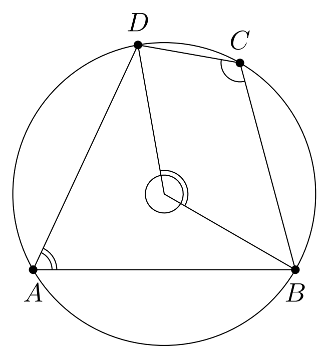
Proof. We begin by proving (i) and (ii) are equivalent. By convexity of $ABCD$, the points CD lie on the same side of AB, and so the equivalence of (i) and (ii) follows from a direct application of Corollary 3.8.



Next we will prove that (i) implies (iii). Assuming the points $ABCD$ lie on a circle, as shown in the following figure, we have $2\angle DAB = \angle DOB$, since the angle DOB is the central angle corresponding to the inscribed angle DAB. Similarly $2\angle BCD = \angle BOD$. Therefore

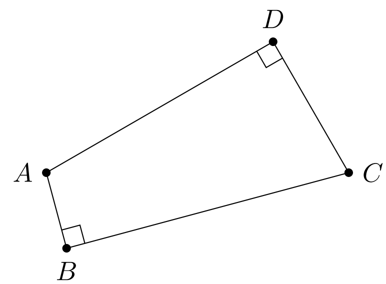
$$2\left(\angle DAB + \angle DOB\right) = \angle DOB + \angle BOD = 360^\circ,$$

which is what we wanted to show.

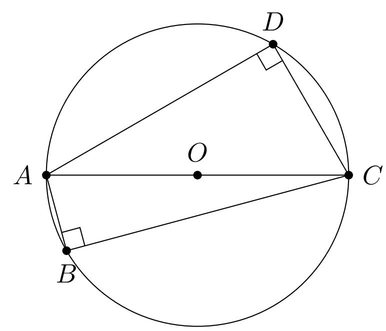


We show that (iii) implies (i). Let O be the centre of the circumscribed circle of ABD. Since $\angle DOB$ is twice $\angle DAB$ and $\angle BOD + \angle DOB = 360^\circ$, we conclude $2\angle BCD = \angle BOD$. However, if we choose any other C' on the arc DB of the circle through ABD we will also have $2\angle BC'D = \angle BOD$. By Corollary 3.8, this implies that $\angle BCD$ is the correct value for C lies on the circle centred on O, and so C lies on the circle through ABD. This completes the proof. $\square$

Corollary 3.17. Let $ABCD$ be a quadrilateral made out of two right angles, as shown below. (So $\angle ABC$ and $\angle CDA$ are $90°$).

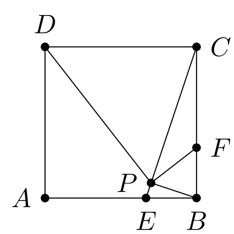


Then $ABCD$ has a circumscribed circle, and furthermore, the circumscribed circle is centred on the midpoint of AC.

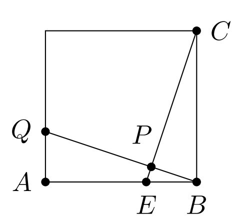


Proof. Left to the reader. □

Problem 3.18. Let $ABCD$ be a square, and pick E and F on sides AB and BC, respectively, so that $|BE| = |BF|$. Let P be the projection of B onto CE. Prove that $\angle DPF$ is $90°$.

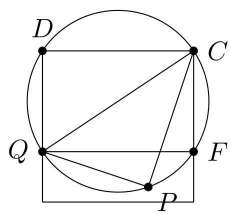


Solution. We begin our solution by extending the segment BP until it intersects AD at a point Q.

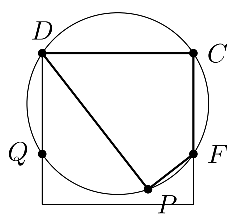


We claim that triangles QAB and EBC are congruent. To prove this claim, we will use the "side-angle-angle" criterion for congruence. Since $\angle BCE = \angle BCP$ and BCP is a right angled triangle (by definition of P) we deduce $\angle BCE + \angle QBC = 90°$. Since ABC is a right angle, we have $\angle ABQ + \angle QBC = 90°$. From these two equalities, we deduce $\angle BCE = \angle ABQ$. Since $\angle QAB = \angle EBC = 90°$, we now know that two angle pairs in triangles QAB and EBC are equal. Since the triangles also have an equal side length pair $|AB| = \angle BC$, we have satisfied the requirements for the "side-angle-angle" criterion, and we conclude QAB and EBC are congruent.

Thanks to this congruence, we conclude $|QA| = |EB| = |BF|$. Thus $|DQ| = |CF|$. Since $DQ \parallel CF$, we conclude $QFCD$ is a parallelogram (by Proposition 2.22) and since angle QDC is a right angle, we deduce $QFCD$ forms a rectangle. Then, since the opposite angles at D and F add up to 180°, we deduce by Corollary 3.17 that $QFCD$ has a circumscribed circle. However, since $\angle QPC = 90°$, and $\angle QDC = 90°$, we deduce by Corollary 3.17 that $QPCD$ also has a circumscribed circle. There is only one circle circumscribing the three points QCD, and so we conclude that the circumscribed circles of $QFCD$ and $QPCD$ are the same, so the five points $QPFCD$ lie on a common circle. This is summarized in the figure below.



But then the quadrilateral $DPFC$ is circumscribed by a circle.

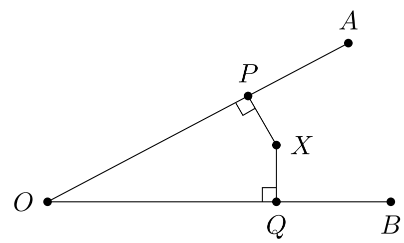


so applying Corollary 3.17 once more, we deduce $\angle DPF + \angle FCD = 180°$, and since $\angle FCD = 90°$ (since it is one of the corners of our original square) we deduce $\angle DFP = 90°$, as desired. This completes the solution. □

Exercise 3.19 (The Simson Line, hard exercise)**.** Let ABC be a triangle and let P lie on the circumscribed circle of ABC. Let K, L, M be the projections of P onto the lines BC, AC and AB, respectively. Then the points K, L, M lie on a common line.

4. Circles inscribed in angles and the "strongest theorem of geometry"

Theorem 4.1. Let X be a point lying inside an acute angle AOB. Let P and Q be the projections of X onto the half-lines OA and OB, respectively. Note that P and Q exist because AOB is an acute angle.



Then the following are equivalent.

(i) X lies on the bisector of the angle AOB.

(ii) $|OP| = |OQ|$.

(iii) $|XP| = |XQ|$.

Proof. To see that (i) implies (ii) and (iii), we use the "side-angle-angle" criterion for congruence to prove OXP and OXQ are congruent. By the assumption (i) we know $\angle POX = \angle QOX$, and, since they are right triangles, we know $\angle XPO = \angle XQO$. They obviously share the side OX, and so we have enough information to apply the "side-angle-angle" criterion. Then (ii) and (iii) follow immediately from congruence of OXP and OXQ. For an alternate proof, we simply reflect through the bisector of AOB. Since O and X lie on this bisector, O and X are fixed by the reflection. By definition[1] of the bisector of an angle, the half lines OA and OB are interchanged by the reflection (since the projections must be interchanged). It follows that P and Q are interchanged by this reflection, and since reflections preserve distances we have $|OP| = |OQ|$ and $|XP| = |XQ|$, as desired.

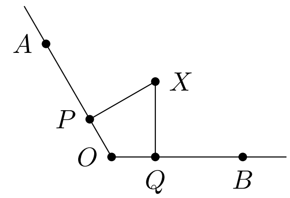
To prove (ii) implies (i) we use Proposition 2.20 which is the "side-side-angle" criterion for congruence when the equal angle pair is a pair of right angles. We can use this proposition to prove triangles OXP and OXQ are congruent, since we have equal side length pairs $|OP| = |OQ|$ and $|OX| = |OX|$, and the equal angle pair $\angle XPO = \angle XQO$ of *right* angles. Then, by this congruence, we have $\angle PXO = \angle OXQ$, and so, indeed X lies on the bisector of the angle POQ (which is the same angle as the angle AOB).

A very similar argument (also using Proposition 2.20) proves the implication (iii) $\Longrightarrow$ (i), and we leave this to the reader. □

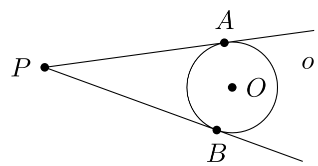
Exercise 4.2. In the setting of Theorem 4.1, prove that as long as the projections P and Q of X onto the lines OA and OB lie on the half-lines OA and OB, then the theorem is true,

[1]some define the bisector of an angle as the line which interchanges the two half-lines defining the angle.

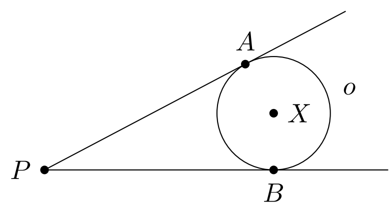
even if the angle AOB is *not* acute.



Exercise 4.3. Let P lie outside a circle o centred at O. Prove that there are exactly two points A and B on o so that PA and PB are tangent to o. *Hint*: Consider the circle whose diameter is OP.



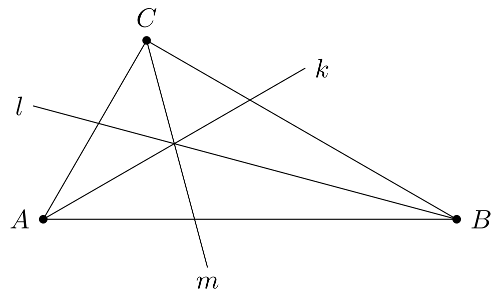
Theorem 4.4 (strongest theorem of geometry)**.** Let P lie outside a circle o centred at a point X, and pick A and B on o so that PA and PB are tangent to o. Then $|PA| = |PB|$



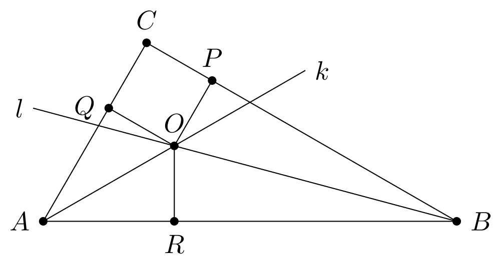
Proof. First we remark that the points A and B are well-defined thanks to Exercise 4.3. Clearly $|AX| = |BX|$ and since PA and PB are tangent to the circle o, we also have $XA \perp PA$ and $XB \perp PB$ (we proved this in Proposition 3.11). Thus we can use (iii) $\Longrightarrow$ (ii) in Theorem 4.1 to conclude $|PA| = |PB|$, as desired. $\square$

Remark. If we want to use Theorem 4.1, then we should require that $\angle APB \leq 90°$. However, as mentioned in Exercise 4.2, as long as the projections remain on the angle, it does not matter if the angle is not acute. Since the points A B always lie on the angle APB in the statement of the "strongest theorem of geometry," we can use Theorem 4.1 without worrying about acuteness of the angle APB.

Theorem 4.5. In a triangle ABC the three bisectors of its angles intersect in a common point.



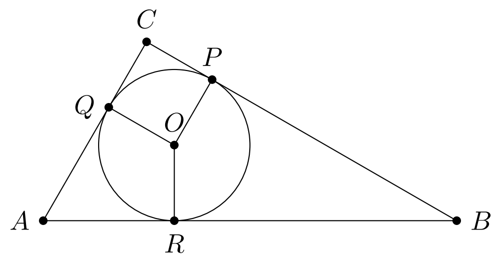
Proof. As in the figure above, let k, l, m be the bisectors of the angles at A, B, C, respectively. Let O be the point of intersection between k and l, and consider the projections P, Q, R of O onto the sides opposite A, B, C, respectively.



Since O lies on the bisector of CAB, we know $\angle CAO$ and $\angle OAB$ are both less than $90°$, and so the projections of O onto the lines AC and AB lie on the half-lines AC and AB, so we may apply Theorem 4.1 to the angle CAB and point O. Similarly, we may apply Theorem 4.1 to the angle ABC and point O (i.e. we do not need to worry about acuteness of angles).

Since O lies on the bisector of CAB we know from Theorem 4.1 that $|QO| = |OR|$, and (similarly) since O lies on the bisector of ABC we know $|OR| = |OP|$. But then $|OQ| = |OP|$ so we conclude, again by Theorem 4.1, that O lies on the bisector of BCA. This completes the proof. □

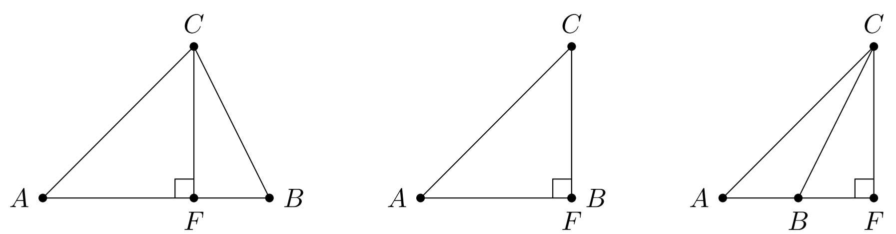
Corollary 4.6. In the setting of the above theorem, the circle o centred at O with radius $|OR| = |OP| = |OQ|$ is the a circle contained in the triangle ABC which is tangent to the edges of ABC. In other words it is an *inscribed* circle of the triangle ABC. We have therefore shown that every triangle admits an inscribed circle.



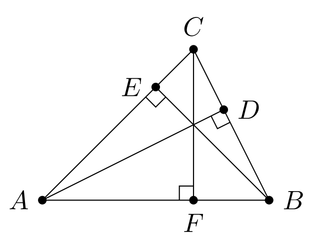
Exercise 4.7. We have shown above that every triangle admits an inscribed circle. However, we have not shown that the inscribed circle is unique. Prove that every triangle has only *one* inscribed circle. *Hint:* What must be the centre of any inscribed circle?

5. Altitudes of Triangles and Escribed Circles

Definition 5.1. The **altitude** in a triangle ABC through the vertex C is the line perpendicular to AB through C. The intersection of the altitude with the line AB is called the **foot** of the altitude. In the figure below we have shown the altitude through C and its corresponding foot F in a few cases.

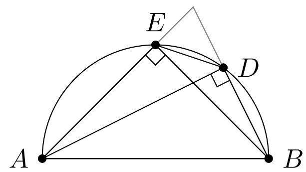


Theorem 5.2. The three altitudes in any triangle intersect in a common point.

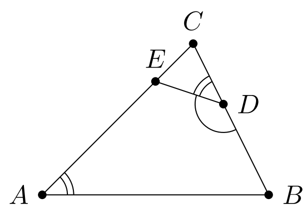


Proof. We will only prove the theorem when all the feet lie inside the triangle, although the theorem is still true if we drop this assumption.

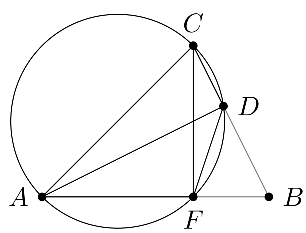
Let D, E, F be the feet of the altitudes of A, B, C, respectively. Since BEA and BDA are both right angles, we conclude that E and D lie on the circle whose diameter is AB.



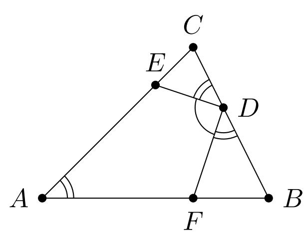
Then the quadrilateral $ABDE$ is inscribed in a circle, so we can apply Theorem to conclude $\angle EAB + \angle BDE = 180°$. Since BDC are collinear, we have $\angle BDE + \angle EDC = \angle BDC = 180°$, and combining this with our previous equality, we deduce $\angle EAB = \angle EDC$.



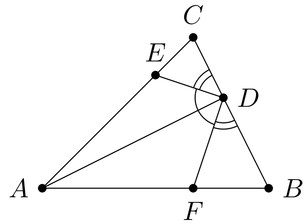
In a similar fashion, we deduce $ACDF$ lie on a common circle.



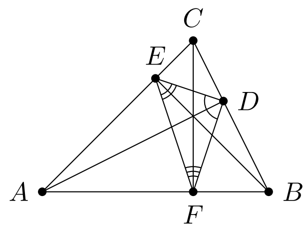
Since $\angle CAF + \angle FDC = 180°$, as we argued above this implies $\angle CAB = \angle DFB$. Combining this with what we have shown already yields $\angle EDC = \angle BDF$.



Since AD is perpendicular to BC we have $\angle EDC + \angle ADE = 90°$ and $\angle BDF + \angle FDA = 90°$, so using our previous equality, we conclude $\angle FDA = \angle ADE$. This implies that AD is the bisector of the angle FDE.

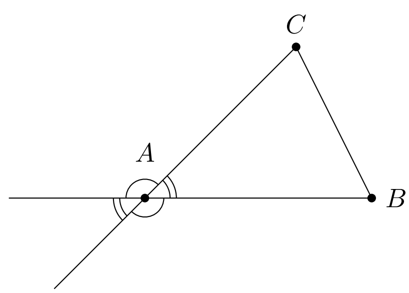


Then, relabeling $A \mapsto B \mapsto C \mapsto A$ and $D \mapsto E \mapsto F \mapsto D$, the same argument we just gave will prove that each altitude of ABC is the bisector of the corresponding angle of the triangle DEF.

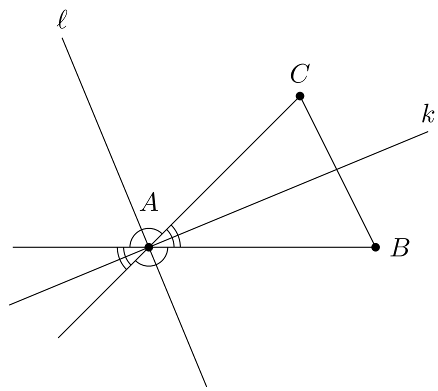


Since we proved in Theorem 4.5 that the bisectors of the angles of a triangle intersect in a single point, we conclude that the altitudes of ABC, which are the bisectors of the angles of DEF intersect in a single point, and this completes the proof. □

Definition 5.3. Given a triangle ABC we define the bisector of the **exterior angle** at the vertex A as follows. The lines AC and AB intersect at A and make 4 angles, which split into two pairs of equal measure, as shown in the figure below.



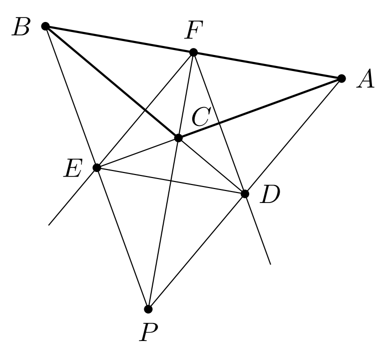
As shown below, there is a unique bisector k of the first (interior) angle pair, and a bisector ℓ of the second (exterior) angle pair.



Exercise 5.4. Let ABC be a triangle, and prove that the exterior bisector and interior bisector at the vertex C are perpendicular. *Hint*: show that the reflection in the interior bisector fixes the lines AB and lines AC, and hence fixes the exterior bisector.

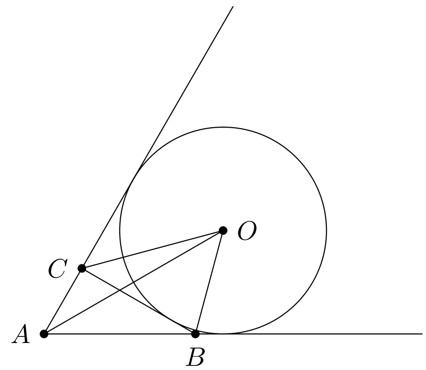
Exercise 5.5. Prove that the exterior bisectors at A and B and the interior bisector at C intersect at a single point. (Obviously we have a similar result when we permute the vertices $A \mapsto B \mapsto C \mapsto A$).

Exercise 5.6. Let ABC be a triangle. Prove that the feet of A and B do not lie inside the triangle if and only if $\angle BCA > 90°$. In thise case, let DEF be the triangle made out of the feet of ABC. Using Execise 5.5, let P be the intersection point of the interior angle bisector at F, and the exterior angle bisectors at D and E. Prove that the altitudes of ABC intersect at P.

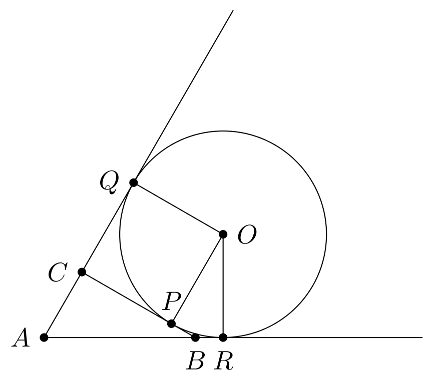


Definition 5.7. Let ABC be a triangle, and let O be the intersection point of exterior angle bisectors at B and C and the interior angle bisector at A (this intersection point is guaranteed by Exercise 5.5). A fairly straightforward application of Theorem 4.1 proves that O lies at an equal distance d from the sides AB, AC and BC. The circle centred at O with

radius d is called the **escribed circle** of ABC opposite A.



Problem 5.8. Let ABC be a triangle and let O be the centre of the escribed circle opposite A. Let P, Q, R be the projections of O onto BC, AC and AB. Then $2\,|AQ|$ is equal to the perimeter of ABC.



Solution. Our solution uses the "strongest theorem of geometry" (Theorem 4.4). The first application of the theorem yields $|AQ| = |AR|$. Therefore, proving $2\,|AQ|$ is equal to the perimeter of ABC is equivalent to proving $|AR| + |AQ| =$ perimeter. Noting that

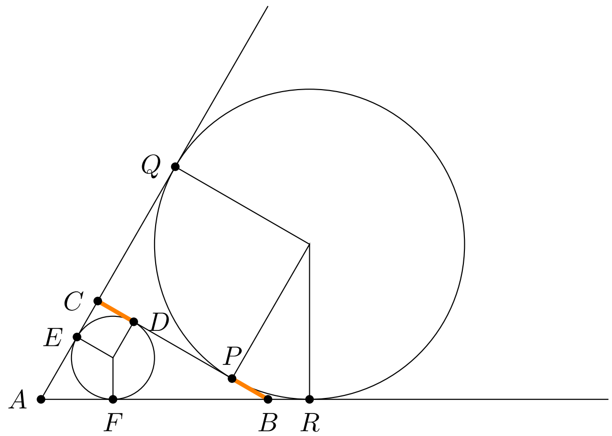
$$|AQ| = |AC| + |CQ| \quad \text{and} \quad |AR| = |AB| + |BR|$$

and that $|CQ| = |CP|$ and $|BR| = |BP|$ by the "strongest theorem of geometry," we obtain

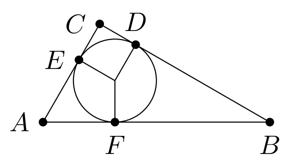
$$|AQ| + |AR| = |AC| + |AB| + |CP| + |BP| = |AC| + |AB| + |BC|\,.$$

This completes the solution. ◻

Problem 5.9. Let ABC be a triangle and consider the inscribed circle and the escribed circle opposite A. Label the tangency points as in the following figure. Show that $|CD| = |BP|$.

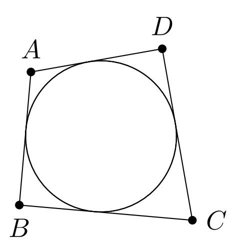


Solution. A simple application of the "strongest theorem of geometry" shows that $|AF| + |FB| + |CD| = |AB| + |CD|$ is half the perimeter of ABC.

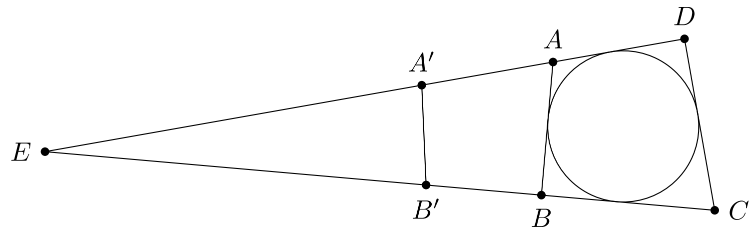


However by the preceeding problem, we also have $|AR| = |AB| + |BP|$ equal to half the perimeter. Since $|AB| + |CD| = |AB| + |BP|$, we deduce $|CD| = |BP|$, as desired. □

Definition 5.10. A circle o is inscribed in a quadrilateral $ABCD$ if o is tangent to each of the sides AB, BC, CD and DA.



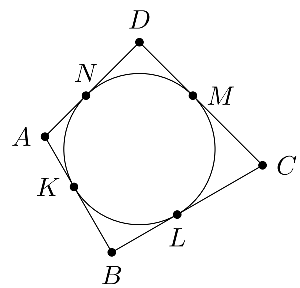
In the figure below we show a quadrilateral $A'B'CD$ which does not have an inscribed circle.



Theorem 5.11. A convex quadrilateral $ABCD$ has an inscribed circle if and only if

$$|AB| + |CD| = |AD| + |BC| .$$

Proof. We will only prove the $\implies$ direction, and the other direction is presented in Exercise 5.12. Suppose that $ABCD$ has an inscribed circle o, and let K, L, M, N be the tangency points of edges AB, BC, CD, DA with o, respectively.



Then the "strongest theorem of geometry" (applied four times) yields

$$|AK| = |AN| , \quad |BK| = |BL| , \quad |CL| = |CM| , \quad |DM| = |DN| ,$$

and since

$$|AB| = |AK| + |BK| \qquad |BC| = |BL| + |CL|$$
$$|CD| = |CM| + |DM| \qquad |DA| = |DN| + |AN|$$

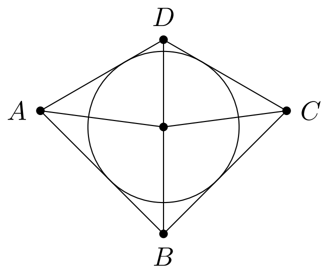
we deduce $|AB| + |CD| = |BC| + |AD|$, as desired. □

Exercise 5.12. The goal of this exercise is to establish the unproved direction in Theorem 5.11: If $ABCD$ is a convex quadrilateral satisfying $|AB| + |CD| = |AD| + |BC|$, then $ABCD$ has an inscribed circle.

(a) Prove that convexity is a necessary assumption.
(b) Suppose that $|AD|$ is the smallest side. Prove that $|BC|$ must be the largest side.
(c) If $|AD| = |BC|$, prove that all sides are equal, and so we have a **rhombus**. Show that, in this case, the diagonals AC and BD are also the bisectors of corresponding angles. Prove that there is a circle centred at the intersection point P of AC and BD

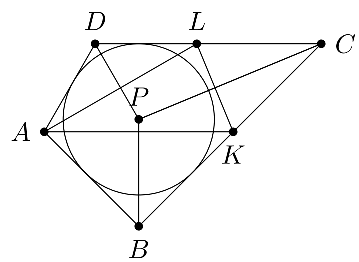
inscribed in $ABCD$. *Hint*: Consider the projections K, L, M, N of P onto the sides of $ABCD$.

(d) If $|AD| = |DC|$, then $|AB| = |BC|$, and so we have a **deltoid** (also known as a **kite**)



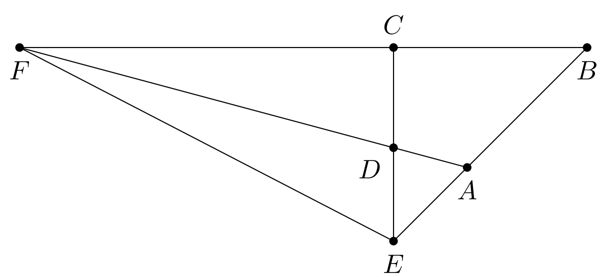
Show that, in this case, DB is the the angle bisector of D and B, and the angle bisectors at A and C intersect on BD. As in part (c) prove that the intersection of all the angle bisectors is the centre of an inscribed circle. Note that (d) implies (c).

(e) Now we assume that $|AD|$ is *not* equal to either adjacent side length $|DC|$ or $|AB|$.



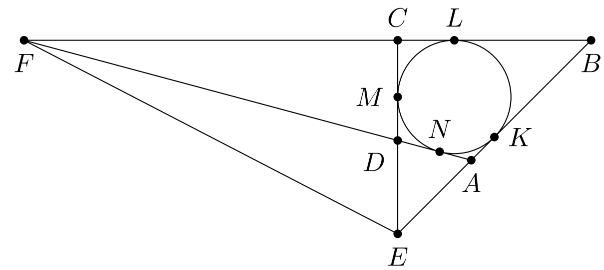
Let K, L be points on BC and DC so that $|DL| = |AD|$ and $|BK| = |AB|$ (why is this possible?). Prove that $|LC| = |KC|$. Prove that, as shown in the figure above, the angle bisectors at B, C, D are the bisectors of the sides AK, KL and LA of the triangle AKL. Therefore the angle bisectors of B, C, D intersect in a single point P. Prove that P is the centre of an inscribed circle of $ABCD$.

Problem 5.13. Consider a triangle EBF and a point D inside the triangle. Let C and A be the intesection points of FD and ED with BE and BF, respectively.



Prove that if $|AB| + |DC| = |AD| + |CB|$, then $|EB| + |DF| = |DE| + |BF|$.

Solution. Using Theorem 5.11, we know that $ABCD$ has an inscribed circle. Let K, L, M, N be the tangency points of the circle on the edges, as shown.



We compute

$$|BE| = |BK| + |EK| = |BL| + |EM| = |BL| + |DE| + |DM| = |BL| + |DE| + |DN|,$$

where we have used the "strongest theorem of geometry" in the second and fourth equalities. We obtain

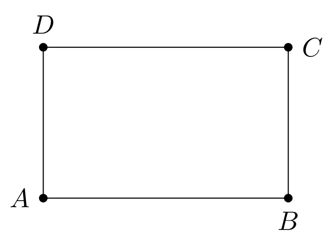
$$|BE| + |DF| = |BL| + |DE| + |FN| = |BL| + |DE| + |FL| = |BF| + |DE|,$$

where we have used the "strongest theorem of geometry" in the second equality to write $|FN| = |FL|$. This completes the solution. ◻

6. Area and Thales' Theorem

Fact 6.1 (the existence of area function)**.** There exists a non-negative **area function** for polygons, satisfying

(i) The area of a rectangle $ABCD$ is $|AB| \cdot |BC|$.



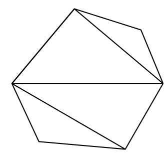
(ii) If polygons P_1 and P_2 have disjoint interiors, then

$$\text{Area}(P_1 \cup P_2) = \text{Area}(P_1) + \text{Area}(P_2).$$

(iii) If polygons P_1 and P_2 are congruent, then $\text{Area}(P_1) = \text{Area}(P_2)$.

We will use the notation $|P| := \text{Area}(P)$.

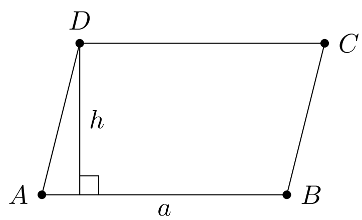
Remark. The usual "proof" of this fact simply defines the area of a triangle, and then defines the area of an arbitrary polygon P by "triangulating" the polygon (as shown below) and then defining the area of P as the sum of the areas of the triangles comprising the triangulation.



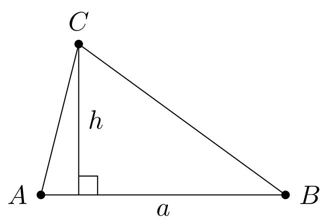
However, there is a major flaw with this "definition," as it *a priori* depends on the particular triangulation of P used. One must then prove that the area does not depend on the triangulation, and this is not easy to do.

There are more abstract proofs of the existence of the area function which use the machinery of "measure theory" (a branch of real analysis).

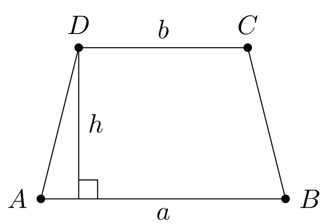
Theorem 6.2. (a) The area of a parallelogram with side length a and height h is ah.



(b) The area of a triangle with base a and height h is $ah/2$.

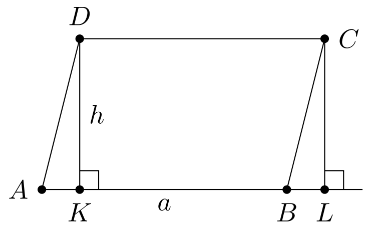


(c) The area of a trapezoid with parallel side lengths a, b and height h is $(a + b)h/2$.



Proof. To prove (a), let $K \in AB$ be the foot of the perpendicular to AB through D, and suppose that K lies between AB. Drop a perpendicular to AB from C, intersecting the line

AB in a point L, as shown below.

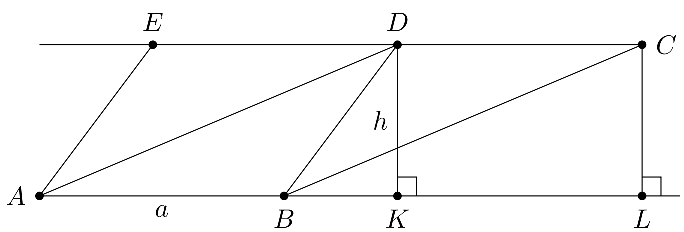


Then ADK and BCL are congruent by side-angle-angle criterion, since $|AD| = |BC|$ (by Definition 2.21, where we defined parallelograms) $\angle KAD = \angle LBC$ (since $AD \parallel BC$), and $\angle AKD = \angle BLC = 90°$. It follows by property (iii) of the area function that $|ADK| = |BCL|$, so

$$|ABCD| = |KBCD| + |AKD| = |KBCD| + |BCL| = |KDCL| = ah,$$

where we have used part (ii) in the first and third equalities, and part (i) in the final equality.

We proceed in the same way of the projection of C lies in AB. Otherwise, after possibly interchanging C with D (and A with B) we can assume that the projection of D separates the projection of C from AB on the line AB. Let E be the point at distance $|DC|$ from D distinct from C on the line CD. Note that $AE \parallel BD$ (show this!). Suppose, without loss of generality, that $|BD| \leq |AC|$.

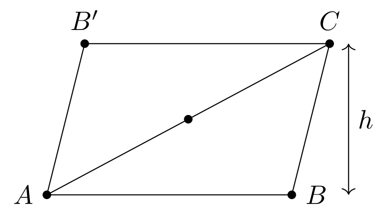


Triangles ADE and BCD are congruent by "side-side-angle" criterion with

$$|ED| = |DC|\,, \quad |AE| = |BC| \quad \text{and} \quad \angle DEA = \angle CDB$$

(prove this) and so a similar argument to the one used in the first part of the proof shows $|ABDE| = |ABCD|$. Repeat this process until either D or C has its projection K inside the segment AB (convince yourself that this will eventually happen).

To prove part (b) of the theorem, we begin with a triangle ABC, and then we rotate $180°$ through the midpoint of AC to obtain a parallelogram $ABCB'$, where B' is the rotated image of B, as shown below.

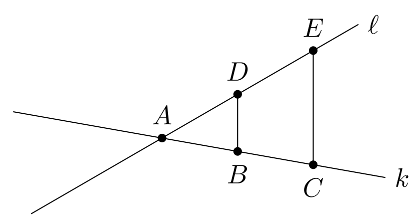


By construction ABC and $AB'C$ are congruent, so they have the same area. It follows from part (a) that $|ABC| = ah/2$.

For part (c), we let the diagonal AC cut the trapezoid into two triangles with the same height and sides lengths a and b, respecitvely. The result now follows from part (b). □

Armed with the area function, we can give a straightforward proof of **Thales' Theorem**.

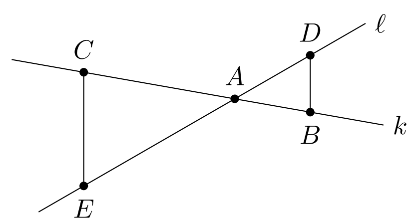
Theorem 6.3. Thales' Theorem If ℓ, k are two lines intersecting at a point A, points D, E and B, C lie on ℓ and k, respectively, and $DB \parallel EC$,



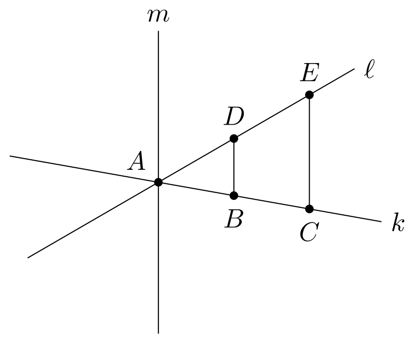
then

$$\text{(i)}\ \frac{|AB|}{|BC|} = \frac{|AD|}{|DE|} \qquad \text{(ii)}\ \frac{|AB|}{|AC|} = \frac{|AD|}{|AE|} \qquad \text{(iii)}\ \frac{|AB|}{|AC|} = \frac{|BD|}{|CE|}.$$

Remark. We note that B, C do need to lie on the same side of A as we have shown above. The theorem is still true if they lie on opposite sides of A, but the figure is different in this case.

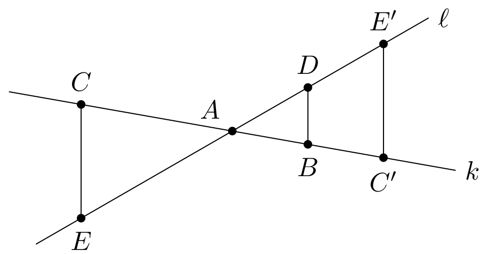


Proof of Thales' Theorem. First we note that if B and C lie on the same side of A, then D and E must also lie on the same side of A. This is intuitively obvious, and it can be shown by letting m be the line through A parallel to BD and CE.



Clearly B, C are on the same half-plane determined by m (by assumption that they lie on the same side of A on k), and since BD and CE do not intersect m, we also conclude D, E are on the same side of m, and, in particular, this implies that A is not between D and E.

Now if A lies in between B and C and between D and E, then we can rotate both C and E by $180°$ through A to obtain points $C' \in k$ and $E' \in \ell$ so that BC' and DE' do not intersect A, as shown. Furthermore, since $180°$ rotations send parallel lines to parallel lines, we deduce that $E'C' \parallel DB$. Finally $|AE| = |AE'|$, $|AC| = |AC'|$ and $|EC| = |E'C'|$.



Therefore, if we prove (ii) and (iii) in the case when A is not between B and C (and so also not between D and E) then we also prove (ii) and (iii) in the case when A is between B and C. Furthermore, if A is between B and C and between C and D then

$$|CB| = |AB| + |AC| \qquad |DE| = |AD| + |AE|$$

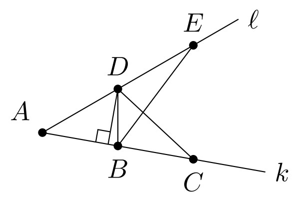
and so if we know (ii) holds, then we have

$$\frac{|CB|}{|AB|} = 1 + \frac{|AC|}{|AB|} = 1 + \frac{|AE|}{|AD|} = \frac{|DE|}{|AD|},$$

which obviously implies (i).

This argument shows that it suffices to prove the theorem in the case when BC and ED do not intersect A.

Consider triangles ABD and BCD.



It is clear the heights of these two triangles through vertex D are the same (since their bases lie on the same line BC) and so we deduce

$$\frac{|ABD|}{|BCD|} = \frac{|AB|}{|BC|}. \tag{$*$}$$

A similar argument yields

$$\frac{|ABD|}{|BDE|} = \frac{|AD|}{|DE|}. \tag{$**$}$$

Since triangles DBC and BDE share a height (namely, the distance between the parallel lines BD and CE) and a base BD, we deduce $|BCD| = |BDE|$, and so combining $(*)$ and $(**)$ we obtain

$$\frac{|AB|}{|BC|} = \frac{|AD|}{|DE|},$$

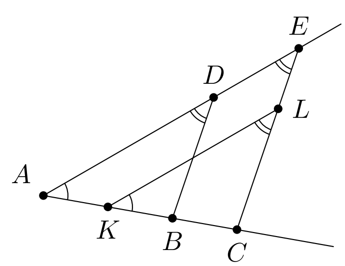
which proves (i).

Part (ii) follows immediately from part (i), since we have

$$\frac{|AC|}{|AB|} = \frac{|AB| + |BC|}{|AB|} = 1 + \frac{|BC|}{|AB|} = 1 + \frac{|DE|}{|AD|} = \frac{|AE|}{|AD|},$$

where we have used part (i) in the third equality.

Part (iii) requires a bit more work. Pick K, L so that CKL is congruent to BAD, as shown.



Since CE cuts KL and AE in equal angles (as shown), we deduce $KL \parallel AE$. Now we apply Thales' Theorem part (ii) to the angle ACE which intersects the parallel lines AE and LK. We obtain

$$\frac{|CK|}{|CA|} = \frac{|CL|}{|CE|}.$$

Since CKL is congruent to BAD, the above equation implies

$$\frac{|BA|}{|CA|} = \frac{|BD|}{|CE|},$$

which is part (iii) as desired. This completes the proof. □

Theorem 6.4 (converse to Thales' Theorem)**.** Let k, ℓ be lines intersecting at a point A, let B, C and D, E lie on k and ℓ, respectively, and (1) suppose that BC and DE both do not contain A or (2) BC and DE both contain A. If

$$\frac{|AB|}{|AC|} = \frac{|AD|}{|AE|}$$

then BD and CE are parallel.

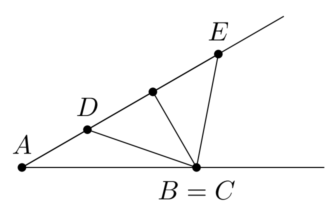
Remark. Note that our converse to Thales' Theorem is not a complete converse, and it only treats the case when we have equality (i) from the statement of Thales' Theorem. Counterexamples to the converse to Thales' when we have (ii) or (iii) are presented in the exercises below.

Exercise 6.5. Prove the converse to Thales' Theorem. *Hint:* Using a 180° rotation (as in the proof of Thales' theorem) argue that it suffices to prove the case (1). In this case, argue that any point $D' \in \ell$ such that (a) $|AD'| \,/\, |AE| = |AB| \,/\, |AC|$ and (b) D' lies on the same side of A as E must satisfy $D' = D$, and argue that the parallel to BC through E intersects ℓ in a point D' satisfying (a) and (b).

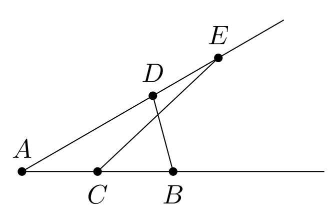
Exercise 6.6. Suppose that A, B, C, D, E are as in Theorem 6.4, but instead of (i) from Thales' theorem we have (iii), namely

$$\frac{|EC|}{|BD|} = \frac{|AC|}{|AB|},$$

Show that BD need not be parallel to CE. *Hint*:

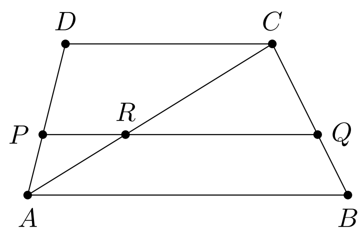


Exercise 6.7. As in the previous exercise, suppose that A, B, C, D, E are as in Theorem 6.4, but suppose instead we have (ii). Prove that BD need not be parallel to CE. *Hint*:



Remark. This counterexample shows that, in general (ii) does not imply $BD \parallel CE$. This remark will show that if we assume that B is between A and C and D is between A and E (or C is between A and B and E is between A and D) then we actually do have (ii) $\Longrightarrow$ parallelism. The details of the straightforward proof of this fact are left to the reader. *Hint*: use Theorem 6.4, and be careful when considering when we can write $|AC| = |AB| + |BC|$ (versus $|AC| = |AB| - |BC|$).

Problem 6.8. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Let P on segment AD and Q on segment BC be such that $|AP| / |PD| = |BQ| / |QC|$.



Prove that $PQ \parallel AB$ and that

$$|PQ| = \frac{1}{1+\lambda}|AB| + \frac{\lambda}{1+\lambda}|CD|,$$

where λ is the common ratio $|AP| / |PD| = |BQ| / |QC|$.

Solution. We will use the converse to Thales' Theorem (Theorem 6.4). Pick R on the diagonal AC so that $|AR| / |RC| = \lambda$ (see the figure above). Then, using the angle DAC with the converse to Thales' Theorem, we have $PR \parallel DC$. Similarly, using the angle BCA in the converse to Thales' Theorem, we have $RQ \parallel AB$. Since there is a unique parallel to $AB \parallel DC$ through R, we deduce the lines RQ and PR are equal and so the line PQ containing R is parallel to AB.

By Thales' Theorem part (iii) (applied to both angles DAC and BCA), we know that

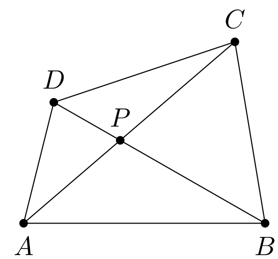
$$|PR| = \frac{|AP|}{|AD|}|DC| \quad \text{and} \quad |QR| = \frac{|CQ|}{|BC|}|AB|.$$

Since $|AD| = |AP| + |PD|$ and $|BC| = |QC| + |QB|$, we deduce

$$|PQ| = |PR| + |QR| = \frac{\lambda}{1+\lambda}|DC| + \frac{1}{1+\lambda}|AB|,$$

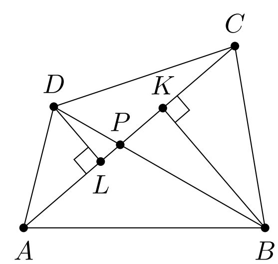
as desired. ◻

Problem 6.9. Let $ABCD$ be a convex quadrilateral with diagonals intersecting at a point P.



Prove that $\dfrac{|ABC|}{|ACD|} = \dfrac{|BP|}{|DP|}$.

Solution. Since the problem concerns the area of triangles ABC and BCD we introduce the feet K and L of altitudes through B and D, respectively.



Then $|ABC| = |AC|\,|BK|\,/2$ and $|ACD| = |AC|\,|DL|\,/2$. This implies

$$(*) \qquad \frac{|ABC|}{|ACD|} = \frac{|BK|}{|DL|},$$

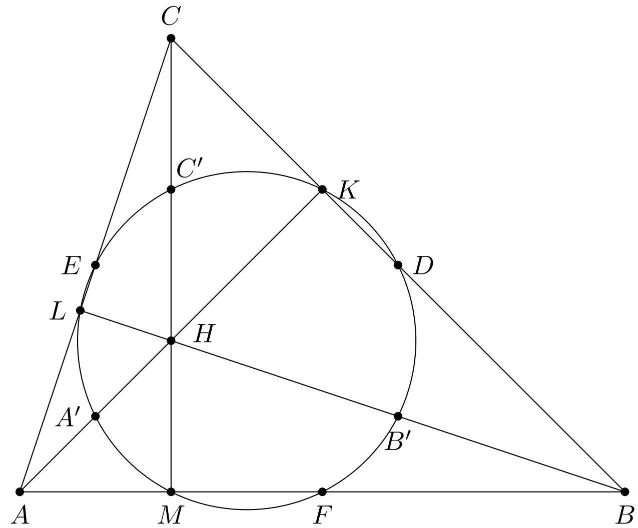
Since lines DL and BK are parallel (since they are both orthogonal to the line AC) we can apply Thales' Theorem to the angle formed at P. We obtain

$$\frac{|BK|}{|DL|} = \frac{|BP|}{|DP|},$$

and combining this with $(*)$ gives us the desired result. ◻

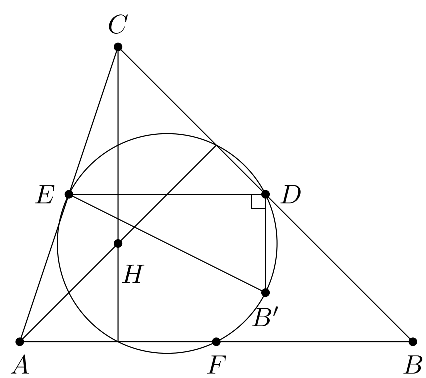
Problem 6.10 (Nine Point Circle)**.** In a triangle ABC with orthocentre H (the orthocentre is the intersection point of the altitudes) there is a circle o containing (i) the feet K, L, M of altitudes through A, B, C, respectively, (ii) the midpoints D, E, F of the edges BC, AC,

AB, respectively, and (iii) the midpoints A', B', C' of segments AH, BH, CH, respectively.



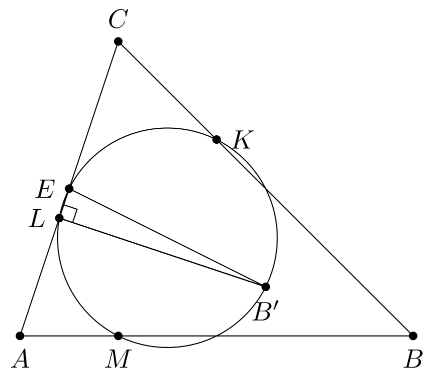
Solution. This problem is fairly daunting without an idea where to begin, and so to start the solution we simply make a guess (the correct guess) that the nine-point circle has the segment EB' as a diameter (we make this guess because it looks like it could be true).

Let o be the circle which has EB' as a diameter. The first step of our solution is to prove that the midpoints D, F of the edges adjacent to B lie on o. By relabelling points it suffices to prove that D lies on o. Since $|CE| / |CA| = 1/2$ and $|CD| / |CB| = 1/2$, we can apply the Converse to Thales' to the angle BCA, to conclude $ED \parallel AB$. Similarly, since $|BD| / |BC| = |BB'| / |BH|$, we can apply the Converse to Thales' to the angle HBC to conclude $B'D \parallel HC$. Since the altitude HC is perpendicular to AB, and $B'D \parallel HC$ and $ED \parallel AB$ we conclude that ED is perpendicular to $B'D$, and so $\angle B'DE = 90°$. By what we have shown in Corollary 3.8, we know that D lies on the circle o. As mentioned above, this argument also works to prove F lies on the circle o.



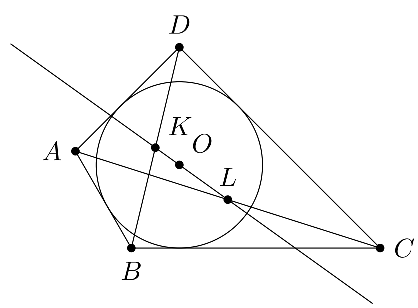
The next step in our solution is to prove that A' and C' lie on o. To show this, we note that in the first step we proved that o is the circumscribed circle of EDF; in other words, we showed that EB' is a diameter of the circumscribed circle of EDF. Since the circumscribed circle of EDF is independent of relabelling the vertices, we conclude (by symmetry) that DA' and FC' are also diameters o, and so, in particular A' and C' lie on o.

The final step in our solution is to prove that the feet of the altitudes lie on o. However this is easy to show using Corollary 3.8. Since $\angle ELB' = 90°$, we know that L lies on the circle o whose diameter is EB'. Again, by symmetry, K, M also lie on o.



This completes the solution. ◻

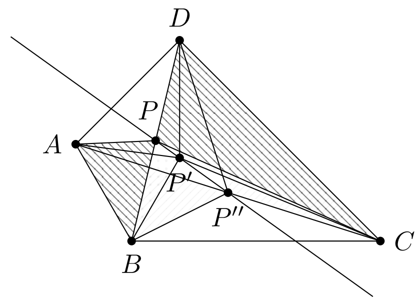
Theorem 6.11 (Newton Theorem)**.** Let $ABCD$ be a convex quadrilateral with an inscribed circle centred at a point O, and let L and K be the midpoints of the diagonals AC and BD, respectively. Then L, O, and K lie on the same line.



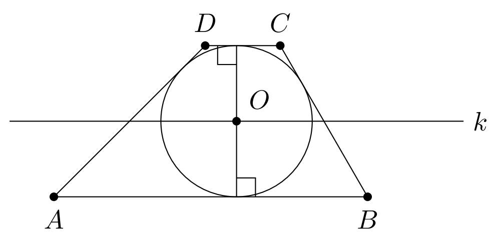
In the proof of Theorem 6.11, we will use a lemma (we will prove the lemma after we prove the theorem).

Lemma 6.12. Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, and choose some $a > 0$. Then the set of points P inside $ABCD$ satisfying $|PAB| + |PCD| = a$ lie on a common line. (In the figure below, we have shown points P, P' and P'' such that

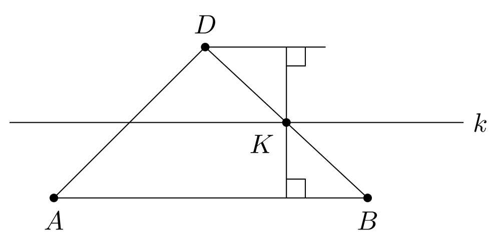
$|PAB| + |PCD| = |P'AB| + |P'CD| = |P''AB| + |P''CD|)$.



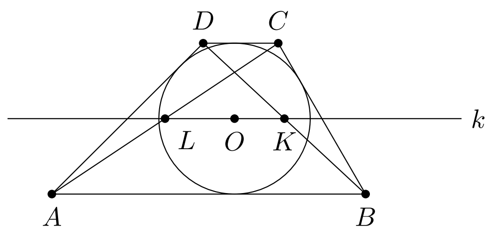
Proof of the Newton Theorem using Lemma 6.12. First suppose that $AB \parallel CD$ (this is the case when we cannot use the Lemma, and so we treat it with a different argument). Let k be parallel to AB and CD so that k is equidistant from AB and CD (more formally, k is defined as all the points which are equidistant from AB and CD). Then the centre of the circle, call it O, lies on the line k, since O bisects the orthogonal segment connecting AB to CD through O.



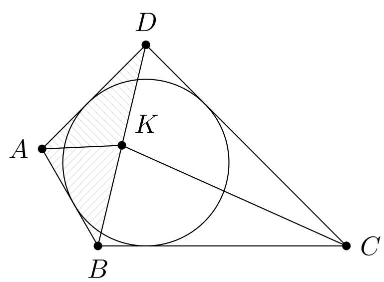
We will show that K and L also lie on k. This follows from Thales' theorem. Consider the angle at K formed between segments DB and the orthogonal segment to AB:



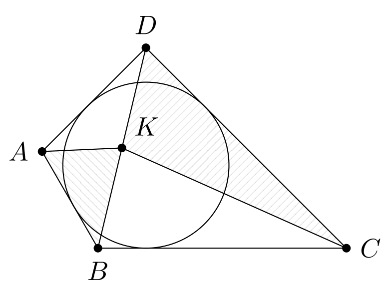
Since K bisects DB, we conclude (by Thales') that K also bisects the orthogonal segment to AB through K, and so K lies on k. A similar argument shows L lies on k, and so we have proved Newton's theorem in the case when $AB \parallel CD$.



In the case where CD is not parallel to AB, we can use Lemma 6.12. Let $a = |ABCD|/2$, and note that triangles ABK and ADK share a height and have a common base length $|BK| = |KD|$, so $|ABK| = |AKD|$.

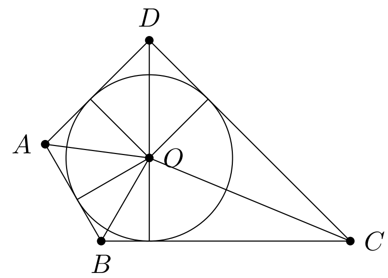


A similar argument proves that $|CDK| = |CKB|$. Therefore $|ABK| + |CKD| = a$.



The same argument shows that $|ALB| + |CLD| = a$.

Let r be the radius of the inscribed circle. We claim that $|AOB| = r|AB|$. This is true because the perpendicular segment joining O to AB intersects AB at the tangency point of the circle (as we have shown in Proposition 3.11), and so the height of AOB is equal to the radius r. A similar argument gives $|COD| = r\,|CD|$, $|AOD| = r\,|AD|$ and $|BOC| = r\,|BC|$.



Since $ABCD$ splits into the four disjoint triangles AOB, BOC, COD and DOA, we have

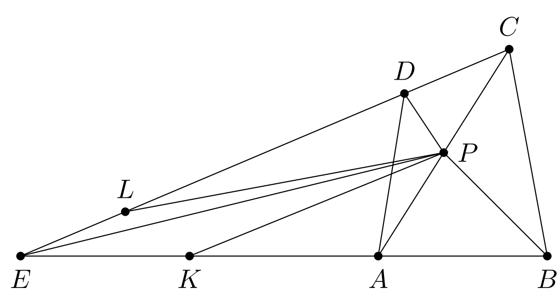
$$|ABCD| = r\left(|AB| + |BC| + |CD| + |DA|\right),$$

but since $|AB| + |CD| = |BC| + |DA|$ (Theorem 5.11), we conclude

$$|ABCD| = 2r\left(|AB| + |CD|\right) = 2\left(|AOB| + |COD|\right),$$

and so $|AOB| + |COD| = a$. By Lemma 6.12, we deduce that L, O, K lie on a common line. □

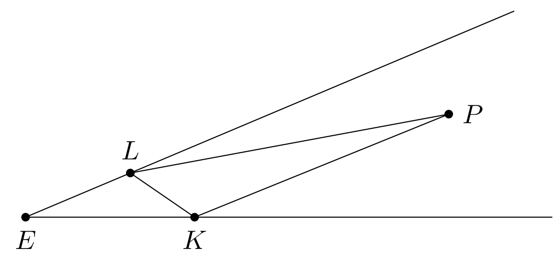
Proof of Lemma 6.12. Since AB and CD are not parallel, we know that they intersect in a point E. Furthermore, suppose P is chosen so that $|APB| + |CPD| = a$. Then choose points L and K on the half-lines ED and EA so that $|EL| = |DC|$ and $|EK| = |AB|$. We assume for simplicity that K and L lie outside the sides AB and DC, as shown below. The other cases are left as exercise for the reader.



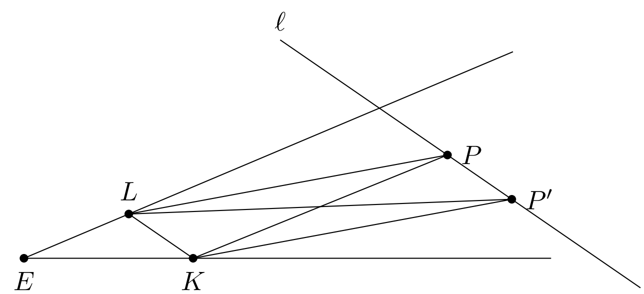
Since triangles EPL and DPC share a base length $|DC| = |LE|$ and share the height through P, they have the same area $|EPL| = |DPC|$. Similarly $|EPK| = |APB|$, and so $|EKPL| = |EPL| + |EPK| = a$.

Splitting the quadrilateral $EKPL$ into two triangles EKL and KPL, we note that $|EKPL| = a$ implies that $|KPL| = a - |EKL|$. Therefore, P lies on the set of all points (on the opposite side of LK to E) for which the triangle LPK has specified area $a - |EKL|$. Conversely, it is easy to show that if we form the triangle LPK and it has area $a - |EKL|$, then $|APB| + |CPD| = a$.

In other words, the set of P such that $|APB| + |CPD| = a$ is precisely the set of P such that $|KPL| = a - |EKL|$.



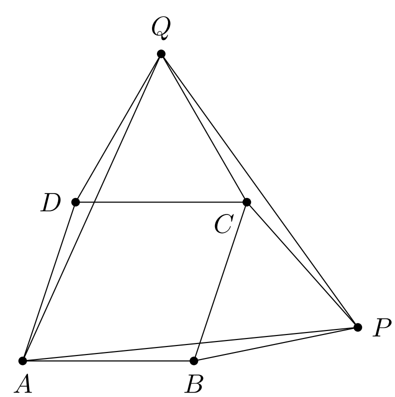
Let ℓ be parallel to LK passing through P, as shown. We claim that $|KPL| = |KP'L|$ if and only if P' lies on ℓ.



This is easy to see since, $|KPL| = |KP'L|$ if and only if the heights of triangles KPL and $KP'L$ through P and P' are equal, which happens if and only if PP' is parallel to base KL, so that $PP' = \ell$. This completes the proof of the lemma. □

Exercise 6.13. Modify the proof of Lemma 6.12 to prove the case when K and L do not lie outside the sides AB and DC.

Problem 6.14. Let $ABCD$ be a parallelogram construct equilateral triangles BPC and DQC outside of $ABCD$. Prove that APQ is equilateral.



Solution. We begin by proving that ABP is congruent to ADQ; this follows from the "side-angle-side" criterion since

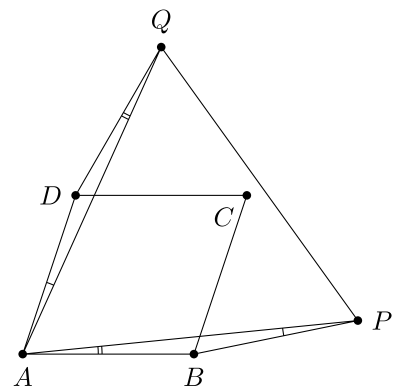
(side) $$|AB| = |DC| = |DQ|,$$

(angle) $$\angle ABP = \angle ABC + 60^\circ = \angle CDA + 60^\circ = \angle QDA,$$

(side) $$|BP| = |BC| = |AD|.$$

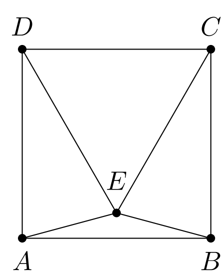
In the (angle) computation, we have used the fact that opposite angles in a parallelogram are equal.

Therefore $|AP| = |AQ|$ and so we deduce that PAQ is isosceles. To prove PAQ is equilateral, it suffices to prove that $\angle PAQ = 60^\circ$. Our proof depends on $\angle CDA$: we assume that $\angle CDA < 60^\circ$, and we leave the case $\angle CDA \geq 60^\circ$ to the reader. Let $\alpha = \angle BPA = \angle DAQ$ and $\beta = \angle PAB = \angle AQD$.

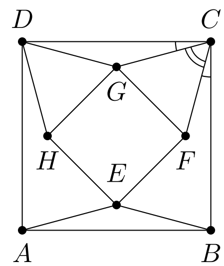


Then, since $\alpha + \angle PAQ + \beta = \angle DAB$, $\alpha + \beta + \angle ABC + 60^\circ = 180^\circ$ (here we have simply added all the angles in triangle ABP), and $\angle DAB + \angle ABC = 180^\circ$, we rearrange and obtain $\angle PAQ = 60^\circ$, as desired. This completes the solution. ◻

Problem 6.15. Let $ABCD$ be a square and ABE be the triangle in its interior satisfying $\angle EAB = \angle ABE = 15^\circ$. Prove that CDE is equilateral.

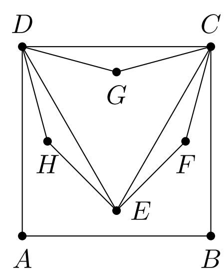


Solution. Let F, G, H be the rotations of E by $90°, 180°, 270°$ around the centre of the square inside $ABCD$, so that $\angle GBC = \angle BCG = 15°$ etc. Clearly $EFGH$ is a square (by its $90°$ rotational symmetry, it must have all sides and all interior angles equal). We claim that FCG and HDG are equilateral triangles.



Proving this claim is not hard. Since $\angle BCD = 90°$ and $\angle BCF = 15°$ and $\angle GCD = 15°$, we deduce $\angle FCG = 60°$. Furthermore, by "angle-side-angle" we know BFG and CGD are congruent (and both are isosceles), so $|CG| = |CF|$. Thus FCG is an isoceles triangle with vertex angle equal to $60°$, so FCG must be equilateral. Similarly GDH is equilateral. Interpreting $EFGH$ as a parallelogram with equilateral triangles FCG and GDH built outside of $EFGH$, we can use the previous problem to deduce EDC is equilateral. This completes the solution. ◻

Exercise 6.16. Solve the previous problem without using Problem 6.14 by proving that triangles CGD, CFE and DHE are all similar, so that $|CE| = |DE| = |CD|$.



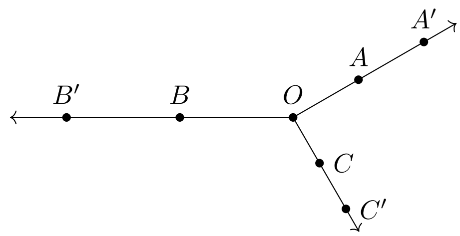
7. Similarities and Similar Triangles

Definition 7.1. Let $k > 0$. A **similarity with scale** k is a transformation of the plane $A \mapsto A'$ satisfying $|A'B'| = |AB|$ for all pairs of points A, B in the plane.

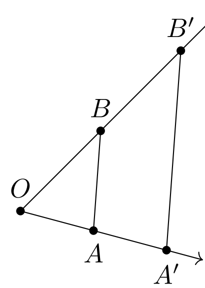
We have seen that all reflections and rotations are similarities with scale $k = 1$. We give scale 1 similarities a special name:

Definition 7.2. An **isometry** is a similarity with scale 1.

Example 7.3. In this example, we will construct the similarity known as a **homothety with centre** O. Fix a point O and a positive scale k. The **homothety centred at** O **with scale** k is the transformation of the plane which (a) fixes O and (b) maps each $A \neq O$ to the unique point A' on the half-line OA with $|OA'| = k\,|OA|$. In the figure below, we have shown the homothety centred at O with scale 2.



We claim that the homothety centred at O with scale k is a similarity with scale k. To prove this, we use both Thales' theorem and its converse. Let A, B be any two points so that A, O, B do not lie on the same line, and let A', B' be their images under the homothety. Since $|A'O| \,/\, |AO| = k$ and $|B'O| \,/\, |BO| \;=\; k$, we apply the converse to Thales' to deduce $A'B'$ is parallel to AB.

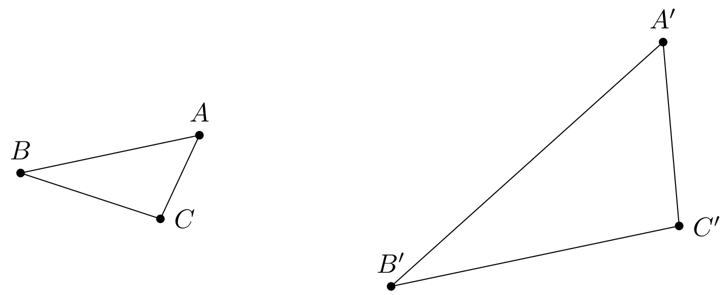


Then we can apply Thales' to deduce $|A'B'| \,/\, |AB| = |A'O| \,/\, |AO| = k$, so that $|A'B'| \;=\; k\,|AB|$, as desired.

If A, O, B lie on a common line, then the proof that $|A'B'| = k\,|AB|$ is an easy application of the equality case in the triangle equality; it is left as an exercise for the reader.

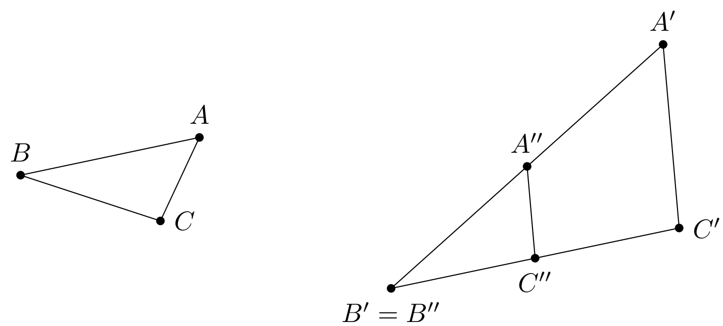
Exercise 7.4. Let O, A, B be three points on a line, and let A', B' be the image of A, B under the homothety with scale k, centred at O. Prove that $|A'B'| = k\,|AB|$. *Hint*: consider two cases: (i) when O is between A, B and (ii) when O is outside A, B. In case (i), prove that $|A'B'| = |A'O| + |OB'|$ and $|AB| = |AO| + |OB|$, so that $|A'B'| = k\,|AB|$. In the second case, use a similar argument.

Definition 7.5. Two triangles ABC and $A'B'C'$ are called **similar** if there is a similarity mapping ABC to $A'B'C'$.



Theorem 7.6. If ABC and $A'B'C'$ are similar then all corresponding angles are equal.

Proof. Suppose that ABC and $A'B'C'$ are similar with scale k. Let A'', B'' and C'' be the images of A', B', C' under the homothety centred at B' with scale $1/k$.



Then A'', B'', C'' is similar to ABC with scale 1, and so, by "side-side-side" criterion for congruence, $A''B''C''$ and ABC are congruent. It follows that $\angle ABC = \angle A''B''C''$. By definition of the homothety centred at B', we know that the half lines $B''A''$ and $B'A'$ and $B''C''$ and $B'C'$ are equal, and so $\angle A''B''C'' = \angle A'B'C'$. Thus $\angle ABC = \angle A'B'C'$. By relabeling the vertices of our triangles, we deduce all angle pairs are equal. This completes the proof. $\square$

Theorem 7.7. If ABC and $A'B'C'$ are two triangles satisfying any one of the following three conditions then ABC and $A'B'C'$ are similar.

(a) $|AB| / |A'B'| = |BC| / |B'C'| = |AC| / |A'C'|$.
(b) $|AB| / |A'B'| = |BC| / |B'C'|$ and $\angle ABC = \angle A'B'C'$.
(c) All corresponding angle pairs between ABC and $A'B'C'$ are equal.

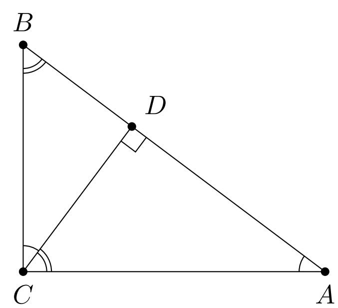
We refer to (a), (b) and (c) as the "side-side-side," "side-angle-side" and "angle-angle-angle" criteria for similarity, respectively.

Proof. Let $A''B''C''$ be the image of $A'B'C'$ under the homothety centred at B' with scale $|AB| / |A'B'|$. If (a) holds, then $A''B''C''$ is congruent to ABC by the "side-side-side" condition for congruency. If (b) holds, then $A''B''C''$ is congruent to ABC by the "side-angle-side" condition for congruency. If (c) holds, then $A''B''C''$ is congruent to ABC by the "side-angle-angle" condition for congruency. In all three cases, there exists a sequence of reflections taking $A''B''C''$ to ABC. Since reflections are isometries, the composition of the sequence of reflections taking $A''B''C''$ to ABC with the initial homothety taking $A'B'C'$ to $A''B''C''$ is a similarity taking $A'B'C'$ to ABC. This completes the proof. □

Corollary 7.8 (Pythagoras)**.** If ABC is a right-angled triangle (with right angle at C) then

$$|AC|^2 + |BC|^2 = |AB|^2 .$$

Proof. Let D be the projection of C onto the side AB. Let $\alpha = \angle CAB$ and $\beta = \angle ABC$. Since the angles in any triangle add up to 180°, we know $\alpha + \beta = 90$. Adding the angles in triangle ADC, we deduce $\angle DCA = \beta$ and, similarly, adding the angles in DBC we deduce $\angle BCD = \alpha$.



Then, by the "angle-angle-angle" criterion for similarity, we know CDB, ADC and ACB are all similar. Thus

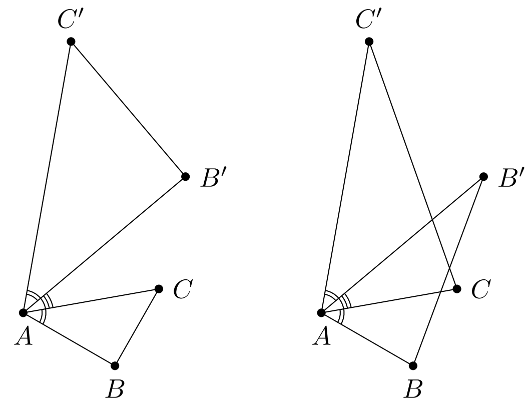
$$\frac{|BC|}{|AB|} = \frac{|BD|}{|BC|} \quad \text{and} \quad \frac{|AC|}{|AB|} = \frac{|DA|}{|AC|},$$

whereby we obtain

$$\frac{|BC|^2 + |AC|^2}{|AB|} = |BD| + |DA| = |AB| ,$$

which gives the desired result. □

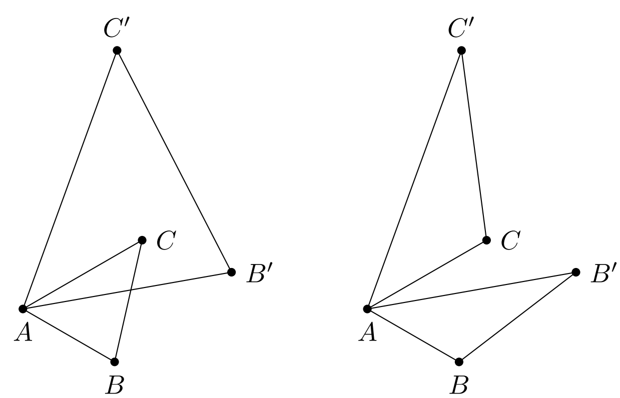
Corollary 7.9. Let ABC and $AB'C'$ be similar triangles, both labelled counterclockwise. Then ABB' and ACC' are similar.



Proof. We will use the "side-angle-side" criterion for similarity presented in part (b) of Theorem 7.7. Because ABC and $AB'C'$ are similar, we have $|AB'| / |AC'| = |AB| / |AC|$, and rearranging we obtain $|AB'| / |AB| = |AC'| / |AC|$. Then to prove the angles $\angle C'AC$ and $\angle B'AB$ are equal, we compute

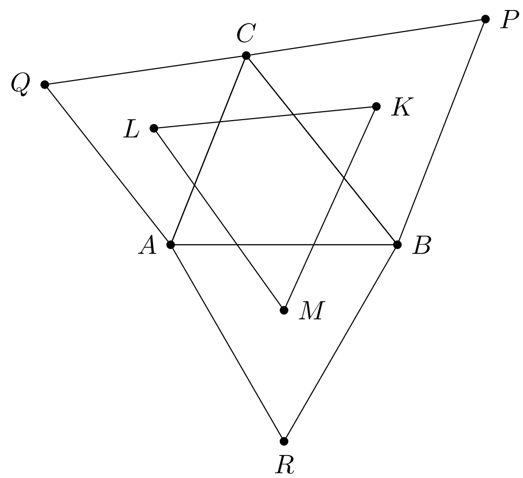
$$\angle C'AC = \angle C'AB' \pm \angle B'AC = \angle B'AC \pm \angle CAB = \angle B'AB,$$

where we have used the fact that $\angle C'AB' = \angle CAB$, which follows by similarity of ABC and $AB'C'$. The $\pm$ depends on whether or not the triangles are interlaced, i.e. we may have a figure of the form pictured below.

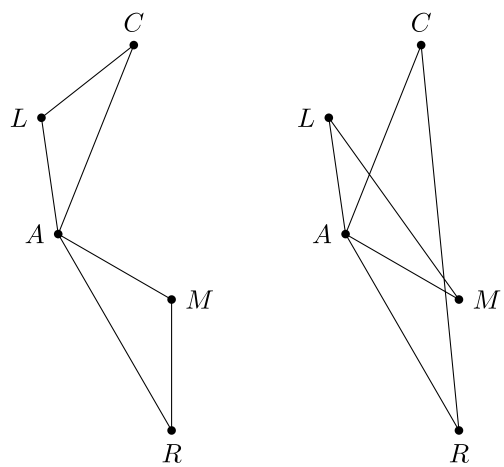


This completes the proof. □

Problem 7.10 (Napoleon's Theorem)**.** Let ABC be a triangle, and build equilateral triangles ARB, BPC and CQA outside of ABC, and let M, K, L be their orthocentres. Prove that KLM is equilateral.



Solution. Using the "angle-angle-angle" criterion for similarity (i.e. part (c) of Theorem 7.7) we deduce AMR is similar to ALC. Hence, by Corollary 7.9, we deduce AML is similar to ARC.



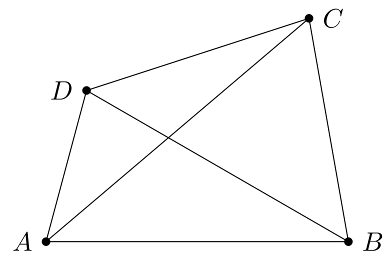
Therefore $|ML| / |CR| = |AM| / |AR|$. Analogously, MBK is similar to RBC, and so $|KM| / |CR| = |BM| / |BR|$. However, $|AM| / |AR| = |BM| / |BR|$, since AMR is congruent to BMR by the "side-side-angle" criterion for congruency, and so $|ML| = |KM|$. But then, by symmetry, all sides are equal (i.e. $|MK| = |KL|$ and $|KL| = |ML|$) and this completes the proof. □

Exercise 7.11. In the solution of Problem 7.10 we used the fact that AMR is congruent to BMR. Prove this using elementary arguments by considering the reflection through the bisector of the segment AB.

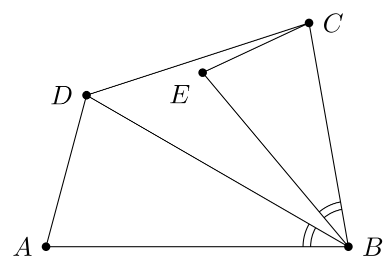
Ptolemy's Theorem. Let $ABCD$ be a convex quadrilateral. Then we have the following inequality

$$|AB|\,|CD| + |AD|\,|BC| \geq |AC|\,|BD|\,,$$

with equality holding if and only if $ABCD$ lie on a circle.



Proof. Let E be the point inside the angle ABC such that $\angle ABD = \angle EBC$ and such that $|BE|\,/\,|BA| = |BC|\,/\,|BD|$. Then, by the "side-angle-side" criterion for similarity, we know ABD is congruent to EBC.



(We remark that it is possible for E to leave the interior of the quadrilateral $ABCD$). By similarity of ABD and EBC, we know

$$(*) \qquad |EC|\,/\,|AD| = |BC|\,/\,|BD|$$

Using Corollary 7.9, we deduce ABE and DBC are similar, and hence

$$(**) \qquad |AE|\,/\,|DC| = |AB|\,/\,|DB|$$

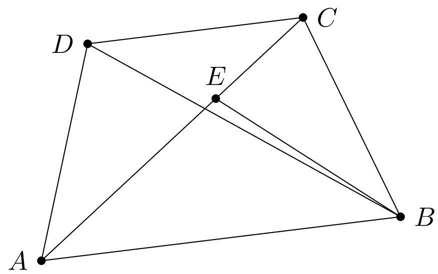
We note that we have found two equalities which involve all of the lengths present in Ptolemy's inequality. Using the triangle inequality, we obtain

$$\text{(T)} \qquad |AC| \leq |AE| + |EC|\,,$$

and multiplying both sides of this equation by $|DB|$, we obtain

$$|AC|\,|DB| \leq |AE|\,|DB| + |EC|\,|DB| = |AB|\,|DC| + |AD|\,|BC|\,,$$

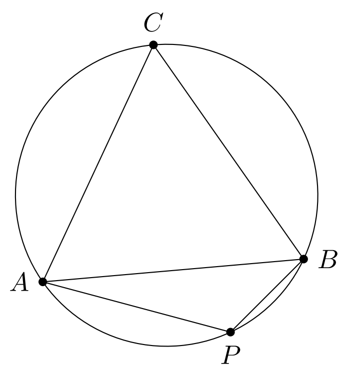
where we have used the equalities $(*)$ and $(**)$ in the second equality. This completes the proof of the inequality, and it only remains to characterize the case when equality holds. Equality will hold in Ptolemy's inequality if and only if equality holds in (T) which happens if and only if E lies on the segment AC. We (cleverly) observe that E lies on AC if and only if $\angle BCE = \angle BCA$.



If this is the case, then similarity of ABD and EBC implies that $\angle BDA = \angle BCE$, and so $\angle BDA = \angle BCA$. By Corollary 3.8 (a long time ago) we know that $ABCD$ lie on a circle.

Conversely, if $ABCD$ lie on a circle, then we have $\angle BDA = \angle BCA$. Since ABD and EBC are similar, We have $\angle BDA = \angle BCE$, and so $\angle BCE = \angle BCA$, so that E lies between A and C. Therefore, we have characterized the case when equality holds in Ptolemy's inequality, and this completes the proof of theorem. □

Example 7.12. Let ABC be an equilateral triangle with circumscribed circle o. Suppose P lies on the arc BA of o. Then $|CP| = |AP| + |BP|$.

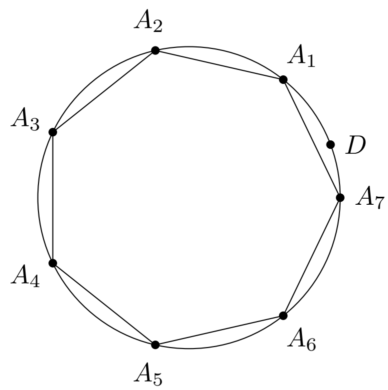


Clearly $APBC$ forms a convex quadrilateral whose vertices lie on a circle, and so we can apply the equality case in Ptolemy's Theorem. We obtain

$$|PC|\,|AB| = |AC|\,|PB| + |BC|\,|AP|\,.$$

However, $|AB| = |AC| = |BC|$, and so we can cancel these common terms in the above equality, and we obtain $|PC| = |PB| + |AP|$, as desired.

Problem 7.13. Let $A_1 \cdots A_7$ be regular heptagon (a **regular n-gon** is a polygon whose vertices form an ordered collection of n points $A_1 \cdots A_n$ on a circle so that the rotation of angle $360°/n$ sends A_j to A_{j+1} and A_n to A_1. A heptagon is just a 7-gon). Let o be the circumscribed circle of $A_1 \cdots A_7$, and pick D on the arc joining vertices A_1A_7.

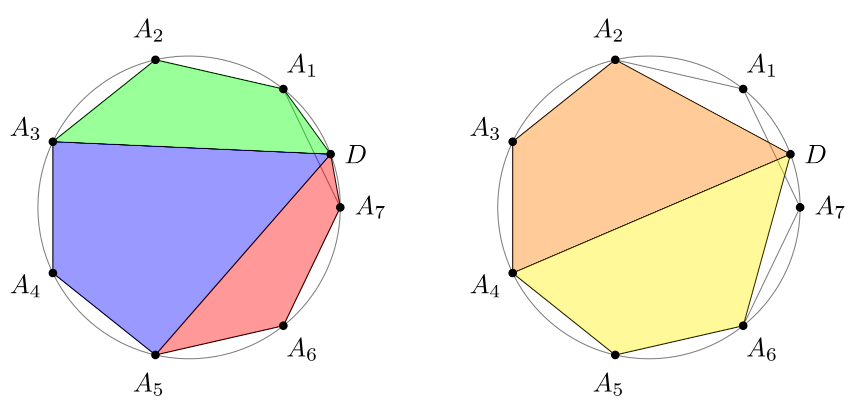


Show that

$$(*) \qquad |DA_1| + |DA_3| + |DA_5| + |DA_7| = |DA_2| + |DA_4| + |DA_6|\,.$$

Solution. The idea for the solution is simple, we will use the equality case in Ptolemy's Theorem seven times applied to various quadrilaterals inscribed in the circle, then we will add the seven resulting equalities together and, after some simplification, we will obtain $(*)$. We remark that this problem can be seen as a generalization of Example 7.12.

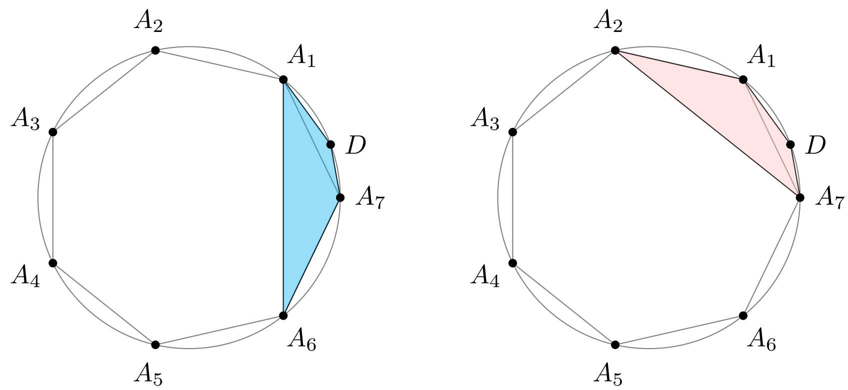
We begin our solution by identifying three side lengths in the heptagon. Let $a = |A_1A_2|$ and $b = |A_1A_3|$. Note that by the rotational symmetry of a regular heptagon, $a = |A_jA_{j+1}|$ and $b = |A_jA_{j+2}|$ (where we add integers modulo 7, so that $6+2 = 1$ etc). Now we consider the quadrilaterals $DA_jA_{j+1}A_{j+2}$, where $j = 1, \cdots, 5$ ($j = 1, 3, 5$ is shown on the left in the figure below, and $j = 2, 4$ is shown on the right).



Since each such quadrilateral is inscribed in a circle, we can apply the equality case in Ptolemy's theorem, and in all cases we obtain

$$(\star) \qquad b\,|DA_{j+1}| = a\,|DA_j| + a\,|DA_{j+2}|\,.$$

This gives us five equations. We still need to apply Ptolemy's theorem twice more. We consider the two quadrilaterals $DA_1A_6A_7$ and $DA_1A_2A_7$, shown below.



Applying Ptolemy to these quadrilaterals yields

$$(\star\star) \qquad a\,|DA_6| = a\,|DA_1| + b\,|DA_7| \quad \text{and} \quad a\,|DA_2| = b\,|DA_1| + a\,|DA_7|.$$

Now we add up the five equations in $(\star)$ and the 2 equations in $(\star\star)$ in a clever way: we gather the "even" terms (terms containing $|DA_j|$ for j even) on the left, and move the "odd" terms (terms containing $|DA_j|$ for j odd) to the right:

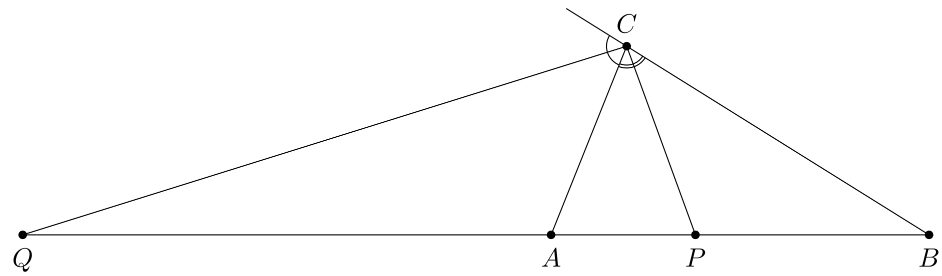
$$(\star \text{ and } \star\star) \qquad \begin{aligned} b\,|DA_2| &= a\,|DA_1| + a\,|DA_3| \\ a\,|DA_2| + a\,|DA_4| &= b\,|DA_3| \\ b\,|DA_4| &= a\,|DA_3| + a\,|DA_5| \\ a\,|DA_2| + a\,|DA_4| &= b\,|DA_5| \\ b\,|DA_6| &= a\,|DA_5| + a\,|DA_7| \\ a\,|DA_6| &= a\,|DA_1| + b\,|DA_7| \\ a\,|DA_2| &= b\,|DA_1| + a\,|DA_7| \end{aligned}$$

Adding all of these up yields

$$(2a+b)(|DA_2| + |DA_4| + |DA_6|) = (2a+b)(|DA_1| + |DA_3| + |DA_5| + |DA_7|),$$

and the common factor of $(2a+b)$ cancels and we obtain $(*)$, as desired. ◻

Theorem 7.14 (angle bisector theorem, cf. Definition 5.3 and Exercise 5.4)**.** Let ABC be a triangle, and let P be the intersection of the interior bisector at C and AB, and suppose that the exterior bisector at C intersects AB in a point Q (note that this is only possible if the exterior bisector at C is not parallel to AB).

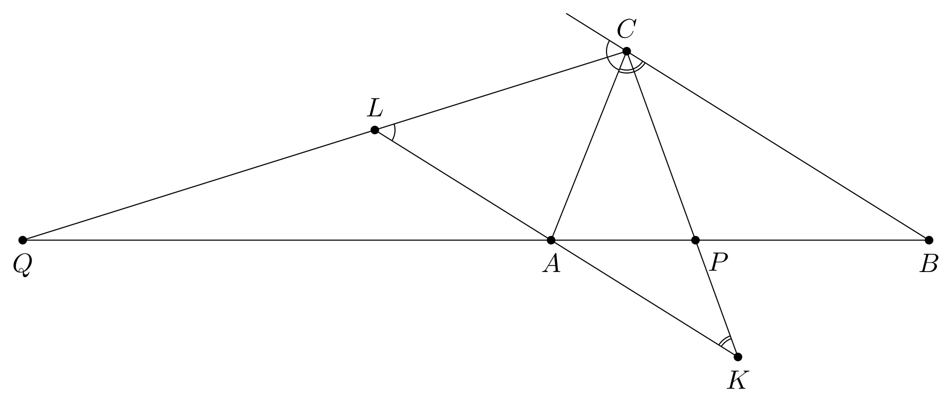


Then P, Q satisfy

$$\frac{|AP|}{|BP|} = \frac{|AQ|}{|BQ|} = \frac{|AC|}{|BC|}.$$

Remark. If $|AC| = |BC|$, then P is the midpoint of AB (as expected) but then Q is actually not defined since the exterior bisector at C is parallel to AB. In this degenerate case we can think of Q as being a point "at infinity."

Proof. Draw a line through A parallel to BC intersecting CP in K and CQ in L.



The line CK cuts the parallel lines LK and CB in equal angles, so

$$\angle AKC = \angle BCK = \angle KCA,$$

where we have used the fact that CK bisects the angle $\angle BCA$. Similarly, the line CQ cuts the lines LK and CB in equal angles, so $\angle CLA = \angle ACL$. In particular, triangles ACK and ALC are isosceles, and so $|AK| = |AC| = |AL|$.

Now we use Thales' Theorem at the angle at P intersecting the parallel lines LK and BC, and we obtain $|AP| / |PB| = |AK| / |BC|$. Using $|AK| = |AC| = |AL|$, this becomes $|AP| / |PB| = |AC| / |BC| = |AL| / |BC|$. Now we use Thales' theorem at the angle at Q to conclude $|AL| / |BC| = |AQ| / |BQ|$, and combining everything, we obtain the desired result

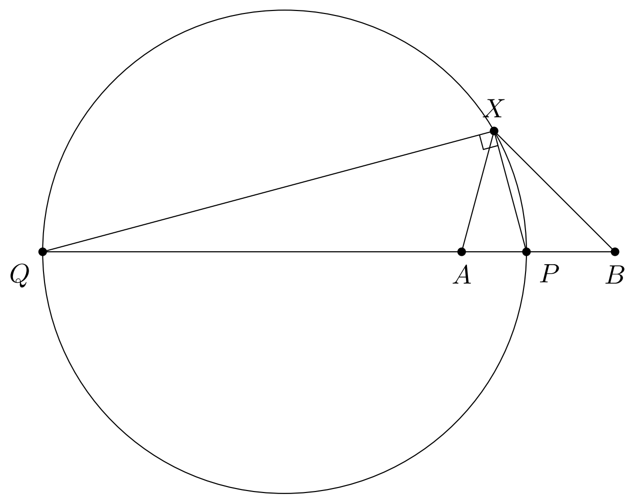
$$\frac{|AP|}{|BP|} = \frac{|AC|}{|BC|} = \frac{|AQ|}{|BQ|}.$$

□

Problem 7.15. Given points $A \neq B$ and a real number $\lambda > 0$, find the set of points such that $|AX| = \lambda\,|BX|$.

Remark. If $\lambda = 1$, we have characterized this set of points as the bisector of AB (Theorem 1.13).

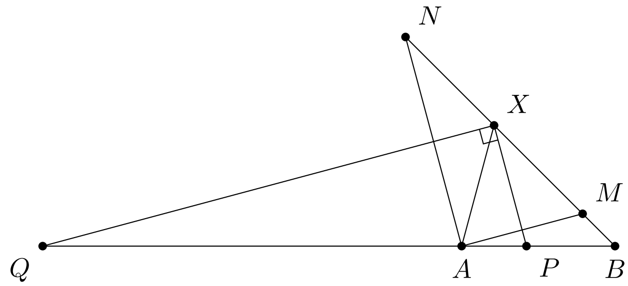
Solution. The idea for the solution is to use the angle bisector theorem (Theorem 7.14). Thanks to the above remark, we may assume $\lambda \neq 1$. Then pick P inside the segment AB and Q outside so that $|AP|\,/\,|BP| = |AQ|\,/\,|BQ| = \lambda$. We claim that the set of points X satisfying $|AX| = \lambda\,|BX|$ is precisely the circle with diameter PQ.



To prove this claim, we begin by supposing that X satisfies $|AX|\,/\,|BX| = \lambda$. Since $|AP|\,/\,|BP| = |AX|\,/\,|BX|$, we know that XP must bisect the angle BXA (here we are using the angle bisector theorem and the fact that there is a unique point P between A and B satisfying $|AP|\,/\,|BP| = |AX|\,/\,|BX|$). Similarly XQ bisects the exterior angle of ABX at X. However, we have proved in Exercise 5.4 that the two angle bisectors at X are perpendicular, and so $\angle PXQ = 90°$, and so X lies on the circle with diameter PQ (applying Corollary 3.8). Thus the set of points satisfying $|AX|\,/\,|BX| = \lambda$ lies on the chosen circle.

It remains to prove that if X lies on the circle with diameter PQ, then X satisfies $|AX|\,/\,|BX| = \lambda$. To show this, we construct two points M, N so that (i) $AN \parallel PX$

and $AM \parallel XQ$ and (ii) M, N both lie on the line BX.



Now we apply Thales' Theorem to the angle QBN intersecting parallel lines $AN \parallel PX$ and $AM \parallel QX$ and obtain the equalities

$$(*) \qquad \frac{|AP|}{|PB|} = \frac{|NX|}{|XB|} \text{ and } \frac{|MX|}{|BX|} = \frac{|AQ|}{|BQ|}.$$

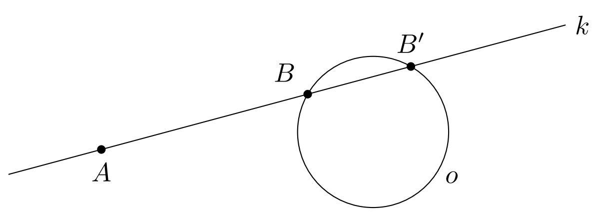
Since $|AQ| / |BQ| = |AP| / |PB| = \lambda$, we deduce $|NX| = |MX|$. Finally, since $\angle PXQ = 90°$ (since X lies on the circle with diameter PQ) and $AN \parallel PX$ and $AM \parallel XQ$, we deduce $\angle NAM = 90°$, and consequently, A lies on the circle with diameter NM. Since X is the midpoint of segment NM, we conclude $|AX| = |NX| = |MX|$ (this is proven in Exercise 3.6). Thus, returning to $(*)$, we obtain

$$\frac{|AX|}{|BX|} = \frac{|AQ|}{|BQ|} = \lambda,$$

as desired. This completes the proof. ∎

8. The Power of a Point with Respect to a Circle

Definition 8.1. The **power of a point** A **with respect to a circle** o is a number computed according to the following rule. Issue a line k from A intersecting o in two distinct points B and B'.



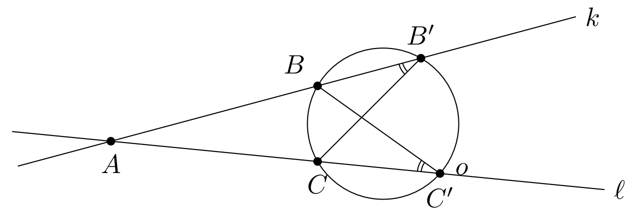
Then the power of A with respect to o is given by the following formula.

$$\text{power of } A \text{ with respect to } o = \begin{cases} |AB|\,|AB'| & \text{if } A \text{ is outside } o \\ 0 & \text{if } A \text{ is on } o \\ -|AB|\,|AB'| & \text{if } A \text{ is inside } o \end{cases}$$

A priori, this definition depends on the line k. However, with a little bit of work, we see that it actually does not:

Theorem 8.2. The definition given for the power of A with respect to o does not depend on the line k.

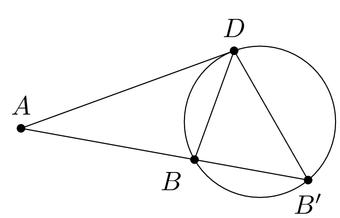
Proof. This result is called a "Theorem" because it is important, but not because it is hard to prove! The proof is an easy application of similar triangles. First suppose that A lies outside the circle o. Suppose k, ℓ are two lines intersecting at A, and suppose $B \neq B'$ and $C \neq C'$ are the intersection points of k and ℓ with o respectively. As in the figure, we assume that B separates A from B' and C separates A from C'.



Since angles $CC'B$ and $CB'B$ are subtended by the same arc CB, we know that $\angle AC'B = \angle CB'B$, and since triangles $AC'B$ and $CB'A$ share a common angle at A, we deduce they are similar by the "angle-angle-angle" criterion. It follows that $|AC| \,/\, |AB'| = |AB| \,/\, |AC'|$ and so $|AC|\,|AC'| = |AB'|\,|AB|$.

The case when A is inside the circle is similar, and it is left to the reader. □

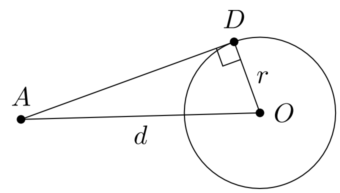
Proposition 8.3. Let A be a point lying outside of o. If $D \in o$ is such that AD is tangent to o, then the power of A with respect to o is $|AD|^2$.



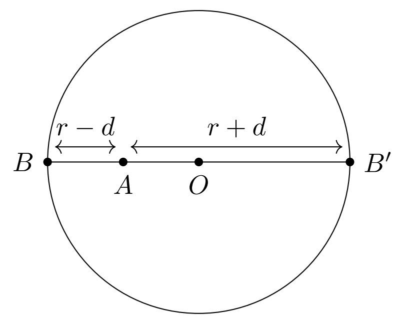
Proof. Let B, B' be as in the definition of the power of a point. Applying Proposition 3.12, we deduce $\angle BDA = \angle AB'D$, and hence $AB'D$ and ADB are similar by the "angle-angle-angle" criterion. In particular, $|AD| \,/\, |AB'| = |AB| \,/\, |AD|$, and hence $|AD|^2 = |AB|\,|AB'|$, and this completes the proof. □

Remark. Let A and D be as in the preceeding proposition, and let O be the centre of the circle o. Recall that AD is perpendicular to the radius OD, and hence we can apply Pythagoras' theorem to write $|AD|^2 = |AO|^2 - |DO|^2$. If we denote $d := |AO|$ and $r := |DO|$

the radius of the circle, then we conclude the power of A with respect to O is given by $d^2 - r^2$.

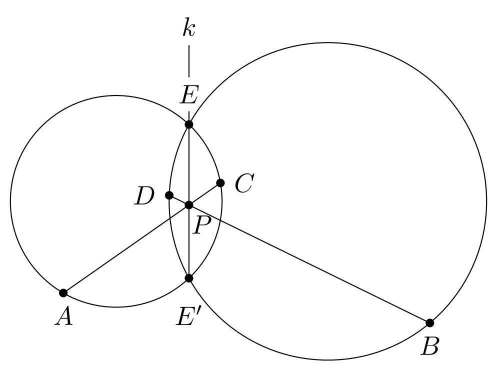


In fact, this formula continues to hold if A is inside of o. To show this, issue a line k passing through A and O, and consider the intersection points B and B' between this line and the circle o. Without loss of generality, suppose that $|AB| \leq |AB'|$. Then we have the configuration shown in the following figure.

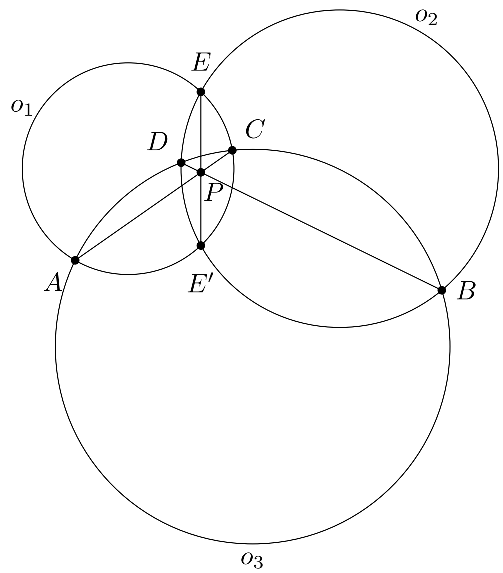


As described in the figure above, we have $-|AB|\,|AB'| = -(r-d)(r+d) = d^2 - r^2$, which is what we wanted to show. We remark that this formula is concise, but the definition given at the beginning of this section is much more useful in practice.

Problem 8.4. Let o_1 and o_2 be two circles intersecting in two points E and E' lying on a line k. Let A, C lie on o_1, and B, D lie on o_2 so that AC intersects BD at a point P on k. Prove that $ABCD$ lie on a circle.



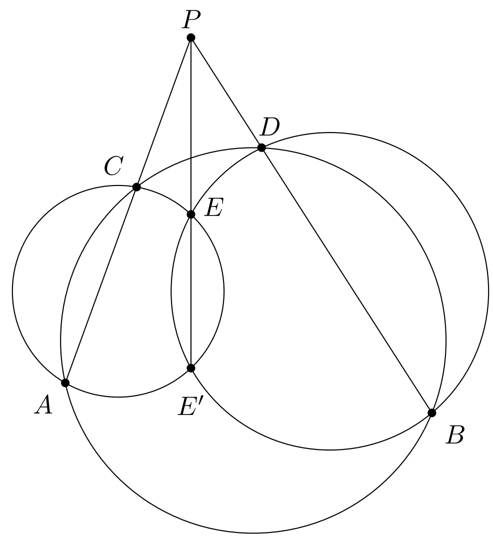
Solution. In our solution we assume that P lies inside the intersection of the interiors of o_1 and o_2 (the other case is when P is outside of both o_1 and o_2). The idea for the solution is simple: we let o_3 be the circumscribed circle of ABC, and we will show that D lies on o_3.



To prove D lies on o_3, we will use an equality which will result from computing the power of P with respect to o_1 and o_2 in various ways.

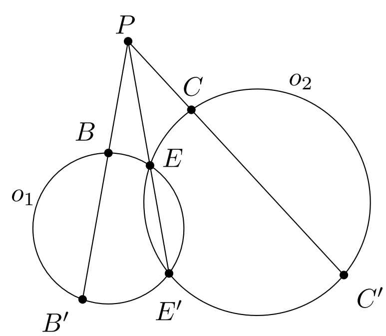
By computing the (negative) power of P with respect to o_1 in two different ways (using the lines AC and EE') we obtain the equality $|AP|\,|PC| = |EP|\,|E'P|$. Similarly, by computing the power of P with respect to o_2 in two different ways, we obtain the equality $|PD|\,|BP| = |EP|\,|E'P|$, and hence $|AP|\,|PC| = |PD|\,|BP|$. Now let o_3 be the circumscribed circle of ABC. We can compute the (negative) power of P with respect to o_3 in two ways, using the line AC and using the line BD. Let B and D' be the intersection points of BD and o_3, so that the power of P with respect to o_3 is equal to $|BP|\,|PD'|$. However, this power is also equal to $|AP|\,|PC| = |PD|\,|BP|$. Hence, keeping track of the various equalities, we have $|PD'| = |PD|$. We claim that D and D' are both on the opposite side of P to B. This is true because P lies on the edge AC, so it must lie inside o_3, and consequently P must separate B, D', and (by our assumption that P lies inside o_2) P separates B from D. Therefore $|PD'| = |PD|$ implies $D = D'$, and so $D \in o_3$, as desired. The case when P lies outside both o_1 and o_2 is similar and is left as an exercise. ◻

Exercise 8.5. Solve Problem 8.4 in the case when P lies outside of o_1 and o_2 (see figure below).



Definition 8.6. If two circles o_1 and o_2 intersect in two distinct points $E \neq E'$, then the line EE' is called the **radical axis** of o_1 and o_2.

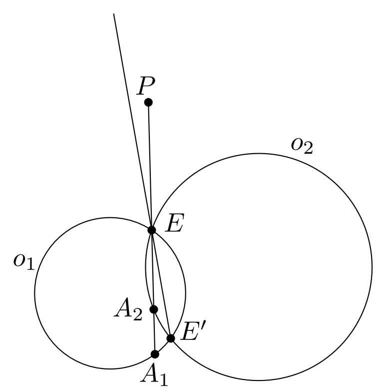
Theorem 8.7. Let o_1 and o_2 intersect in points $E \neq E'$. Then P lies on the radical axis of o_1 and o_2 if and only if the power of P with respect to o_1 is equal to the power of P with respect to o_2.



Proof. Let $B \neq B'$ be on o_1 and $C \neq C'$ on o_2 so that lines BB' and CC' both contain P. First we show that if P lies on the radical axis EE' then the powers of P with respect to o_1 and o_2 are equal. Case 1: if P lies outside o_1 and o_2. Proving equality of the powers in this case is easy to since $|PE|\,|PE'| = |PB|\,|PB'|$ (by computing the power of P with respect to o_1 in two different ways) and $|PC|\,|PC'| = |PE|\,|PE'|$ (by computing the power of P with respect to o_2 in two different ways). Thus $|PC|\,|PC'| = |PB|\,|PB'|$, as desired. Case 2: if P lies inside both o_1 and o_2. The same argument as in Case 1 shows $-\,|PC|\,|PC'| = -\,|PB|\,|PB'|$, as desired.

We note that since P lies on the radical axis, it cannot be inside o_1 and outside o_2 (or vice-versa) and so we have exhausted all cases.

Now we assume that P has equal powers with respect to o_1 and o_2, and we want to show that P lies on the radical axis EE'. We will prove this by contradiction, so suppose that P does not lie on the radical axis. First we note that P cannot be inside o_1 and outside o_2 (or vice-versa), or else the powers with respect to o_1 and o_2 would have different signs. Therefore there are only two cases: if P is outside both o_1 and o_2 or P is inside both o_1 and o_2 (the third case where P lies on the intersection of o_1 and o_2 is trivial, for then P is either E or E'). Assume that P lies outside both o_1 and o_2. Consider the line PE, and let PE intersect o_1 in a point A_1 and o_2 in a point A_2.



Since P does not lie on EE', we know that $A_1 \neq A_2$ (since $A_1 = A_2$ implies $A_1 = A_2 = E'$ is the other intersection point between o_1 and o_2). But since the power of P with respect to o_1 is the same as with respect to o_2, we have $|PE|\,|PA_1| = |PE|\,|PA_2|$, and hence $|PA_1| = |PA_2|$. Since A_1 and A_2 both lie on the ray PE, we deduce $A_1 = A_2$, which contradicts our assumption. A similar argument works when P lies inside both o_1 and o_2. This completes the proof of the theorem. □

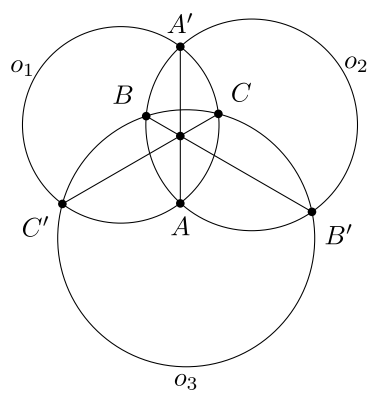
Exercise 8.8. Prove that for any two circles o_1, o_2 with different centres, the set of points P for which its power with respect to o_1 equals its power with respect to o_2 forms a line. *Hint*: let O_1 and O_2 be the centres of o_1, o_2 and issue a perpendicular to O_1O_2 through a point P on O_1O_2 which has equal powers with respect to o_1 and o_2.

Remark. By virtue of Exercise 8.8, we define the **radical axis** of two circles o_1 and o_2 with distinct centres to be the set of points P with equal power with respect to o_1 and o_2.

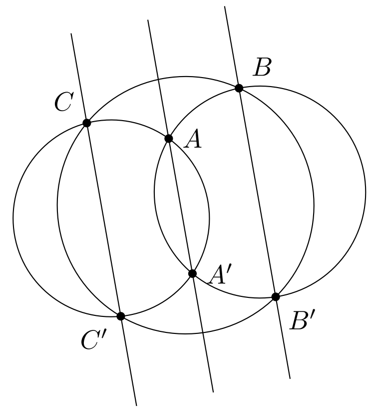
Problem 8.9. Let o_1, o_2 and o_3 be circles so that

$$o_1 \cap o_2 = \{A, A'\},\ o_2 \cap o_3 = \{B, B'\} \text{ and } o_1 \cap o_3 = \{C, C'\}.$$

Then the three radical axes AA', BB' and CC' either intersect in a point or are parallel.



or



Solution. In our solution we may obviously assume that the lines are not parallel, since if they are parallel, then we are already done!

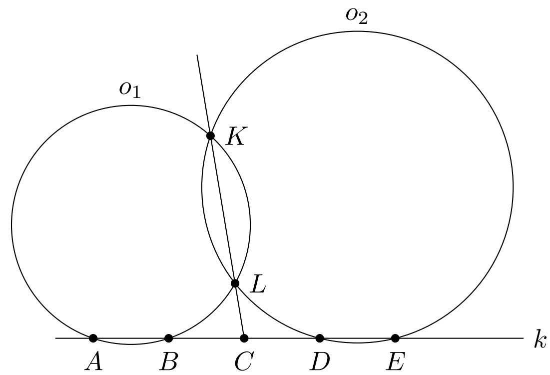
Therefore one pair of the radical axes must intersect, and so, without loss of generality, we assume that AA' and BB' intersect in a point P. Then we apply Theorem 8.7 to deduce that the power of P with respect to o_1 is equal to the power of P with respect to o_2 (since P lies on AA') and the power of P with respect to o_2 is equal to the power of P with respect to o_3 (since P lies on BB'). But then (applying Theorem 8.7 once again) we deduce that the power of P with respect to o_1 equals its power with respect to o_3, hence P lies on CC'. This completes the proof. □

Exercise 8.10. Show that the argument given above does not require the circles to intersect, and deduce that the three radical axes (as defined in the remark preceeding Exercise 8.8) which can be made out of three arbitrary circles o_1, o_2 and o_3 intersect in a common point (or are parallel).

Problem 8.11. Let A, B, C, D, E be distinct points on a line k so that

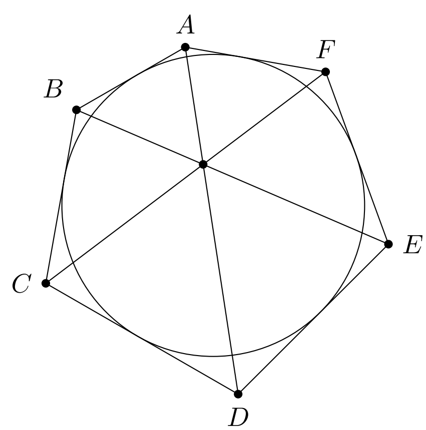
$$|AB| = |BC| = |CD| = |DE| = a.$$

Prove that, given any circle o_1 containing A, B and any circle o_2 containing D, E so that o_1 and o_2 intersect in distinct points K, L, we have that K, L, C lie on a common line.

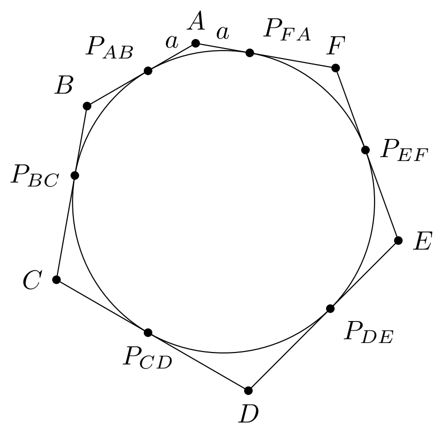


Solution. Without our knowledge of the power of a point, this problem seems like it would be difficult. However, it is easily solved using Theorem 8.7. Using the fact that $|AC| = |CE| = 2a$ and $|BC| = |CD| = a$, we compute that the powers of C with respect to o_1 and o_2 are both equal to $2a^2$. Hence C lies on the radical axis of o_1 and o_2, and thus C lies on the line KL, as desired. ◻

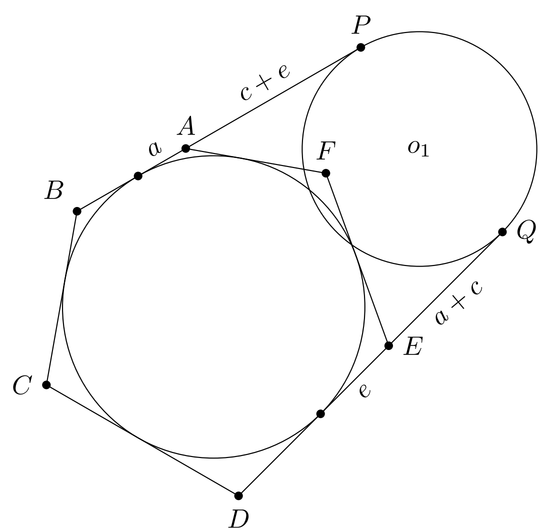
Problem 8.12 (Brianchon's Theorem)**.** Let $ABCDEF$ be a convex hexagon with an inscribed circle o; here **convex** means each diagonal is contained inside the hexagon. Then the lines AD, BE and CE intersect in a single point.



Solution. The idea for the solution is to find three circles o_1, o_2, o_3 so that AD, BE and CF are the three radical axes constructed from o_1, o_2, o_3. Then we can apply Exercise 8.10 to deduce AD, BE and CF intersect in a single point (a straightforward argument shows that they cannot be parallel). To do this, we introduce some convenient notation: let P_{AB} be the tangency point between the line AB and the inscribed circle, and denote the other tangency points by $P_{BC}, P_{CD}, P_{DE}, P_{EF}$ and P_{FA} in a similar fashion. Let a denote the distance $|AP_{AB}|$, and recall that the "strongest theorem of geometry" states that a is also equal to $|AP_{FA}|$. Define distances b, c, d, e, f similarly.

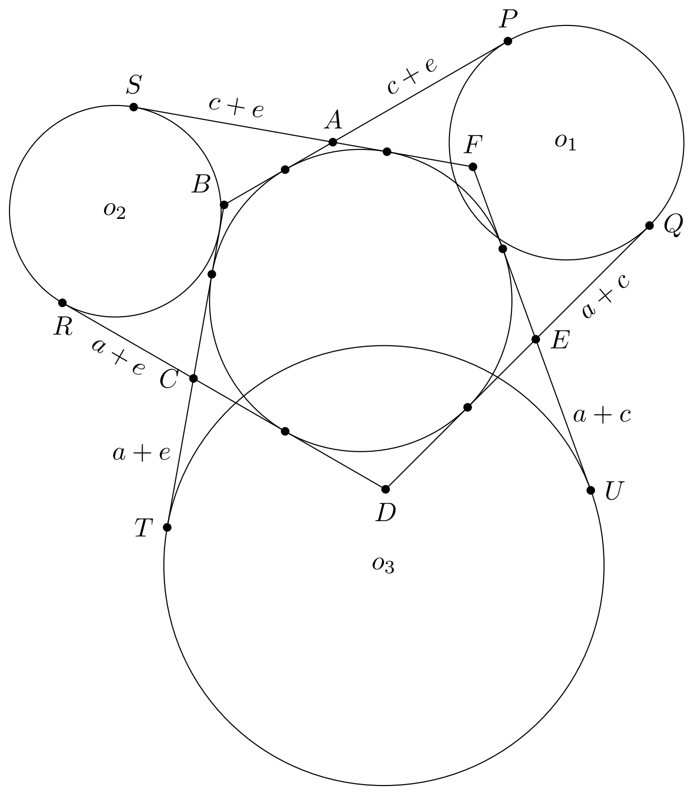


Let P be the point on the line AB at a distance of $c+e$ from A on the opposite side of B, and let Q be the point on DE at a distance $a+c$ from E on the opposite side of D. We claim that there is a circle o_1 tangent to lines AB and DE at the points P and Q.



To prove this claim, we consider the intersection point S between lines AB and ED. Since S is equidistant to the tangency points P_{AB} and P_{DE}, and the distance from P_{AB} is $a+c+e$ and the distance from Q to P_{DE} is also $a+c+e$, we deduce that P and Q are equidistant from S. A simple application of Theorem 4.1 proves that there is a circle tangent to the angle at DSB at points P and Q.

In a similar fashion, and by consulting the figure below, define points R, S, T and U. Then circles o_2 and o_3 (as shown below) exist, and this details of the proof of this assertion are left to the reader.



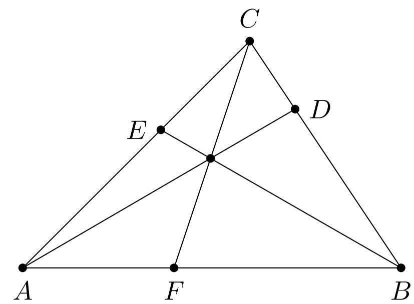
Now we claim that AD is the radical axis of o_1 and o_2. To prove this, note that $|AP| = |AS| = c + e$ and AP and AS are tangent to o_1 and o_2 respectively. Therefore the power of A with respect to o_1 is equal to its power with respect to o_2. A similar computation proves that the power of D with respect to o_1 equals its power with respect to o_2, and so AD is the radical axis of o_1 and o_2. Similar arguments prove that BE is the radical axis of o_1 and o_3, and CF is the radial axis of o_2 and o_3. Therefore, by Exercise 8.10, the lines AD, BE and CF intersect in a common point or they are parallel. However, convexity of $ABCDEF$ implies that the segments AD and BE intersect, and so we must have AD, BE and CF intersecting in a common point. This completes the solution. ◻

Exercise 8.13 (challenging)**.** Let $ABCD$ be a quadrilateral with an inscribed circle o. Let E (respectively F, G, H) be the point of tangency between AB and o (respectively BC, CD, DA and o). Prove that AC, BD, EG and FH intersect in a common point. *Hint*: use the ideas in the proof of Brianchon's Theorem by treating $AEBCGD$ as a degenerate hexagon.

9. Ceva's Theorem

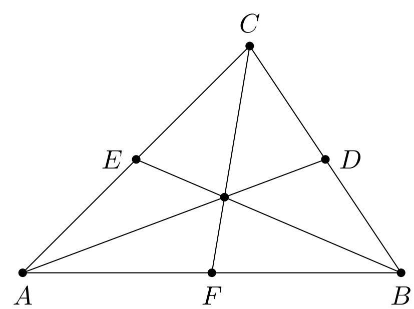
Theorem 9.1 (Ceva)**.** Let ABC be a triangle and let D, E, and F be arbitrary points on the sides BC, CA, and AB, respectively. Then the lines AD, BE and CF intersect at a single point if and only if

$$(*) \qquad |AF|\,|BD|\,|CE| = |BF|\,|CD|\,|AE|\,.$$



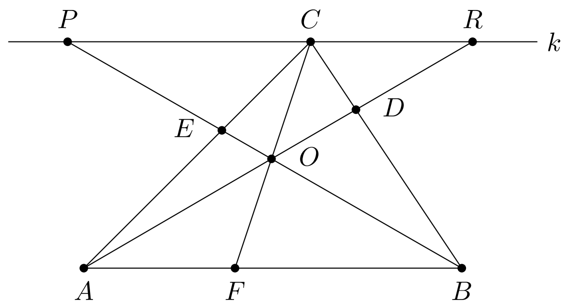
Before we give the proof of Ceva's theorem, we demonstrate how useful it is with a corollary.

Corollary 9.2. If ABC is a triangle and D, E, F are the midpoints of the sides BC, AC and AB, respectively, then the **medians** AD, BE and CF intersect in a single point (recall that this point is called the **centroid** of the triangle ABC).



Proof. Using Ceva's theorem, we see that it suffices to prove that $|AF|\,|BD|\,|CE| = |BF|\,|CD|\,|AE|$, but this is immediate, since we have equalities $|AF| = |BF|$, $|BD| = |CD|$ and $|AE| = |CE|$! □

Proof of Ceva's Theorem. First we show that if CF, AD and BE intersect at a point then $(*)$ holds. This is a fairly straightforward consequence of Thales' theorem. Let O denote the intersection of CF, AD and BE, let k be the line through C parallel to AB, and consider the points P and R, obtained by extending BE to k and AD to k, as shown.



Then we apply Thales' to the angle at D formed by the lines BC and AR to conclude

$$\frac{|BD|}{|CD|} = \frac{|AB|}{|CR|}, \tag{1}$$

while applying Thales' theorem to the analogous angle at E yields

$$\frac{|CE|}{|AE|} = \frac{|CP|}{|AB|}. \tag{2}$$

Applying Thales' theorem to the two angles at O formed by the lines CF, AR and CF, BP we conclude

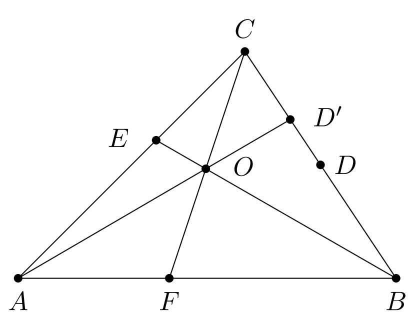
$$\frac{|AF|}{|CR|} = \frac{|FO|}{|OC|} = \frac{|BF|}{|CP|}, \text{ hence } \frac{|AF|}{|BF|} = \frac{|CR|}{|CP|}. \tag{3}$$

Taking the product of (1), (2) and (3) yields

$$\frac{|AF|\,|BD|\,|CE|}{|BF|\,|CD|\,|AE|} = 1,$$

which is equivalent to $(*)$. This completes the first half of the proof.

Now we establish the converse, and so we assume that equality $(*)$ holds and wish to prove that AD, BE and CF intersect in a common point. Let O denote the point of intersection between CF and BE, and let D' be the intersection of AO with the segment BC.



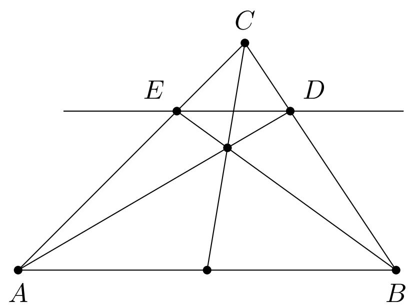
By applying the first part of Ceva's theorem, we know that

$$\frac{|BD'|}{|D'C|} = \frac{|BF|\,|AE|}{|AF|\,|CE|} = \frac{|BD|}{|DC|}, \tag{4}$$

where the second equality holds because we assume (∗) holds for the point D. But there is a unique point between B and C with the ratio given by (4), and so $D = D'$, which completes the proof. □

9.1. **Applications of Ceva's theorem.** In this subsection we present an assortment of problems which we solve using Ceva's theorem.

Problem 9.3. Let ABC be a triangle and let Let D, E be points on ABC, as shown. Prove that AD and BE intersect on the median from C if and only if ED is parallel to AB.



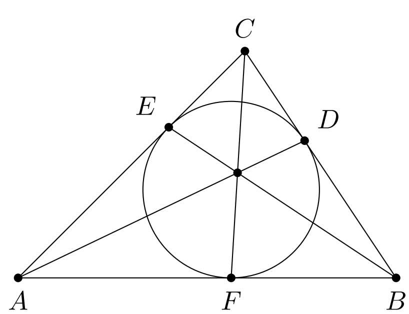
Solution. First suppose that AD and BE intersect at a point O on the median through C. Let F be the midpoint of AB; then we apply Ceva's theorem to the points D, E, F to conclude

$$|AF|\,|BD|\,|CE| = |FB|\,|CD|\,|AE|\,. \tag{1}$$

Since $|AF| = |FB|$, we conclude $|CE| / |AE| = |CD| / |DB|$, and hence Thales' theorem implies ED is parallel to AB.

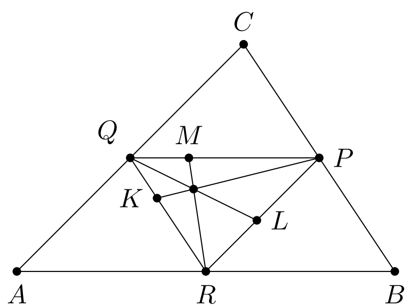
For the converse, we assume ED is parallel to AB, and use Thales' theorem at the angle at C to conclude $|CE| / |AE| = |CD| / |DB|$, and since $|AF| = |BF|$, we conclude (1). Then Ceva's theorem implies BE and AD intersect on CF, which is the median through C. ⌑

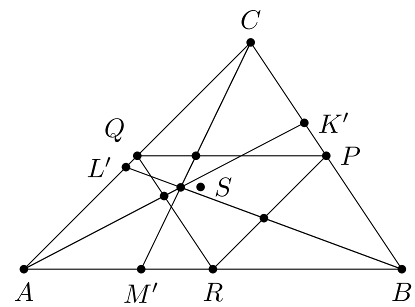
Problem 9.4. Let ABC be a triangle with inscribed circle o, and let D, E, F be the tangency points between o and the edges BC, CA and AB. Then AD, BE and CF intersect in a common point. **Remark**: this point of intersection is called the **Gergonne point**.



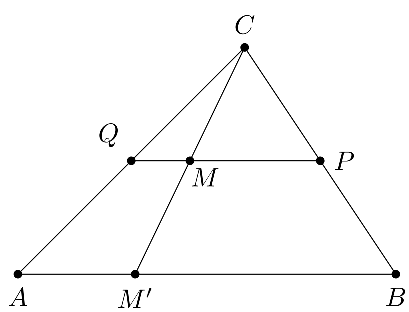
Solution. By the "strongest theorem of geometry," $|AE| = |AF|$, $|BF| = |BD|$ and $|CD| = |CE|$, and multiplying these three equalities together yields $|AE|\,|BF|\,|CD| = |AF|\,|BD|\,|CE|$, hence Ceva's theorem implies AD, BE and CF intersect in a common point. □

Problem 9.5. Let ABC be a triangle, and let P, Q, R be the centres of the sides BC, CA and AB. Let S be a point inside the triangle PQR, and let K, L, M be the intersections of the lines PS, QS and RS with the sides QR, RP, PQ, respectively. Prove that AK, BL and CM intersect in a single point.





Solution. As in the above figure, let K', L', M' be the intersections of AK, BL and CM with the opposite side BC, AC and AB (respectively). Since Q, P are the midpoints of AC and BC, Thales' theorem guarantees that QP is parallel to AB, and so we can apply Thales' theorem to the angle at C formed by the lines CM' and CA to deduce $2\,|QM| = |AM'|$.



Similar argument yield $2\,|AL'| = |RL|$, $2\,|RK| = |BK'|$, $2\,|PM| = |BM'|$, $2\,|QK| = |CK'|$ and $2\,|PL| = |CL'|$. Therefore, by multiplying all these equalities we deduce

$$|AM'|\,|BK'|\,|CL'| = |BM'|\,|CK'|\,|AL'| \text{ if and only if } 8\,|QM|\,|RK|\,|PL| = 8\,|PM|\,|QK|\,|RL|,$$

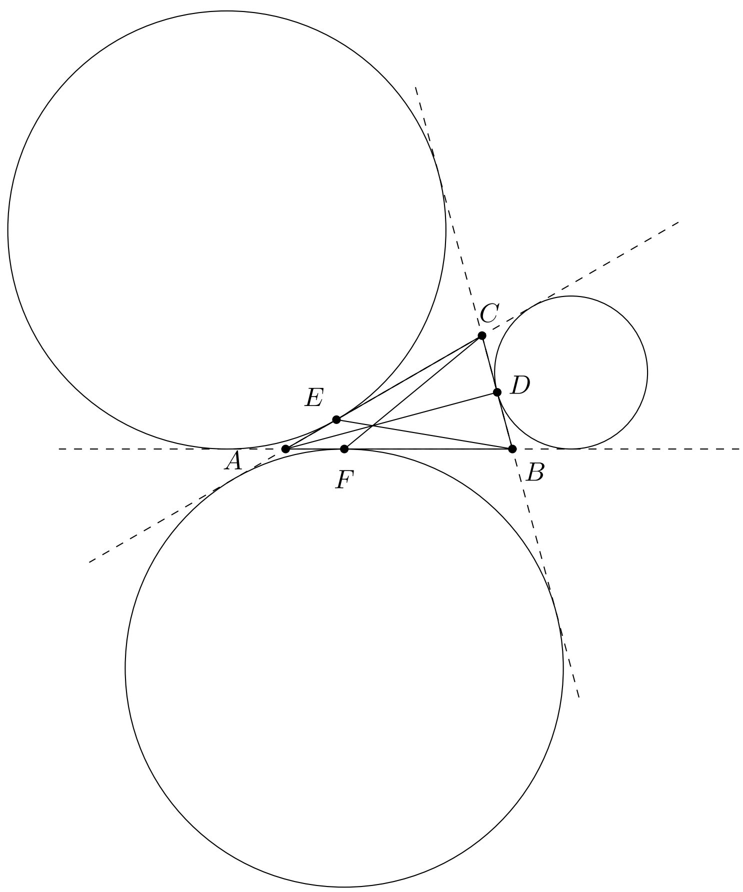
and the latter equality is true by Ceva's theorem, since QL, RM and PK intersect at S. Applying Ceva's theorem once again allows us to conclude AK', BL' and CM' intersect in a common point. This completes the solution. □

Problem 9.6. Let D, E, F be the tangency points of the escribed circles to the triangle A, B, C. Then the lines AD, BE and CF intersect in a common point.

Solution. The idea for the solution is fairly simple: we will use the "strongest theorem of geometry" to produce the equalities needed to invoke Ceva's theorem. By the "strongest theorem of geometry" we have

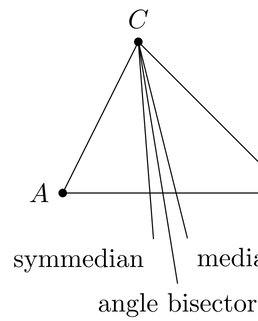
$$|AB| + |BD| = \frac{1}{2}(\text{perimeter of } ABC) = |AB| + |AE|\,, \tag{1}$$

so $|AE| = |BD|$ (we proved (1) in the solution to Problem 5.8). Similarly $|CD| = |AF|$ and $|FB| = |CE|$. Multiplying these three equalities yields $|AF|\,|BD|\,|CE| = |CD|\,|AE|\,|FB|$, as desired. This point of intersection is called the **Nagel point**. ◻

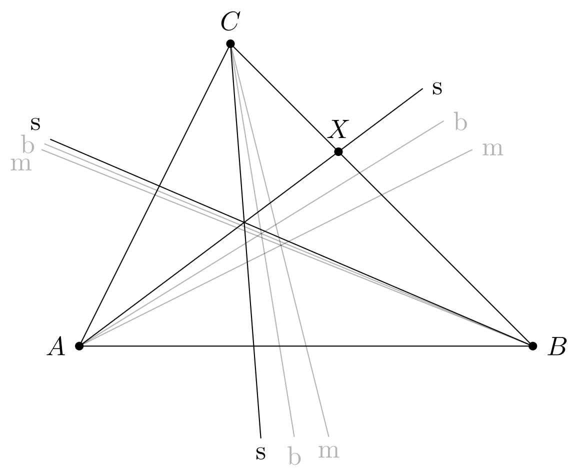


Problem 9.7 (The intersection of symmedians)**.** This application of Ceva's theorem requires a new notion: If ABC is a triangle, then the **symmedian** through C is the reflection of C's

median through the angle bisector at C.



Prove that the three symmedians of a triangle intersect at a common point.



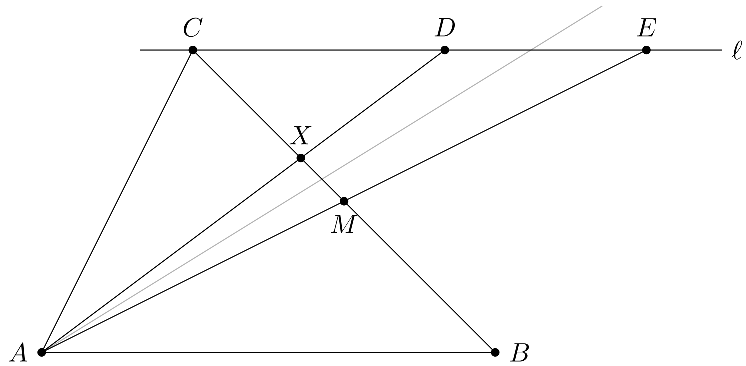
Solution. Let X, Y, Z be points on sides BC, AC and AB, respectively, so that AX, BY and CZ are the symmedians through A, B, C, respectively. The key to the solution is the fact that

$$\text{(1)} \qquad \frac{|BX|}{|CX|} = \left(\frac{|BA|}{|CA|}\right)^2, \quad \frac{|CY|}{|AY|} = \left(\frac{|CB|}{|AB|}\right)^2, \text{ and } \frac{|AZ|}{|BZ|} = \left(\frac{|AC|}{|BC|}\right)^2.$$

To see why this implies the solution, we compute

$$\frac{|BC|\,|CY|\,|AZ|}{|CX|\,|AY|\,|BZ|} = 1,$$

which, by Ceva's theorem, implies that AX, BY and CZ intersect in a single point. To prove (1), it suffices to prove the first equality $|BX| / |CX| = |BA|^2 / |CA|^2$. To do this, we issue a line ℓ through C parallel to AB, and let D denote the point of intersection of the symmedian AX and ℓ, and let E denote the point of intersection of the median AM and ℓ.



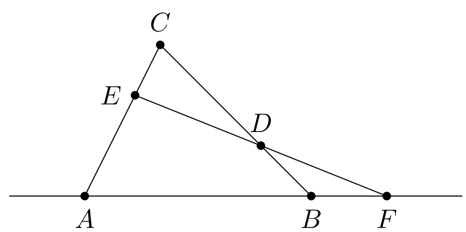
By Thales' theorem, (i) $|CE| = |AB|$, and (ii) $|BX| / |CX| = |AB| / |CD|$.

Since the bisector through A bisects both CAB and DAE (by definition) we deduce that $\angle CAD = \angle EAB$. But since CE is parallel to AB, $\angle EAB = \angle AEC$. Thus triangles AEC and DAC share two angles, and so they are similar. Therefore $|CE| / |AC| = |AC| / |DC|$, and combining this with (i) and (ii) we deduce $|BX| / |CX| = |BA|^2 / |CA|^2$, as desired. ◻

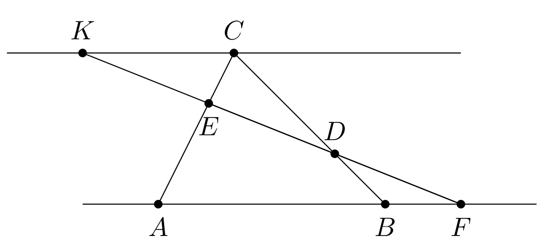
9.2. **Menelaus' Theorem.** Suppose that we have a triangle ABC, points D, E on sides BC and AC, and a third point F lying on the line AB but *not* between A and B. In a similar spirit to Ceva's theorem, Menelaus' theorem gives a necessary and sufficient criterion for when D, E, F lie on a common line.

Theorem 9.8 (Menelaus)**.** With ABC a triangle and D, E, F as described above, D, E, F are collinear if and only if the following equality holds

$$|AF|\,|BD|\,|CE| = |BF|\,|CD|\,|AE|\,.$$

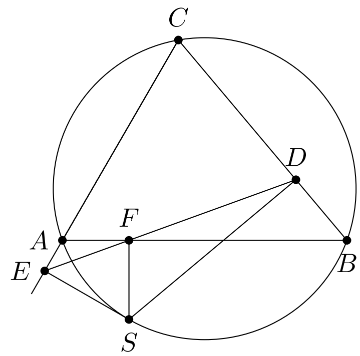


Exercise 9.9. Prove Menelaus' theorem by considering the following figure



Hint: Apply Thales' theorem to the angles formed at E and D.

Problem 9.10 (The Simson line)**.** Let ABC be a triangle and let S lie on the circumscribed circle of ABC. Let D, E, F be the projections of S onto the lines BC, AC and AB, respectively. Then the points D, E, F lie on a common line. **Remark**: this problem was already asked in Exercise 3.19, but here we will solve it as an application of Menelaus' theorem.

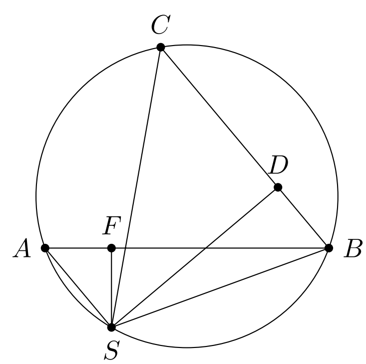


Solution. Without loss of generality, we can assume that S and C lie on opposite sides of AB. We assume that F lies on the segment AB (there is another case, where F does not lie on AB). Then, since $ASBC$ admits a circumscribed circle, we know that $\angle CAS + \angle SBC = 180°$ (Theorem). Thus, if $\angle CAS = 90°$ then $\angle SBC = 90°$, so we have $E = A$ and $D = B$, and so D, E, F lie on the line AB. Therefore, we may assume without loss that $\angle CAS > 90°$, so that $\angle SBC < 90°$. It follows that E lies outside AC while D lies inside BC (as in the figure above).

Therefore, we are in the setting of Menelaus' theorem, since F, D lie on the edges of the triangle, while E lies outside of its corresponding segment. We will establish the desired equality

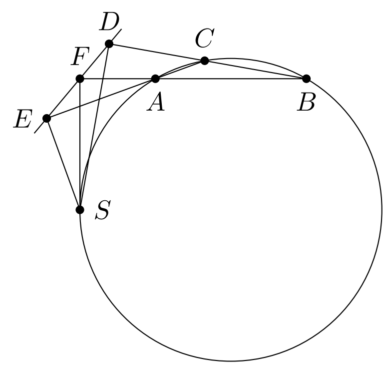
$$(*) \qquad \frac{|AE|\,|BF|\,|CD|}{|CE|\,|AF|\,|BD|} = 1$$

by proving that various triangles are similar. We claim that SFA is similar to SDC.



This follows by the "angle-angle" criterion for similarity, since $\angle BAS = \angle BCS$, since BAS and BCS both are subtended by the arc BS. Therefore, (i) $|CD| / |AF| = |CS| / |AS|$. Similar arguments show that SEC is similar to SFB, so (ii) $|BF| / |CE| = |BS| / |CS|$, and SDB is similar to SEA, so (iii) $|AE| / |BD| = |AS| / |BS|$. Multiplying together (i), (ii) and (iii) we obtain $(*)$, and so we can apply Menelaus' theorem to conclude D, E, F are collinear. This completes the proof when F lies on AB. The proof of the other case is left to the reader (Exercise 9.11). ☐

Exercise 9.11. Suppose that we are in the setting of the previous problem, with AB separating C from S, but that F does lies outside of the segment AB. Prove that E, F, D still lie on a common line.



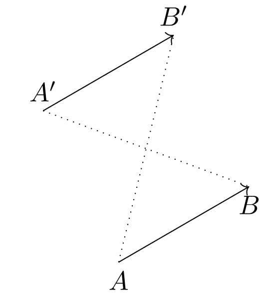
If the reader wishes to prove solve this problem using the same arguments as the solution to Problem 9.10, she/he should prove an analogous theorem to Menelaus' theorem which applies when the points D, E, F lie on the lines CB, AC, AB, respectively, but none of D, E, F lie inside their corresponding interval.

10. Isometries of the Euclidean plane

Recall that an **isometry** is a similarity with scale 1, i.e. a transformation of the plane which preserves distances. Since isometries are similarities, they also preserves angles. The goal of this section is to try to classify all isometries.

We have already seen that reflections and rotations are isometries. Another class of isometries are the "parallel translations," which we will define shortly. The concept of a parallel translation is closely related to the concept of a "vector," which is hopefully well known to the reader. For completeness, we now give the formal definition of a "vector."

Definition 10.1. We say that two pairs of points A, B and A', B' define the same **vector**, written $\overrightarrow{A'B'} = \overrightarrow{AB}$, if and only if the point reflection (i.e. 180° rotation) through the midpoint of AB' sends B to A'. The reader will show below that $\overrightarrow{A'B'} = \overrightarrow{AB}$ defines an equivalence relation, and hence we can define a **vector** as an equivalence class of pairs of points.



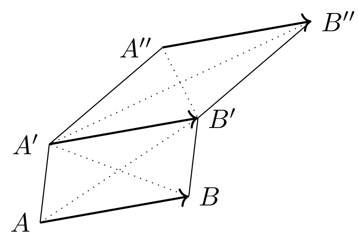
Exercise 10.2. Prove the following facts about vectors:

(a) If $\overrightarrow{AB} = \overrightarrow{A'B'}$, then $|AB| = |A'B'|$, and the point reflection through the midpoint of BA' interchanges A and B'.

(b) If $A \neq B$ and A, B, A' do not lie on a common line, then $\overrightarrow{AB} = \overrightarrow{A'B'}$ implies $ABB'A'$ forms a parallelogram.

(c) $\overrightarrow{AB} = \overrightarrow{A'B'}$ if and only if $\overrightarrow{AA'} = \overrightarrow{BB'}$.

(d) If $\overrightarrow{AB} = \overrightarrow{A'B'}$ and $\overrightarrow{A'B'} = \overrightarrow{A''B''}$, then $\overrightarrow{AB} = \overrightarrow{A''B''}$.



Hint for (d): first show this (i) when A, B, A', B', A'', B'' all lie on a common line, and (ii) when A', B', A'', B'' lie on a common line and A, B lie on a different line. Then using (ii), show that the general case can be reduced to the case when A, A', A'' lie on a common line.

Thus the relation $\overrightarrow{AB} = \overrightarrow{A'B'}$ is an equivalence relation, and so we may define a vector as an equivalence class of this relation.

Exercise 10.3. The goal of this exercise is to prove that vectors form a vector space.

(a) For any vector v and any point A, prove there is unique B so that $\overrightarrow{AB} = v$.

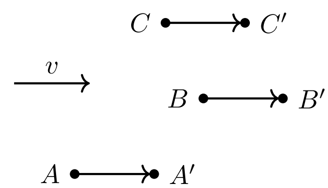
(b) If $v = \overrightarrow{AB}$, and $w = \overrightarrow{BC}$, define $v + w = \overrightarrow{AC}$, and show that this not depend on the choice of the pair A, B defining v.

(c) If $v = \overrightarrow{AB}$, $\lambda \geq 0$ (resp. $\lambda < 0$) is a real number, and C is the unique point on AB so that $|AC| = |\lambda|\,|AB|$ and C lies on the ray $[A, B\rangle$ (resp. C lies on the other ray $AB - [A, B\rangle$), define $\lambda v = \overrightarrow{AC}$. Show this does not depend on the pair AB defining v.

(d) Prove that with these operations of addition and scalar multiplication, the set of vectors forms a vector space.

(e) Prove that this vector space is two-dimensional.

Definition 10.4. If v is a vector, then the **translation by** v is a transformation of the plane which takes A to A', where A' is the unique point so that $v = \overrightarrow{AA'}$ (the proof of uniqueness was asked in Exercise 10.3).



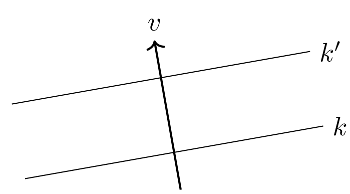
Proposition 10.5. The translation by a vector v is an isometry whose inverse is the translation by the vector $-v$.

Proof. Let A, B be two points in the plane. Since $\overrightarrow{AA'} = \overrightarrow{BB'}$, part (b) of Exercise 10.2 implies that $\overrightarrow{AB} = \overrightarrow{A'B'}$, so part (a) of the same exercise guarantees that $|AB| = |A'B'|$.

Referring to the definition of scalar multiplication given in Exercise 10.3, a simple argument shows $\overrightarrow{AA'} = v$ if and only if $\overrightarrow{A'A} = -v$, which proves that the inverse of the translation by v is the translation by $-v$. □

Exercise 10.6. Let v and w be two vectors. If A' is the translation of A by v, and A'' is the translation of A' by w, then A'' is the translation of A by $v + w$. (This is easy, once we recall the definition of $v + w$ given in Exercise 10.3).

Exercise 10.7. Let k, k' be parallel lines, and let d be the distance between k and k'. Prove that the composition of the reflection through k and the reflection through k' is the translation by a vector perpendicular to k with length $2d$.



Groups of transformations. The main goal in this section is to classify all isometries of the Euclidean plane. It is therefore useful to consider the set of all isometries. The first observation one makes about the set of all isometries is that they form a **group**, which, informally, is a collection of transformations which can be *composed* and which have *inverses*. Here are the formal definitions.

Definition 10.8. Recall that if f, g are two transformations of the plane, $f \circ g$ is the composite transformation, which sends a point A to $f(g(A))$.

A transformation f is **invertible** if there is another transformation g (an **inverse** of f) such that

$$f \circ g = g \circ f = \mathrm{id}$$

where id is the **identity transformation**, which maps each point to itself. As a corollary to part (c) of Exercise 10.9 (see below), if f has an inverse, it has a unique inverse, which we denote by f^{-1}.

If g satisfies $f \circ g = \text{id}$, then we say g is a **right-inverse** for f, while if g satisfies $g \circ f = \text{id}$ then we say g is **left-inverse** for f. Therefore, an inverse is both a right and left-inverse.

We say that f is **onto** (or **surjective**) if every point B is the image of some point A.

We say that f is **one-to-one** (or **injective**) if distinct points $A \neq B$ map to distinct points $f(A) \neq f(B)$.

Exercise 10.9. Let f be a transformation of the plane.

(a) Show that f has a right-inverse if and only if f is onto. *Hint*: to prove the existence of a right-inverse, for each point A define $g(A)$ to be any point B so that $f(g(A)) = B$. Prove that $A \mapsto g(A)$ is the desired right-inverse. Why does this argument fail if f is not surjective?

(b) Show that f has a left-inverse if and only if f is one-to-one. *Hint* Pick a point Q. If $B = f(A)$ for some A, let $g(B) = A$. If $B \neq f(A)$ for any A, let $g(B) = Q$. Prove that g is the desired left-inverse.

(c) Suppose that f has left-inverse h and right inverse g. Prove that $g = h$, and hence f has a *unique* inverse.

(d) Suppose that f is one-to-one and onto. Prove that f has a unique inverse.

Definition 10.10. A set G of transformations of the plane is a **group** if (i) $\text{id} \in G$, (ii) $f \in G$ implies that f is invertible and $f^{-1} \in G$, and (iii) $f, g \in G$ implies that $f \circ g \in G$.

Example 10.11. Proposition 10.5 and Exercise 10.6 imply that the set of all translations forms a group. Indeed, (i) it is clear that the identity transformation equals the translation by the zero vector, (ii) if T is the translation by v, then T^{-1} is the translation by $-v$ (Proposition 10.5) and (iii) if T, S are translations by v, w, respectively, then $T \circ S$ is the translation by $v + w$.

Exercise 10.12. Prove that the group of all translations has the structure of a vector space. Indeed, show that the group of all translations is isomorphic to the vector space of vectors. Thanks to this result, we can 'think' of a vector as a translation, rather than an equivalence class of pairs of points.

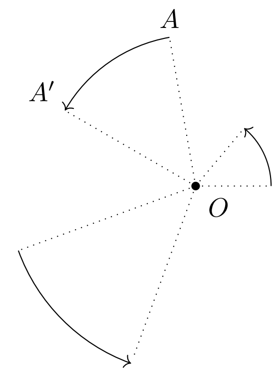
Exercise 10.13. Fix a non-zero vector v.

(a) Consider the set $\mathbb{N}v = \{v, 2v, 3v, 4v, \cdots\}$, and for each natural number $n \in \mathbb{N}$, let T_n be the translation by nv. Let $G = \{T_n : n \in \mathbb{N}\}$. Prove that G satisfies part (iii) of the definition of a group, but show that it does not satisfy (i) or (ii).

(b) Consider the set $\mathbb{Z}v = \{\cdots, -2v, -v, 0, v, 2v, 3v, \cdots\}$, and for each integer $n \in \mathbb{Z}$, let T_n be the translation by nv. As in part (a), let $G = \{T_n : n \in \mathbb{Z}\}$. Prove that G is a group.

(c) Fix two non-zero vectors v, w, and consider the set $\mathbb{Z}v + \mathbb{Z}w = \{nv + mw : n, m \in \mathbb{Z}\}$. Prove that the set of all translations by elements in $\mathbb{Z}v + \mathbb{Z}w$ forms a group.

Example 10.14. Fix a point O, and consider the set G of all rotations around O.



Then G is a group: to show this, we let $R_\alpha \in G$ denote the rotation by angle α, and then note that (i) $R_{0^\circ} = \mathrm{id}$, (ii) $R_\alpha \circ R_{-\alpha} = R_{-\alpha} \circ R_\alpha = R_{0^\circ} = \mathrm{id}$, and (iii) $R_\alpha \circ R_\beta = R_{\alpha+\beta}$.

Exercise 10.15. (a) Pick some natural number $n > 0$, and let α be the angle $(360/n)^\circ$. Let G be the set of all rotations by angle $k\alpha$ with $k = 1, 2, 3, \cdots$ (i.e. $G = \{R_\alpha, R_{2\alpha}, R_{3\alpha}, \cdots\}$). Prove that G is a group. In fact, prove that G is a *finite* group.

(b) Pick an irrational number $\lambda \in (0, 1)$. Let α be the angle $(\lambda\, 360)^\circ$. Prove that

$$G = \{R_\alpha, R_{2\alpha}, R_{3\alpha}, \cdots\}$$

is not a group. However, prove that

$$\widetilde{G} = \{\cdots, R_{-2\alpha}, R_{-\alpha}, \mathrm{id}, R_\alpha, R_{2\alpha}, R_{3\alpha}, \cdots\}$$

is a group. Unlike the group in part (a), show that $\widetilde{G}$ is not finite.

Exercise 10.16. Let G be the set of all reflections of the plane (there is one reflection for every line). Prove that G is not a group.

Proposition 10.17. Isometries form a group.

Before we give the proof, we ask the reader to solve the following exercise, whose conclusion will be used in the proof of the proposition.

Exercise 10.18. (a) Let $A \neq B$ be two points, and let $d = |AB|$. Suppose a, b are non-negative real numbers so that one of the following relations is satisfied:

$$d = a + b \quad \text{or} \quad a = d + b \quad \text{or} \quad b = a + d.$$

Show there is a *unique* point C on the line AB so that $|AC| = a$ and $|CB| = b$.

(b) Let f be an isometry, and suppose that C is a point lying on the line AB. Use the equality case in the triangle inequality to prove that $f(C)$ lies on the line $f(A)f(B)$.

(c) If Y is any point on $f(A)f(B)$, use part (a) to show that there is a unique point C on AB so that $|CA| = |Yf(A)|$ and $|CB| = |Yf(B)|$. Conclude that $f(C) = Y$.

(d) Combine parts (b) and (c) to conclude that any isometry f maps the line AB to the line $f(A)f(B)$ in a one-to-one and onto fashion.

Proof of the Proposition. It is clear that the identity transformation is an isometry, so we have shown that isometries satisfy property (i) from the definition of a group.

Now we prove that isometries satisfy property (iii) for groups. Suppose f, g are both isometries, and we want to show that the distance between $f \circ g(A)$ and $f \circ g(B)$ equals $|AB|$. But this is easy:

$$|f(g(A))f(g(B))| = |g(A)g(B)| = |AB|$$

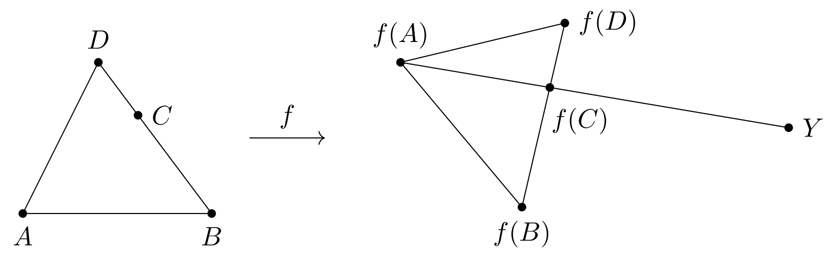
where the first equality follows since f is an isometry, and the second equality follows g is an isometry. Therefore $f \circ g$ is an isometry.

The hardest part of the proposition is establishing that isometries satisfy property (ii) for groups. Our argument is broken into three steps: first we fix some isometry f, and show that it is one-to-one and onto, which guarantees the existence of an inverse f^{-1} (see Exercise 10.9). Then we show that this inverse f^{-1} is also an isometry.

If $A \neq B$ are distinct points, then $|AB| > 0$, and so $|f(A)f(B)| > 0$, so $f(A) \neq f(B)$. Therefore f is one-to-one.

To prove that f is onto, we will use the result of Exercise 10.18. Let Y be an arbitrary point in the plane, and we want to find C so that $f(C) = Y$. Take a (non-degenerate) triangle A, B, D, and, without loss of generality, suppose that the line $Yf(A)$ intersects the line $f(B)f(D)$. By the triangle inequality, $f(A)f(B)f(D)$ is also a non-degenerate triangle (e.g. no vertex lies on the line joining the other two).

Let the line $Yf(A)$ intersect the line $f(B)f(D)$ at a point C'. Since C' lies on $f(B)f(D)$, we know there is C on BD so that $C' = f(C)$. Then, since Y lies on the line $f(A)f(C)$, we may apply Exercise 10.18 know $Y = f(X)$ for some X on AC.



This proves that f is onto. Therefore, by Exercise 10.9 we know that f^{-1} exists. Now it is easy to show that f^{-1} is an isometry, since for any A, B we have

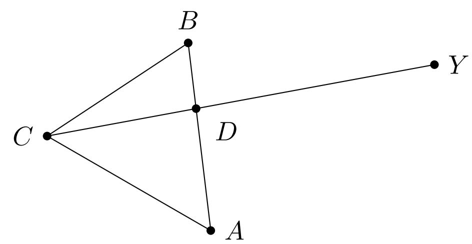
$$|AB| = \left|f(f^{-1}(A))f(f^{-1}(B))\right| = \left|f^{-1}(A)f^{-1}(B)\right|,$$

as desired. Since we have shown that the set of isometries satisfies (i), (ii) and (iii), we have proved the proposition. □

Theorem 10.19. If an isometry f fixes the vertices of a triangle ABC, then f is the identity.

Proof. First we will prove that f fixes every point D on the line AB. Since f fixes A, B, $|f(D)A| = |DA|$, $|f(D)B| = |DB|$, and so part (a) of Exercise 10.18 implies that $f(D) = D$.

Now suppose that Y is an arbitrary point. Without loss of generality, assume that the line YC intersects the line AB in a point D.



By the first part of our proof, we know that f fixes C and D. Therefore, applying the same argument as in the first paragraph, we know that f fixes every point on the line CD, so f fixes Y. It follows that $f = \text{id}$. □

Corollary 10.20. Let f, g be isometries, and assume there is a triangle ABC so that

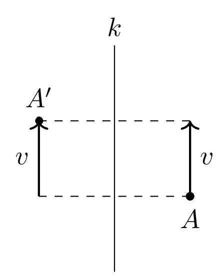
$$f(A) = g(A),\ f(B) = g(B) \text{ and } f(C) = g(C).$$

Then $f = g$.

Proof. Apply Theorem 10.19 to the isometry $g^{-1}f$. □

10.1. **The classification of isometries of the plane.** So far we have encountered three types of isometries: reflections, rotations and translations. There is one more kind of isometry we have not seen yet, the so-called **glide-reflection**.

Definition 10.21. Let k be a line and v a vector parallel to k. The **glide reflection** through k along v is the composition of the reflection R through k and a translation by v. It is easy to see that the image of A under the glide reflection does not depend on the order with which we translate and reflect (in other words, R commutes with the translation by v).



Note that the reflection through k is the glide reflection with vector $v = 0$.

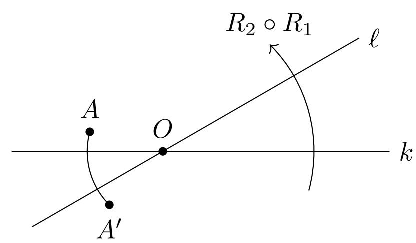
Theorem 10.22 (classification of isometries)**.** Any isometry f is a rotation or a translation or a glide reflection.

Proof. Let ABC be a triangle. Then the triangle $f(A)f(B)f(C)$ is congruent to ABC by the "side-side-side" criterion. Hence, by definition of congruence, there is a sequence $R_1, R_2, \cdots, R_n$ of reflections taking ABC to $f(A)f(B)f(C)$. By Corollary 10.20 we conclude that $f = R_n \circ \cdots \circ R_1$. In fact, the proof of Theorem 2.15 shows that there is a sequence of *at most three* reflections taking ABC to $f(A)f(B)f(C)$, so we may assume that $n = 1$, 2 or 3.

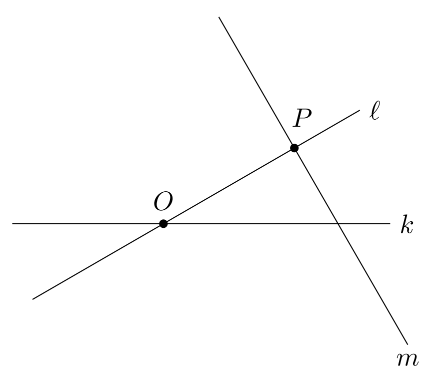
If $n = 1$, so that $f = R_1$, then we are obviously done.

If $n = 2$, then let R_1 be the reflection through k and R_2 be the reflection through ℓ. If k and ℓ intersect at a point O, then $R_2 \circ R_1$ is a rotation through O (see Theorem 2.2). If k and ℓ are parallel, then $R_2 \circ R_1$ is a translation (see Exercise 10.7). In either case, we have classified what f is.

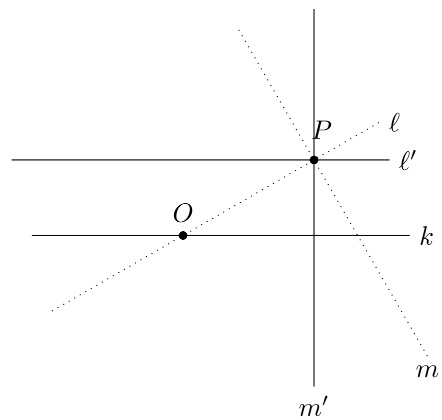
The hardest part of the proof is analyzing the case $n = 3$. Let R_1, R_2 and R_3 be the reflections through lines k, ℓ and m, respectively. We will prove the case when k and ℓ intersect in a point O (the case when they are parallel is left as an exercise, see Exercise 10.23). Suppose that the angle between k, ℓ is α, so that $R_2 \circ R_1$ is the rotation around O of angle 2α.



The key observation is that if we have any other pair of lines k', ℓ' so that (i) k' and ℓ' intersect at O and (ii) k', ℓ' are separated by an angle α, then the reflection R_1' through k' and the reflection R_2' through ℓ' satisfy $R_2' \circ R_1' = R_2 \circ R_1$ (since they are both the rotation around O by angle 2α). Therefore, without loss of generality, we may rotate the pair of lines k, ℓ so that ℓ is perpendicular to m. Let P denote the intersection point of ℓ and m.



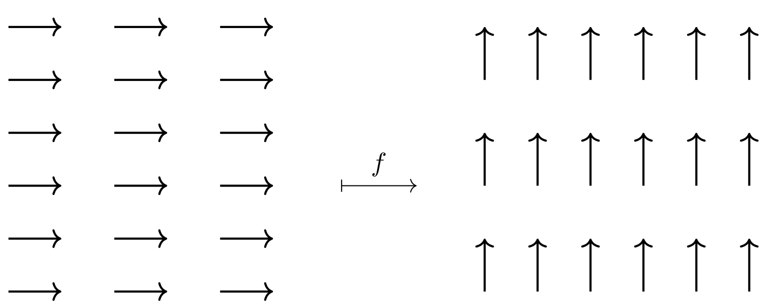
As we did above with lines k and ℓ, we may now rotate the lines ℓ and m to lines ℓ' and m' still intersecting at P so that $R_3' \circ R_2' = R_3 \circ R_2$ (where R_2', R_3' are the reflections through ℓ' and m'). Therefore, replacing ℓ and m by ℓ' and m' we may assume that m is perpendicular to k.



Since m is perpendicular to both k and ℓ, k and ℓ parallel, and so $R_3 \circ R_2 \circ R_1$ is a glide reflection ($R_2 \circ R_1$ is a translation, and R_3 is a reflection through a line parallel to the translation vector). This completes the proof of the theorem. $\square$

Exercise 10.23. Prove the above theorem in the case when k and ℓ are parallel.

10.2. **Bonus subsection on the linear part of an isometry.** A key idea in the study of isometries is that isometries transform vectors in a natural way. If $\overrightarrow{AB}$ is a vector and f is an isometry, we can consider the transformed vector $\overrightarrow{f(A)f(B)}$. What is important about this transformation is that $\overrightarrow{f(A)f(B)}$ only depends on the vector $\overrightarrow{AB}$ - in other words, if $\overrightarrow{AB} = \overrightarrow{A'B'}$, then $\overrightarrow{f(A)f(B)} = \overrightarrow{f(A')f(B')}$. In fact, the transformation $\overrightarrow{AB} \mapsto \overrightarrow{f(A)f(B)}$ is a linear transformation of vectors. Exercise 10.24 presents the necessary results.



Exercise 10.24. Let f be an isometry.

(a) If $\overrightarrow{AB} = \overrightarrow{A'B'}$, show that $\overrightarrow{f(A)f(B)} = \overrightarrow{f(A')f(B')}$. *Hint*: $\overrightarrow{AB} = \overrightarrow{A'B'}$ if and only if the midpoint of AB' is the midpoint of $A'B$. The latter statement is preserved under isometries.

(b) Show that for each vector there is a corresponding vector $f^\flat(v)$ so that $\overrightarrow{AB} = v$ implies $\overrightarrow{f(A)f(B)} = f^\flat(v)$. *Hint*: use part (a)!

Then $v \mapsto f^\flat(v)$ is a transformation of the set of vectors.

(c) Prove that for any positive real number $\lambda > 0$, $f^\flat(\lambda v) = \lambda f^\flat(v)$. *Hint*: it may be convenient to prove the cases $\lambda > 1$ and $\lambda < 1$ separately.

(d) Suppose that v, w are two vectors. Prove that $f^\flat(v + w) = f^\flat(v) + f^\flat(w)$. *Hint*: Let A, B, C, D be chosen so that $\overrightarrow{AB} = v$ and $\overrightarrow{BD} = w$, so $\overrightarrow{AD} = v + w$. Then

$$\overrightarrow{f(A)f(D)} = f^\flat(v + w).$$

However, we also have $\overrightarrow{f(A)f(B)} = f^\flat(v)$ and $\overrightarrow{f(B)f(D)} = f^\flat(w)$, so

$$\overrightarrow{f(A)f(D)} = f^\flat(v) + f^\flat(w).$$

(e) Use part (d) to show that $f^\flat(-v) = -f^\flat(v)$, and then use part (c) to show that $f^\flat(\lambda v) = \lambda f^\flat(v)$ for any real number λ.

Therefore, for each isometry f, $f^\flat$ is a **linear operator** on the set of all vectors (a "linear operator" is a transformation of a vector space which preserves addition and scalar multiplication).

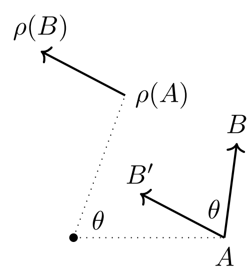
(f) If g is another isometry, then $(g \circ f)^\flat = g^\flat \circ f^\flat$.

Definition 10.25. Following the notation introduced in Exercise 10.24, we define the **linear part** of an isometry f to be the linear operator $f^\flat$. This operator acts on the space of vectors in such a way that if A, B are points in the plane, then $f^\flat(\overrightarrow{AB}) = \overrightarrow{f(A)f(B)}$.

Exercise 10.26. Give an example of two different isometries $f \neq g$ so that $f^\flat = g^\flat$.

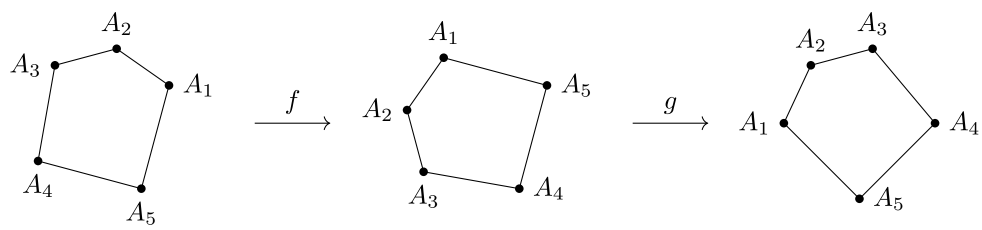
Exercise 10.27. Show that $f^\flat(v) = v$ for all vectors v if and only if f is a translation. *Hint:* for the "only if" direction it suffices to show that, for all A, B, $\overrightarrow{Af(A)} = \overrightarrow{Bf(B)}$.

Exercise 10.28. If ρ is any rotation by an angle α, then $\rho^\flat(\overrightarrow{AB}) = \overrightarrow{AB'}$, where B' is the rotation of B around A by the angle α.



Hint: Let O be the centre of rotation of ρ and let T be a translation taking A to O. Then show that $T^{-1} \circ \rho \circ T$ is a rotation around A by angle α. Finally, note that $(T^{-1} \circ \rho \circ T)^\flat = \rho^\flat$.

10.3. **Bonus subsection on orientation.** The goal of this subsection is to make precise the statement: "the isometry f preserves (or reverses) orientation." Intuitively, an isometry f "preserves orientation" if it takes a polygon $A_1 \cdots A_n$ whose vertices are oriented "clockwise" to another polygon whose vertices are oriented "clockwise," while f reverses orientation if the vertices of $f(A_1)\cdots f(A_n)$ are oriented "counter-clockwise." In the figure below, f preserves orientation, while g reverses it.



It is rather awkward to make the notion of "clockwise" precise, but it is possible to give a rigourous definition of orientation which agrees with our intuition.

First let us write down a few properties any reasonable definition of orientation should satisfy.

Requirements. Let f and g be isometries of the plane.

(I) If f preserves orientation and g preserves orientation, then $f \circ g$ also preserves orientation.

(II) If f preserves orientation and g reverses orientation, then $f \circ g$ and $g \circ f$ reverse orientation.

(III) If f reverses orientation and g reverses orientation, then $f \circ g$ preserves orientation.

(IV) All reflections reverse orientation.

We note that properties (I), (II), (III) can be packaged into the shorter statement: for every isometry f there is a corresponding number $\text{sgn}(f) \in \{-1, +1\}$, so that

$$\text{sgn}(f \circ g) = \text{sgn}(f)\text{sgn}(g),$$

with the convention that $\text{sgn}(f) = +1$ means f preserves orientation and $\text{sgn}(f) = -1$ means f reverses orientation. Then property (IV) implies that $\text{sgn}(R) = -1$ whenever R is a reflection.

Therefore, the problem of giving a rigourous definition of orientation reduces to the problem of constructing the function sgn.

Proposition 10.29. There is *at most one* function $f \mapsto \text{sgn}(f) \in \{-1, +1\}$, defined on isometries, so that $\text{sgn}(f \circ g) = \text{sgn}(f)\text{sgn}(g)$ and $\text{sgn}(R) = -1$ for all reflections R.

Proof. Let f be an isometry of the plane. Following the proof of the classification of all isometries, write f as a composition of reflections $f = R_n \circ \cdots \circ R_1$. Then $\text{sgn}(f) = (-1)^n$, so the value of $\text{sgn}(f)$ is already determined for us. This proves the proposition. □

It is tempting to define $\text{sgn}(f) = (-1)^n$ whenever $f = R_n \circ \cdots \circ R_1$ for reflections $R_1, \cdots, R_n$, but this definition is, a priori, not well-defined, since it is certainly conceivable that we could also write $f = R'_{n+1} \circ \cdots \circ R'_1$ for some different reflections $R'_1, \cdots, R'_{n+1}$. In short, the value of $\text{sgn}(f)$ seems to be over-determined.

Theorem 10.30. There exists a function sgn : {isometries} $\to \{\pm 1\}$ so that

$$\mathrm{sgn}(f \circ g) = \mathrm{sgn}(f)\mathrm{sgn}(g)$$

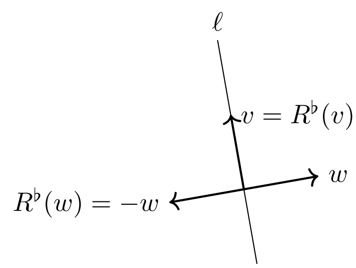
and $\mathrm{sgn}(R) = -1$ for all reflections R.

Proof. A key ingredient in the contruction is the linear part of an isometry. As we did above, we write $f^\flat$ for the linear part of an isometry f. Then $f^\flat$ is a linear operator on a (two-dimensional) vector space, and so its determinant $\det f^\flat$ is well-defined. Furthermore, basic properties of $f^\flat$ and det imply that $\det(f \circ g)^\flat = \det f^\flat \cdot \det g^\flat$. We define

$$\mathrm{sgn}(f) = \det f^\flat.$$

Then $\mathrm{sgn}(f \circ g) = \mathrm{sgn}(f)\mathrm{sgn}(g)$.

If R is a reflection through a line ℓ, pick v, w basis vectors so that $v \parallel \ell$ and $w \perp \ell$. By picking $v = \overrightarrow{AB}$ with $A, B \in \ell$ we clearly see that $R^\flat(v) = v$. Write $w = \overrightarrow{AC}$ with $A \in \ell$ and $C \notin \ell$. Then $R^\flat(w) = \overrightarrow{AC'}$ with $C' \neq C$. Since $AC \perp \ell$, A is the midpoint of $C'C$, so $|CA| = |AC'|$. Thus $|AC'| + |AC| = 0$, and so $R^\flat(w) = -w$.



The matrix of $R^\flat$ with respect to the basis $\{v, w\}$ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $\det R^\flat = -1$. Since any isometry f is a composition of reflections, $\mathrm{sgn}(f) \in \{+1, -1\}$, and this completes the proof. □

We will use orientation as a tool to distinguish isometries; the following exercise demonstrates how this can be done.

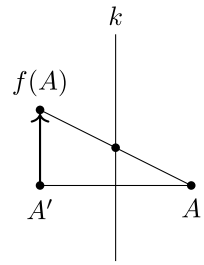
Exercise 10.31. Suppose that $R_1, \cdots, R_n$ are rotations (whose centres of rotation may be different). Prove that $R_1 \circ \cdots \circ R_n$ is either a translation or a rotation.

10.4. **Applications of the classification of isometries.**

Problem 10.32. Let f be a glide reflection parallel to the line k. Then for any point A, the centre of the segment $Af(A)$ lies on k.

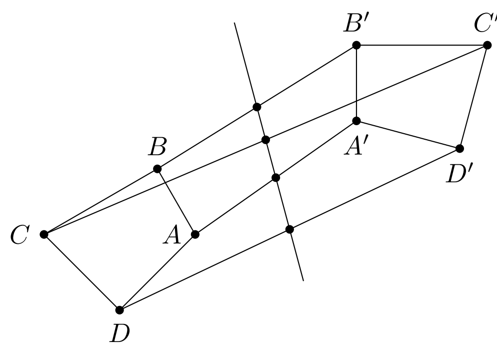
Solution. The case when the translation vector is zero is immediate, and so we suppose that the translation vector of the glide reflection is non-zero. Let A' denote the reflection of

A through k, so that $f(A)$ is the translation of A' along a vector parallel to k.



Then k is parallel to $A'f(A)$, and so we may apply Thales' theorem to the angle $A'Af(A)$ to deduce that the intersection of $Af(A)$ with k is the midpoint of $Af(A)$. ◻

Problem 10.33. Let $ABCD$ and $A'B'C'D'$ be congruent quadrilaterals, but suppose the vertices of one quadrilateral are labelled clockwise, while the vertices of the other quadrilateral are labelled counter-clockwise. Then the centres of the segments AA', BB', CC' and DD' all lie on a common line.



Solution. This problem is a nice application of our classification of isometries. The fact that $ABCD$ and $A'B'C'D'$ are congruent implies there is some isometry f which takes $ABCD$ to $A'B'C'D'$. Since $A'B'C'D'$ has a different orientation to $ABCD$, we know that the isometry f must reverse orientation. Thus f cannot be a rotation or a translation. By our classification of isometries, f is therefore a glide reflection through some line k. Then Problem 10.32 implies that the midpoints of the segments AA', BB', CC' and DD' all lie on k, and this completes the solution. ◻

Theorem 10.34. Let $f = T_n \circ R_n \circ \cdots \circ T_1 \circ R_1$ be a composition of translations and rotations. Suppose the rotation angle of R_j is α_j. Then f is a translation if $\alpha_1 + \cdots + \alpha_n = 0$ and otherwise f is a rotation by angle $\alpha_1 + \cdots + \alpha_n$.

Proof. The idea of the proof is fairly simple: start with some line ℓ, analyze how ℓ transforms as we apply $R_1, T_1, \cdots, R_n, T_n$.

We have shown that $R_1(\ell)$ intersects ℓ at an angle α_1 (see Theorem 2.9)); if $\alpha_1 = 0°$ or $= 180°$, then $R_1(\ell)$ doesn't actually intersect ℓ, but we will interpret "intersecting at an angle of 0° or 180°" to mean "parallel."

Then $T_1 \circ R_1(\ell)$ is parallel to $R_1(\ell)$, and hence $T_1 \circ R_1(\ell)$ also intersects ℓ at an angle of α_1. Iterating this argument, we deduce that $f(\ell) = T_n \circ R_n \circ \cdots \circ T_1 \circ R_1(\ell)$ intersects ℓ at an angle of $\alpha = \alpha_1 + \cdots + \alpha_n$. If α is not 0° or 180°, then $f(\ell)$ actually intersects ℓ in a unique point point P. It follows that f cannot be a translation, since translations send lines to parallel lines. Since f preserves orientation, f cannot be a glide reflection. By the classification of isometries, it follows that f is a rotation. Furthermore, since $f(\ell)$ intersects ℓ at an angle α, we know (from Theorem 2.9 again) that the angle of rotation of f must be α or $\alpha + 180°$ (this indeterminacy is because there are always two angles formed between a pair of lines). To resolve this indeterminacy we will look at the linear part of f. Furthermore when $\alpha = 0°$ or $\alpha = 180°$, then $f(\ell)$ is parallel to ℓ and so we cannot detect whether f is rotation by 180° or a translation simply by comparing $f(\ell)$ with ℓ. However, instead of keeping track of how the line ℓ transforms under $R_1, T_1 \cdots, R_n, T_n$, if we keep track of how a vector v transforms, then we will be able to differentiate the angles α and $\alpha + 180°$ and also classify the cases when $\alpha = 0°$ and $\alpha = 180°$. In other words, we will look at the linear part of the composition $R_n \circ T_n \circ \cdots \circ R_1 \circ T_1$. Since $T^\flat$ = identity for any translation T (Exercise 10.27) we have

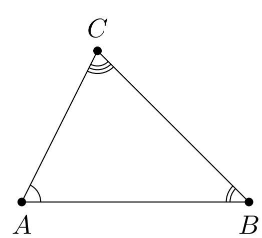
$$(R_n \circ T_n \circ \cdot \circ R_1 \circ T_1)^\flat = R_n^\flat \circ T_n^\flat \circ \cdots \circ R_1^\flat \circ T_1^\flat = R_n^\flat \circ \cdots \circ R_1^\flat.$$

Then, by Exercise 10.28, we know that $R_n^\flat \circ \cdots R_1^\flat$ simply takes a vector $\overrightarrow{AB}$ and sends it to $\overrightarrow{AB'}$, where B' is the rotation of B around A by an angle $\alpha_1 + \cdots + \alpha_n$.

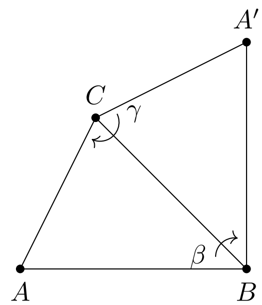
If $\alpha = \alpha_1 + \cdots + \alpha_n \neq 0°, 180°$, then we already know that f is a rotation, so the fact that $f^\flat$ rotates by α implies f also rotates by α. This proves the case when $\alpha \neq 0°, 180°$.

If $\alpha_1 + \cdots + \alpha_n = 180°$, every vector v is mapped to its inverse $-v$. In this case, we conclude that $R_n \circ T_n \circ \cdot \circ R_1 \circ T_1$ is not a translation, and hence must be a rotation by angle 180°. On the other hand, if $\alpha_1 + \cdots + \alpha_n = 0°$, then the above discussion implies that $f^\flat$ is the identity transformation, so f is a translation (by Exercise 10.27). ☐

Problem 10.35. Let R_A, R_B and R_C be rotations around the vertices of a triangle ABC by angles α, β and γ, respectively. Supposing that $R_C \circ R_B \circ R_A = \text{id}$, find the interior angles of ABC.



Solution. To begin, let us see how the point A transforms when we apply the rotations R_A, R_B, R_C. Clearly $R_A(A) = A$. Define $A' = R_B(A)$; then $R_C \circ R_B \circ R_A = \text{id}$ implies $R_C(A') = A$.



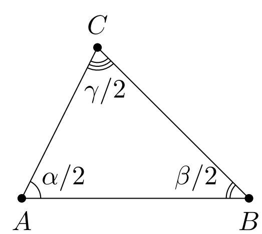
Then $|A'C| = |AC|$ and $|A'B| = |AB|$, so that CB is the bisector of AA'. This fact uniquely determines A', and also tells us that BC bisects the angles $\angle ABA'$ and $\angle A'CA$. Therefore

$$\angle BCA = \gamma/2 \text{ and } \angle ABC = \beta/2.$$

To find the interior angle at A, we use a similar argument. First we note that $R_C \circ R_B \circ R_A = \text{id}$ implies

$$R_B \circ R_A = R_C^{-1}, \text{ and thus } R_B \circ R_A \circ R_C = \text{id}.$$

Then we may apply the same argument used in the first paragraph but starting with vertex C to conclude $\angle CAB = \alpha/2$. This completes the solution.

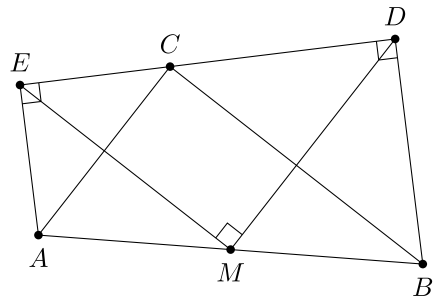


□

Exercise 10.36. Let R_A, R_B be rotations by angles α, β around points A, B, respectively. Assuming that $\alpha + \beta < 360°$, find a rotation R_C by angle γ around a point C so that $R_B \circ R_A = R_C^{-1}$. *Hint*: Use Problem 10.35.

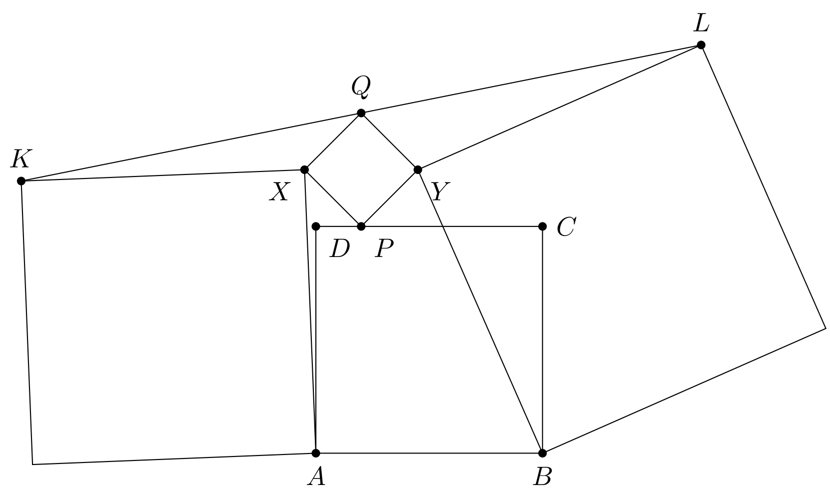
Problem 10.37. Let E, C, D be three points lying on a common line with C between E and D. Suppose A and B are chosen on the same side of the line ED so that the triangles AEC and BDC are isosceles and right, and let M be the midpoint of AB. Prove that EMD

is isosceles and right.



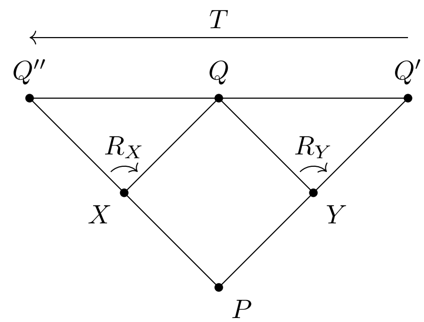
Solution. Interestingly, we will solve this problem by invoking the result of Problem 10.35. Let R_D be the rotation around D sending B to C, and let R_E be the rotation around E sending C to A. Then the composition $R_E \circ R_D$ sends B to A. By Theorem 10.34, $R_E \circ R_D$ must be a rotation by 180°, and since $R_E \circ R_D$ sends B to A, it must be the 180° rotation centred at M, which we denote by R_M. Then $R_M \circ R_E \circ R_D = \text{id}$, and so, by Problem 10.35, $\angle EMD = 90°$, $\angle DEM = \angle MDE = 45°$, hence EMD is isosceles and right. ◻

Problem 10.38. Let A, B, C, D be a square with side length a, and choose P on the side CD. Let $PYQX$ be another square with diagonal $QP \perp DC$ and diagonal length $a/2$. Let K, L be constructed as in the figure below (so that LYB and AXK are corners of squares). Then Q is the midpoint of the segment KL.



Solution. Let R_Y and R_X be 90° rotations around Y, X sending L to B and A to K, respectively, and let T be the translation by vector $\overrightarrow{BA}$. Then $R_X \circ T \circ R_Y(L) = K$, and, further, by Theorem 10.34 we know $R_X \circ T \circ R_Y$ is an 180° rotation. We claim that $R_X \circ T \circ R_Y$

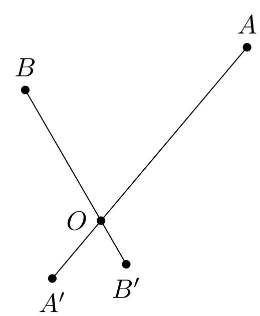
fixes the point Q. To see this, consider the following figure.



As above, let $Q' = R_Y(Q)$ and $R_X(Q'') = Q$. As is clear in the above figure, QQ' is orthogonal to QP, and hence QQ' is parallel to T. Similarly QQ'' is parallel to T, and $|Q'Q''| = |QQ'| + |QQ''| = a$, so T takes Q' to Q''. Thus $R_X \circ T \circ R_Y(Q) = Q$, and so Q is the centre of the 180° rotation taking L to K, which means Q is the midpoint of KL. ◻

11. Homotheties with negative scale

Definition 11.1 (Homothety with negative scale)**.** In Example 7.3 we constructed homotheties with positive scale $k > 0$, and showed that they were similarities with scale k. We can extend this definition: the **homothety with scale** $k < 0$ centred at O is the transformation of the plane fixing O and sending each point $A \neq 0$ to the point A' lying on the line AO so that (a) O separates A from A' and (b) $|A'O| / |AO| = |k|$.



Using the fact that homotheties with positive scale $|k|$ are similarities with scale $|k|$, it is straightforward to show that homotheties with scale $k < 0$ are also similarities with scale $|k|$ (see Exercise 11.2 below).

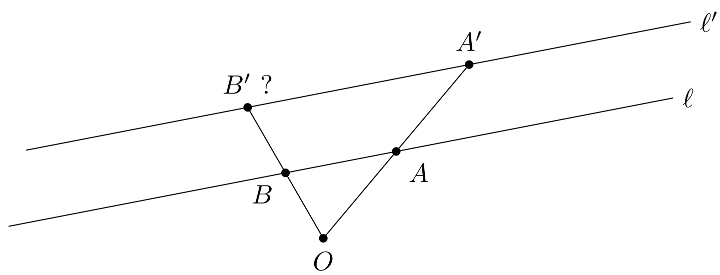
Exercise 11.2. (a) Show that the homothety with scale -1 centred at O is the point reflection (i.e. 180° rotation) around O.

(b) Using part (a), prove that the homothety of scale $k < 0$ centred at O is the composition of a homothety of scale $|k| > 0$ and an isometry. Conclude that homotheties of scale $k < 0$ are similarities with scale $|k| > 0$.

Exercise 11.3. Prove that the inverse of a homothety of scale k (k may be positive or negative) centred at O is the homothety of scale $1/k$ centred at O.

Theorem 11.4. Homotheties map lines to parallel lines. In other words, if ℓ is a line and ℓ' is its image under a homothety, then ℓ' is a line parallel to ℓ.

Proof. Let O and k denote the centre and scale of the homothety. We consider the case $k > 0$. If ℓ contains O, then the homothety fixes ℓ, so there is nothing to prove. Consider now a line ℓ disjoint from O. Pick a point A on the line ℓ, and let ℓ' be the line parallel to ℓ passing through the image A' of A under the homothety. If B lies on ℓ, we want to show that B' lies on ℓ'.

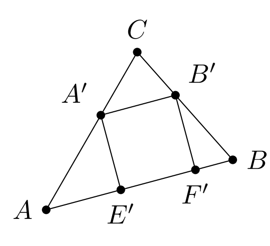


To prove that B' lies on ℓ', we use the converse to Thales' Theorem. Since $|OA'| \,/\, |OA| = |OB'| \,/\, |OB|$, and O does not separate A from A' or B from B', we deduce that AB and $A'B'$ are parallel, and so $A'B'$ must be the line ℓ', hence $B' \in \ell'$. We have shown that the homothety maps all points on ℓ onto the line ℓ'.

We still need to prove that every point on ℓ' is the image of a point from ℓ. To show this, fix $C' \in \ell'$, and pick C on the ray OC' so that $|OC'| \,/\, |OC| = k$ (such a C' exists by continuity). Applying the converse to Thales' Theorem again, we deduce that $\ell' = A'C'$ is parallel to AC, and so C must lie on ℓ. By our construction, the image of C under the homothety is C' (justifying our notation) and this completes the proof. $\square$

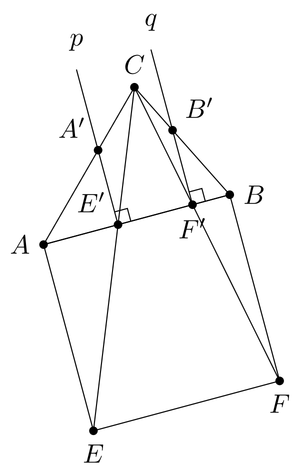
Exercise 11.5. Prove Theorem 11.4 in the case $k < 0$.

Problem 11.6. Given an acute triangle ABC, find an inscribed square $A'B'F'E'$ such that the edge $E'F'$ lies on the side AB, and so that $A' \in AC$ and $B' \in BC$.



Solution. The idea for the solution is to begin by constructing a square which satisfies some of the requirements of the problem (but not all of the requirements) and then use a homothety to transform the square into a new one which satisfies all the requirements of the problem.

We begin by defining E, F so that $ABFE$ is a square and $ABFE$ lies inside the angle BCA (this determines E and F uniquely), and we define E' and F' to be the intersections of EC and FC with AB, respectively. Then we raise perpendiculars p and q to AB from the points E' and F', respectively, and define A' to be the intersection of p and AC and B' the intersection of q with BC.

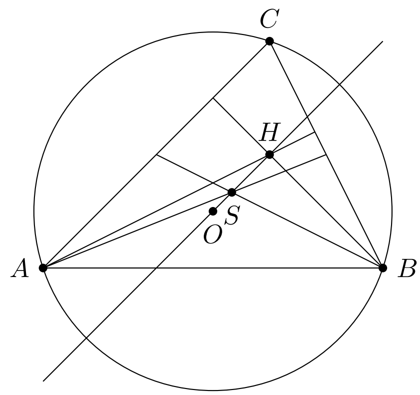


Consider the homothety centred at C of scale $|CE'|\,/\,|CE|$. We claim that this homothety sends A to A', B to B', E to E' (obviously) and F to F'. Since EF is parallel to $E'F'$ (since $E'F' = AB$, and $ABFE$ is a square), we can apply Thales' Theorem to deduce that $|CF'|\,/\,|CF| = |CE'|\,/\,|CE|$, and so we know that the homothety maps F to F'.

Since homotheties preserve parallel lines, we deduce that the line EA is mapped to the line p (show this!), and so the image of A under the homothety must lie on p. However, since the homothety is centred at C, we also know that the image of A lies on the ray CA, and so the image of A must be the intersection point A' of p and AC. A similar argument proves that the image of B is B'.

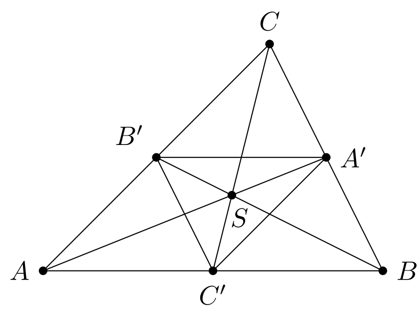
We claim that $A'B'E'F'$ is a square. This is easy to see since $ABEF$ is a square and homotheties are similarities (so they preserve angles and the ratios of sides). This completes the solution. ◻

Problem 11.7 (Euler line)**.** Let ABC be a triangle, and let O be the intersection point of the side bisectors (i.e. O is the centre of the circumscribed circle), let H be the orthocentre of ABC (recall that the orthocentre is the intersection of the altitudes), and let S be the centroid of ABC (recall that the centroid is the intersection of medians). Prove that S lies on OH and divides it in a $1:2$ ratio.

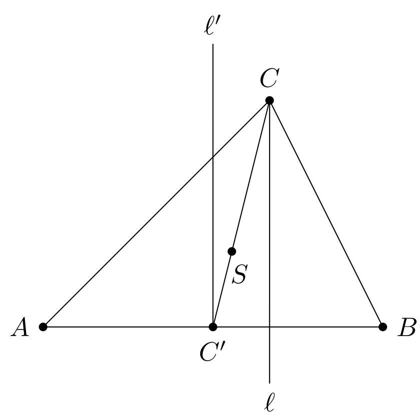


Solution. The idea is to consider the homothety with scale $-1/2$ with centre S, and show that H gets mapped to O, which will prove that S divides OH in a $1:2$ ratio.

Let A', B', C' be the midpoints of the edges BC, AC and AB, respectively. It can be shown that S divides $A'A$, $B'B$ and $C'C$ in $1:2$ ratio (see Exercise 11.8 below). It follows that the homothety of scale $-1/2$ through S sends the triangle ABC to the triangle $A'B'C'$.



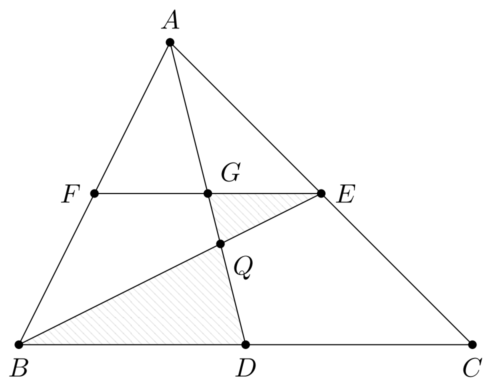
Furthermore, by Theorem 11.4, the altitude ℓ through C gets mapped to a parallel line ℓ' passing through C', and since ℓ is perpendicular to AB, ℓ' is also perpendicular to AB. Since C' is the midpoint of AB and ℓ' is perpendicular to AB, ℓ' is the bisector of AB.



A similar argument shows that the altitude through A is mapped to bisector of BC. It follows that the intersection point H of the two altitudes is mapped to the intersection point O of the two bisectors. Thus HSO lie on a common line. Since the homothety has negative scale, we know S separates HO, and since it has scale $-1/2$, we know $|SO| / |SH| = 1/2$, so S splits OH in a $1 : 2$ ratio, as desired. ▢

Exercise 11.8. Let D, E, F be the centres of the sides BC, AC, AB of a triangle ABC. Prove that the segments AD, BE, DF intersect at a common point that divides each of them in the ratio $2 : 1$.

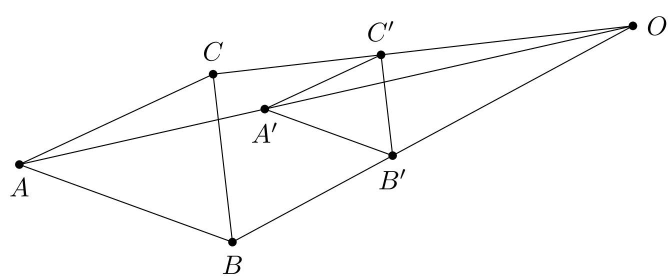
Hint: Consider the following figure, and prove that $|AQ| / |DQ| = 2$.



Theorem 11.9. Let ABC and $A'B'C'$ be triangles with $AB \parallel A'B'$, $BC \parallel B'C'$ and $AB \parallel A'C'$. Then there is a homothety or a translation mapping ABC to $A'B'C'$.

Before we prove this theorem, we remark that it has a nice corollary of independent interest.

Corollary 11.10. With ABC and $A'B'C'$ with parallel edges (as in Theorem 11.9), we deduce that AA', BB' and CC' are parallel (corresponding to a translation in Theorem 11.9) or the intersect in a single point (corresponding to a homothety in Theorem 11.9).

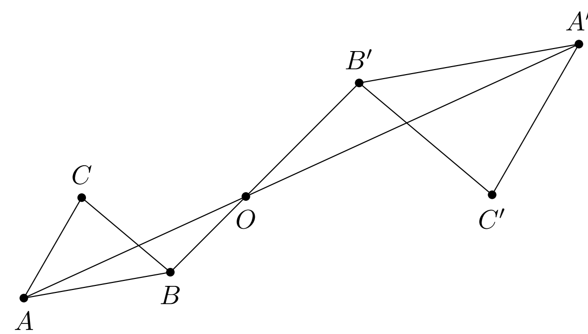


Exercise 11.11. Prove Corollary 11.10 using Theorem 11.9.

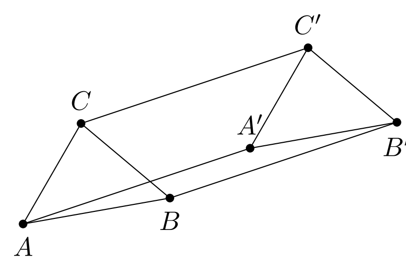
Proof of Theorem 11.9. Assume first that AA' and BB' intersect in a point O. Since AB and $A'B'$ are parallel, there are only two possibilities, either O lies outside of the segments

AA' and BB' or O separates A from A' and B from B' (for the proof of this assertion, refer to the proof of Thales' theorem). Assume first that O lies outside of the segments AA' and BB'. Consider the homothety with centre O and (positive) scale $|A'B'|\,/\,|AB|$. A straightforward application of Thales' theorem proves that the image of A is A' and the image of B is B'. Since the line $A'C'$ is parallel to the line AC, and (by Theorem 11.4) the line AC is mapped to a parallel line under the homothety, we deduce that the image of the line AC is the line $A'C'$. Similarly, the image of the line BC is the line $B'C'$, and so the intersection point C of the lines AC and BC is mapped to the intersection point C' of the lines $A'C'$ and $B'C'$. Therefore we have found a homothety taking ABC to $A'B'C'$.

In the case where O separates A from A' and B from B', we consider the homothety centred at O with scale $-\,|A'B'|\,/\,|AB|$. Using a similar argument to the one in the previous case, we deduce that this homothety takes C to C'. The details are left to the reader.



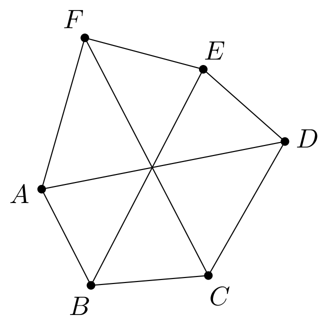
Now we assume that AA' and BB' are parallel. Then $ABB'A'$ forms a parallelogram, and so the translation by vector $\overrightarrow{AA'} = \overrightarrow{BB'}$ takes A to A' and B to B'. Since translations map lines to parallel lines, we deduce that AC is mapped to $A'C'$ and BC is mapped to $B'C'$, and so C must be mapped to C', as desired. This completes the proof of the theorem.



□

Problem 11.12. Let $ABCDEF$ be a convex hexagon with area a, and so that each diagonal AD, BE and CF cuts $ABCDEF$ into two quadrilaterals of equal area $a/2$. Prove that AD,

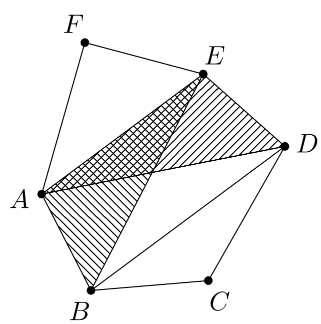
BE and CF intersect in a single point.



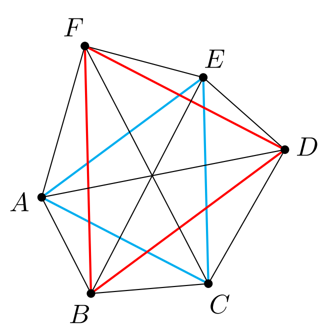
Solution. First we remark that $|ABE| = |ADE|$, since $|ADEF| = |ABEF| = a/2$, and

$$|ADEF| = |AEF| + |ADE| \text{ and } |ABEF| = |AEF| + |ABE|.$$

Then since ABE and ADE share the base AE, they must also share a height, so D and B are the same distance from AE, thus $DB \parallel AE$.

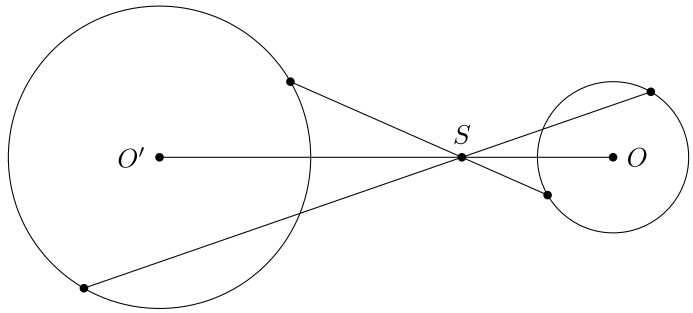


Analogously, $BF \parallel CE$ and $AD \parallel DF$. Therefore we may apply Corollary 11.10 to the triangles ACE and DFB (since they have parallel side pairs) so that AD, CF and EB intersect in a single point, as desired.



□

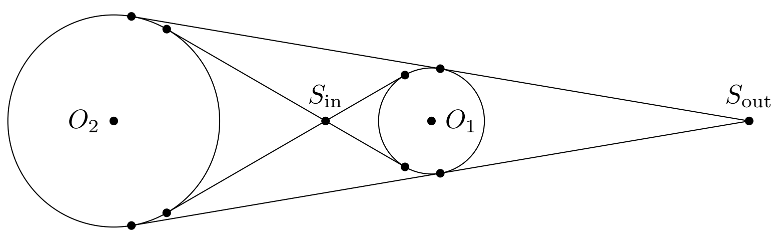
Theorem 11.13. The homothety with centre S and scale k maps the circle o with centre O and radius r to the circle with centre O' (the image of O) and radius $|k|\, r$.



Proof. If X lies on o and X' is the image of X under the homothety, then we know that $|X'O'| = k\,|XO|$, and so X' lies on o'. This proves that every point on o is mapped onto o'.

If Y' lies on o', then we let Y be the image of Y' under the homothety centred at S of scale $1/k$, and note that Y must lie on o; this proves that every point on o' is the image of a point from o, and this completes the proof. □

Theorem 11.14. Let o_1 and o_2 be two circles with radii $r_1 \neq r_2$. Then there are exactly two homotheties mapping o_1 to o_2 (one with scale r_2/r_1 and one with scale $-r_2/r_1$).



Proof. It is clear that the scale of any homothety sending o_1 to o_2 must be $\pm r_2/r_1$, and so it remains to find the centre of the homotheties. Let O_1 and O_2 be the centres of o_1 and o_2. First we will find the centre of the homothety with scale $-r_2/r_1$. Pick S in the segment O_1O_2 so that $|SO_2|\,/\,|SO_1| = r_2/r_1$. Such an S always exists, by continuity (since the ratio $|XO_2|\,/\,|XO_1|$ goes to 0 as X goes to O_2 and to ∞ as X goes to O_1). Then the homothety of scale $-r_2/r_1$ around S sends O_1 to O_2, and, by Theorem 11.13, sends the circle of radius r_1 to the circle of radius r_2. Thus the chosen homothety sends o_1 to o_2. Note that we did not need to assume $r_1 \neq r_2$ in this case.

Now we will find the centre of the homothety with scale r_2/r_1. Without loss of generality, assume that $r_1 < r_2$. Note that the ratio $|XO_2|\,/\,|XO_1|$ goes to ∞ as X approaches O_1 and asymptotically decreases to 1 as X gets further from O_1 (while remaining on the opposite side of O_2). Therefore, by continuity, we can find an S on the line O_1O_2 so that O_1 separates S from O_2, and so that $|SO_2|\,/\,|SO_1| = r_2/r_1 > 1$. Then an analogous argument to the one in the previous paragraph shows that the homothety of scale r_2/r_1 centered at O takes o_1 to o_2.

In Exercise 11.15 below, the reader will prove that the two homotheties we have constructed are the only ones taking o_1 to o_2, and so we are justified in saying that there are *exactly* two homotheties taking o_1 to o_2. □

Exercise 11.15. Prove that two homotheties we have constructed in the proof to Theorem 11.14 are unique using the following outline: Suppose there were a third homothety h sending o_1 to o_2, and let H be the centre of h.

(a) Prove that the scale of h is $\pm r_2/r_1$.

(b) If the scale of h is $-r_2/r_1$, prove that H lies between O_1 and O_2, and furthermore that $|O_2H| \,/\, |O_1H| = r_2/r_1$. In this case, prove that $H = S_{\text{in}}$ (using the notation in the figure accompanying Theorem 11.14).

(c) If the scale of h is r_2/r_1, and $r_2 > r_1$, prove that O_1 lies between H and O_2, and that $|O_1H| \,/\, |O_2H| = r_2/r_1$. In this case, prove that $H = S_{\text{out}}$.

(d) Repeat part (c) when $r_1 < r_2$.

(e) Conclude that h is one of the two homotheties we constructed in the proof to Theorem 11.14.

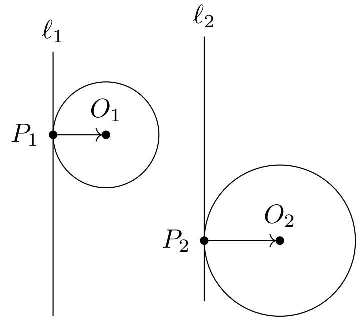
Exercise 11.16. Let o_1 and o_2 be two circles, and let ℓ_1 be a line tangent to o_1. Prove that there are exactly two lines, $\ell_{2,+}$ and $\ell_{2,-}$, which are parallel to ℓ_1 and tangent to o_2, and prove that they are the image of ℓ_1 under the homotheties (or translations) from Theorem 11.14.

Remark. The proof of Theorem 11.14 uses a continuity argument to guarantee the existence of the two centres of homothety (these are denoted S_{in} and S_{out} in the accompanying figure). It is natural to ask whether there is a more concrete description of S_{in} and S_{out}.

We will be able to describe S_{in} and S_{out} as the intersection of various tangent lines to o_1 and o_2. However, before we begin, we will need a preliminary exercise concerning tangent lines to circles.

Exercise 11.17. Given two circles o_1, o_2 with centres O_1, O_2, respectively, and two parallel lines ℓ_1 and ℓ_2 tangent to o_1 at P_1 and o_2 at P_2, respectively, we say that ℓ_1 and ℓ_2 are tangent to o_1 and o_2 **from the same side** if the vectors $\overrightarrow{P_1O_1}$ and $\overrightarrow{P_2O_2}$ differ by a positive scalar (since $\overrightarrow{P_1O_1}$ and $\overrightarrow{P_2O_2}$ are both perpendicular to $\ell_1 \parallel \ell_2$, we know they differ by some scalar but it could be negative).

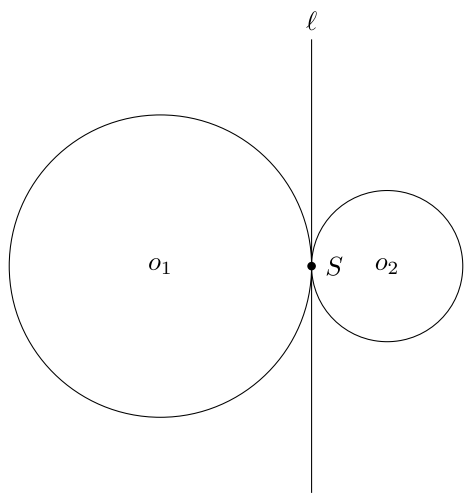
If $\overrightarrow{P_1O_1}$ and $\overrightarrow{P_2O_2}$ differ by a negative scalar then we say that ℓ_1 and ℓ_2 are tangent to o_1 and o_2 **from opposite sides**.



Show that (a) Given a line ℓ_1 tangent to a circle o_1 and a second circle o_2, there is a *unique* line ℓ_2 tangent to o_2 so that ℓ_1 and ℓ_2 are tangent to o_1 and o_2 from the same side.

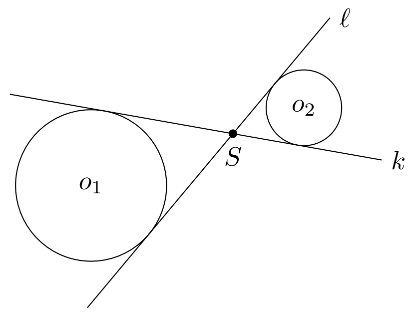
(b) If there is a homothety with positive scale (resp. negative scale) taking the pair (o_1, ℓ_1) to the pair (o_2, ℓ_2), where ℓ_1 is tangent to o_1, then ℓ_2 is tangent to o_2 from the same (resp. opposite) side as ℓ_1 is to o_1. (*Hint*: consider how homotheties transform vectors.)

With the results of this preliminary exercise at our disposal, we will describe S_{in} in various cases. First, if we assume that o_1 and o_2 are externally tangent at a point S, then S_{in} must be the tangency point S.



To see why this is so, let ℓ denote the tangent line to o_1 and o_2 through S and consider the homothety with scale $-r_2/r_1$ centered at S. This homothety preserves ℓ and S, and so, by part (b) of Exercise 11.17, we know that the image of o_1 is tangent to ℓ at S and lies on the opposite side as o_1. Since o_2 is the unique circle of radius r_2 tangent to ℓ at S lying on the opposite side of o_1, the image of o_1 must be o_2.

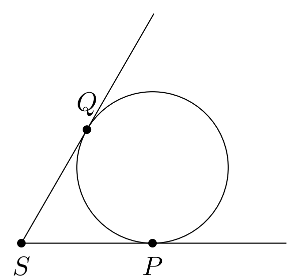
Now let us assume that o_1 and o_2 are tangent to opposite angles formed by a pair of lines k and ℓ which intersect at a point S, as shown below.



Considering the homothety of negative scale mapping o_1 to o_2, part (b) of Exercise 11.17 guarantees that the image of k is a parallel line which is tangent to o_2 from the opposite side as k is tangent to o_1 - in this case, it is easy to see that this implies k is preserved by the homothety. Similarly ℓ is preserved by the homothety, and so their intersection point must be the centre of the homothety.

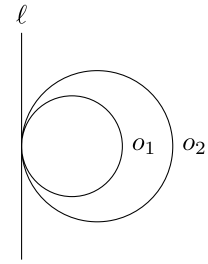
Similar arguments allow us to find the centre of the positive scale homothety in the special cases when o_1 and o_2 are internally tangent, and when they are both tangent to the same angle (see Exercise 11.19).

Exercise 11.18. Let P and Q be two points on opposite sides of an angle (less than 180°), both at the same distance from the centre S. Show that there is a unique circle tangent to angle with tangency points P and Q. (*Hint*: Consider the perpendicular raised from P).

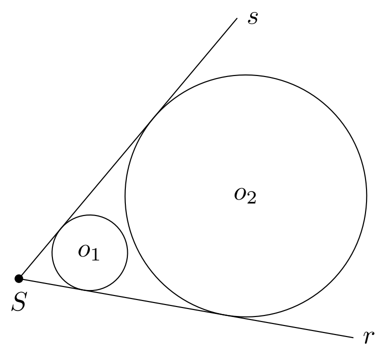


Exercise 11.19. Let o_1 and o_2 be two circles with different radii.

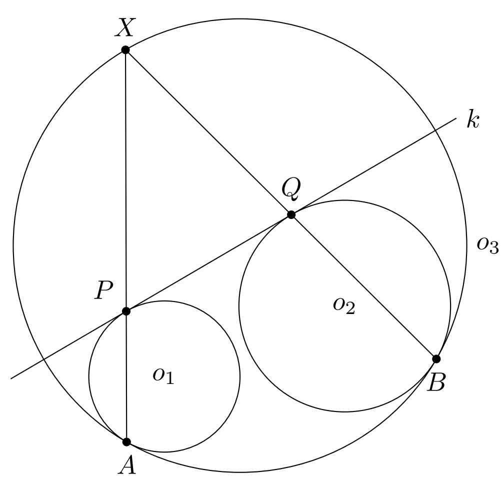
(a) If o_1 and o_2 are internally tangent at a point S (this means that they are both tangent to a line ℓ at a point S, and they both lie on the same side of the line), then S is the centre of the positive scale homothety taking o_1 to o_2.



(b) If o_1 and o_2 are both tangent to an angle with vertex S (the **vertex** of an angle formed by two half-lines r, s is the common end point shared by the two half-lines), then S is the centre of the positive scale homothety taking o_1 to o_2.

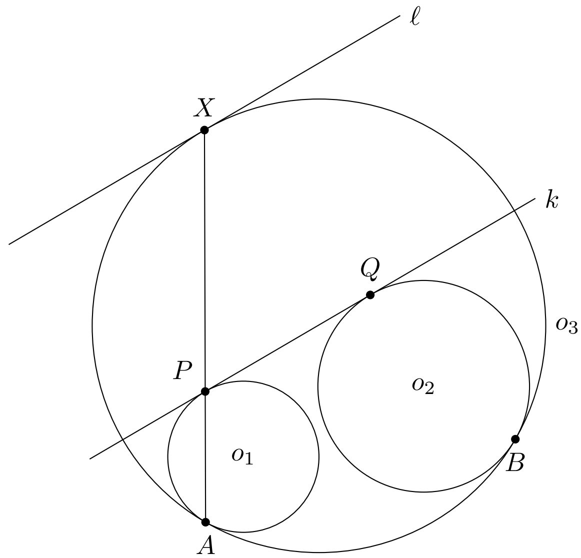


Problem 11.20. Let o_1 and o_2 be two circles internally tangent to a third circle o_3 at points A and B, respectively. Let k be a common tangent line to o_1 and o_2 in points P and Q which does not separate o_1 from o_2. Prove that AP and BQ intersect on o_3.



Solution. Consider the homothety h_1 with positive scale taking o_1 to o_3. As proven in Exercise 11.19 h_1, is centred at the point A. Let X be the image of P under h_1, and let ℓ

be the image of k under h_1. Since k is tangent to o_1 at P, ℓ is tangent to o_3 at X. Since the homothety is centred at A, we know A, P, X lie on a common line.



Now consider the homothety h_2 centred at B with positive scale taking o_2 to o_3, and let ℓ' be the image k under h_2. We claim that $\ell' = \ell$. Using the notions and results introduced in Exercise 11.17, we know that ℓ is tangent to o_3 from the same side as k is to o_1 and ℓ' is tangent to o_3 from the same side as k is to o_2. By our assumption k is tangent to o_1 from the same side as it is to o_2. Therefore ℓ is tangent to o_3 from the same side as ℓ' is to o_3, and so $\ell = \ell'$. As in the previous paragraph, it follows that h_2 sends Q to X, and so B, Q, X lie on a common line. Thus BQ and AP intersect at $X \in o_3$, and this completes the solution. ⧉

Theorem 11.21 (Classification of similarities)**.** Every similarity is (i) a translation, (ii) a glide reflection, or a (iii) composition of a homothety with centre O and a rotation around O or a reflection through a line through O.

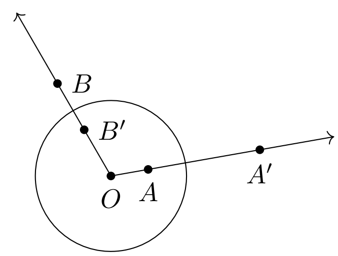
Exercise 11.22 (challenging)**.** Prove Theorem 11.21. To make the exercise a bit easier, you may assume the fact that every similarity with scale $0 < \lambda < 1$ has a fixed point.

12. Inversion in a circle

In this section, we add to the plane one point "at infinity," denoted ∞. By definition, every line contains ∞. The reason for this addition is that it allows us to define the following transformation of the plane:

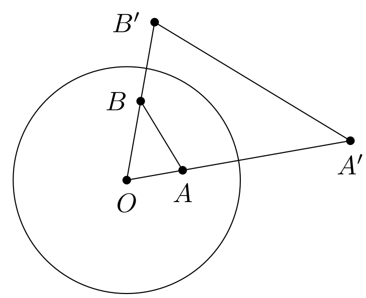
Definition 12.1. Let o be a circle with radius r and centre O. The **inversion in** o is the the transformation of the plane (with our added point!) which sends (a) O to ∞, (b) ∞ to

O and (c) any point $A \neq O, \infty$ to the point A' lying on the ray OA so that $|OA|\,|OA'| = r^2$.



Note that (i) If $A \in o$, then $A' = A$ (i.e. the inversion fixes every point in the circle), (ii) the inside of o is swapped with the outside of o, and (iii) the inversion satisfies $(A')' = A$.

Theorem 12.2. If A', B' are the images of A, B, respectively, under the inversion through a circle o with centre O, then OBA is similar to $OA'B'$.

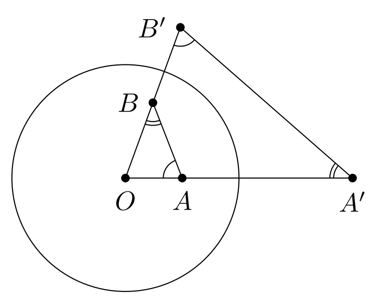


Proof. We will use the "side-angle-side" criterion to show that OBA and $OA'B'$ are similar. Since the two triangles share the angle at O, it suffices to show that $|OA'| \,/\, |OB| = |OB'| \,/\, |OA|$, but this follows immediately from the fact that

$$|OA|\,|OA'| = (\text{radius of } o)^2 = |OB'|\,|OB|\,,$$

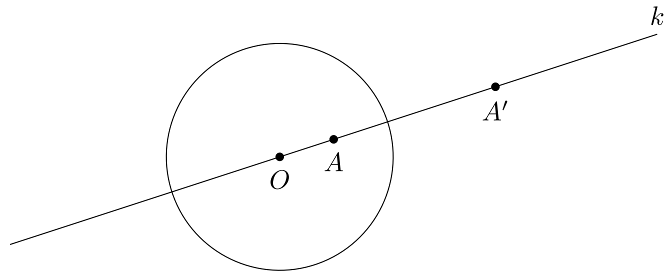
and this completes the proof. □

Corollary 12.3. In the setting of Theorem 12.2, we have equalities $\angle ABO = \angle OA'B'$ and $\angle OAB = \angle A'B'O$.

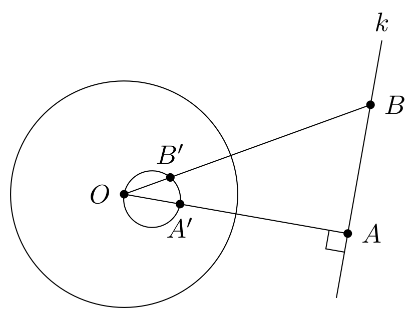


Theorem 12.4. Inversions map circles and lines to circles and lines. (i.e. it is possible for a circle/line to be mapped to either a line or a circle, and these are the only possibilities.)

Proof. Let o be the inversion circle. Consider first the case where a line k passes through the centre O. If $A \in k$, then the entire ray OA lies in k, and hence $A' \in k$. Thus k is fixed by the inversion.



Now we consider the complementary case when k does not contain O. Let A be the orthogonal projection of O onto k, and let A' be the image of A under the inversion. We claim that the image of k under the inversion is the circle with diameter OA'. To see this, fix any point $B \in k$, with $B \neq A, \infty$, and let B' be the image of B under the inversion through o. Since $\angle OAB = 90°$, Corollary 12.3 implies $\angle OB'A' = 90°$, and so, by Corollary 3.8, B lies on the circle with diameter OA'.

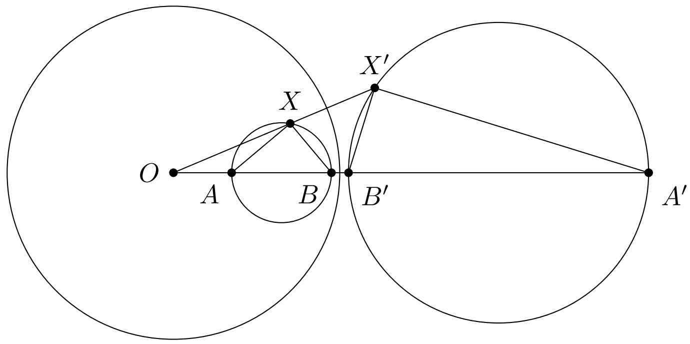


Finally, it is clear that O and ∞ are interchanged by the inversion, and so the entire line k is mapped into the circle with diameter OA'. We observe that the image of k is tangent to the line through O parallel to k.

Repeating the argument in reverse shows the the circle with diameter OA' is mapped into the line k, and hence the line k and the circle with diameter OA' are interchanged under the inversion.

The preceeding argument shows that a circle containing O is mapped to a line disjoint from O. The final case we consider is when we begin with a circle c disjoint from O. Let C be the centre of c, let A, B be the intersection points of the line OC with c, and let A', B' be their images under the inversion. We will prove that c' is the circle with diameter $A'B'$.

To see this, fix any point $X \neq A, B$ on c, and let X' be its image.



Now we apply Corollary 12.3 two times:

$$\angle AXO = \angle OA'X' \text{ and } \angle BXO = \angle OB'X'$$

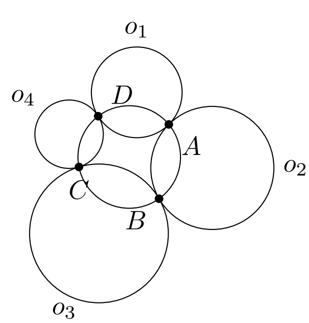
and using the fact that $\angle BXA = \angle BXO - \angle AXO$ and $\angle OB'X' = \angle A'X'B' + \angle OA'X'$ (by adding the angles in triangle $B'A'X'$), we conclude

$$\angle BXA = \angle OB'X' - \angle OA'X' = \angle A'X'B',$$

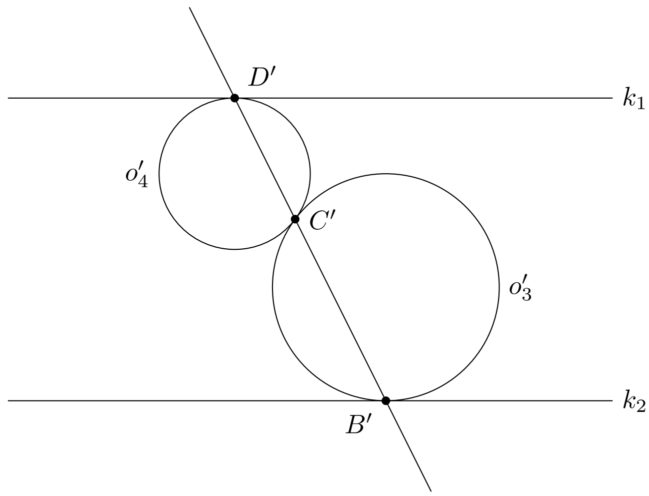
and since $\angle BXA = 90°$, we have $\angle A'X'B' = 90°$, and hence X' lies on the circle with diameter $A'B'$. Therefore the entire circle c is mapped to the circle with diameter $A'B'$. By symmetry, we know that the circle with diameter $A'B'$ is mapped to c, and hence the two circles are interchanged by the inversion. This completes the proof of the theorem. □

Exercise 12.5. Let o and c be circles with centres O and C, respectively, and suppose that $O \notin c$. Let C' and c' denote the images of C and c under the inversion through o (the proof of Theorem 12.4 establishes that c' is indeed a circle). Show that the center of c' is C' if and only if c and o are concentric.

Problem 12.6. Let o_1, o_2, o_3, o_4 be circles so that o_1, o_2 are externally tangent at A, o_2, o_3 are externally tangent at B, o_3, o_4 are externally tangent at C and o_4, o_1 are externally tangent at D. Prove that A, B, C, D lie on a common circle.



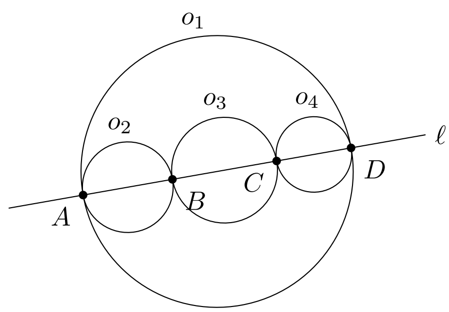
Solution. Without loss of generality we may assume that A, B, C, D are distinct, since we can always find a circle containing the vertices of a triangle. The idea for the solution is to apply an inversion in any circle centered at A. Since o_1 and o_2 are tangent at A, they get mapped to lines $k_1 = o_1'$ and $k_2 = o_2'$. Further, since $o_1 \cap o_2 = A$, $k_1 \cap k_2 = A' = \infty$, so k_1 and k_2 are parallel. Since o_3 is tangent to o_2 at B, o_2 cannot contain A, and similarly o_4 also cannot contain A. Therefore, following the proof of Theorem 12.4, we conclude that o_3 and o_4 are mapped to circles o_3' and o_4'.



The idea now is to show that B', C', D' lie on a common line, for then B', C', D', ∞ lie on a line ℓ, and when we reapply the inversion in the circle centred at A, we will deduce (by Theorem 12.4) that B, C, D, A lie on a common circle (or line, but then we will show that they cannot actually lie on a line).

To show that B', C', D' lie on a common line, we consider the homothety with negative scale with centre C' mapping o_4' to o_3'. Since homotheties maps tangent lines to tangent lines (see Exercise 11.17), we conclude that the homothety maps k_1 to k_2 and D' to B'. Therefore B', C', D' lie on a common line, and, as described in the preceeding paragraph, this shows A, B, C, D lie on a common circle or line.

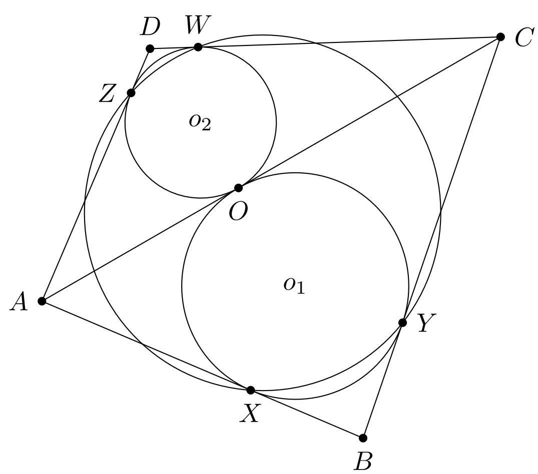
However, if A, B, C, D lie on a common line ℓ, then the points A, B, C, D can be ordered on ℓ; without loss, suppose that $A < B < C < D$ on ℓ.



Then since the diameter to the circle o_2 is AB and the diameter of o_1 is AD, and AB is entirely contained in AD, we conclude that o_1 contains o_2 in its interior, which means they are not *externally* tangent.

Therefore, A, B, C, D must lie on a common circle. □

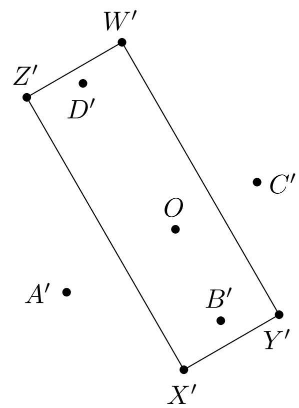
Problem 12.7. Let $ABCD$ be a convex quadrilateral, and assume that the inscribed circles of ABC and ACD are tangent at a common point $O \in AC$. Prove that the other tangency points of o_1 and o_2 lie on a common circle.



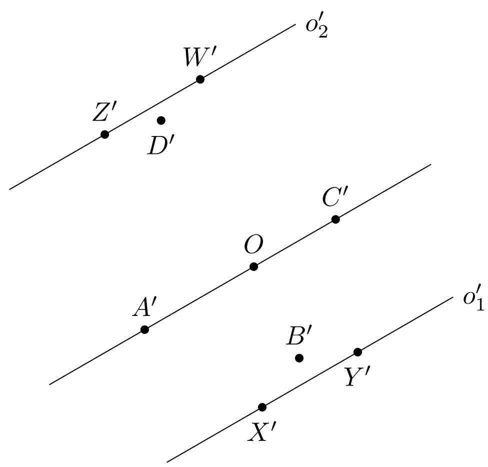
Solution. Following the figure above, let X, Y and Z, W denote the other tangency points of o_1 and o_2 with ABC and ACD, respectively.

The idea for the solution is to apply the inversion through any circle centered at O, and then show that the images X', Y', W', Z' (of X, Y, W, Z) form a rectangle, and hence lie on a common circle. Then when we apply the inversion again, we will deduce that X, Y, Z, W lie on a circle or a line. It is impossible for a single line to intersect every edge on a non-degenerate quadrilateral, and hence X, Y, Z, W cannot lie on a common line, and we will be able to conclude that X, Y, W, Z lie on a common circle.

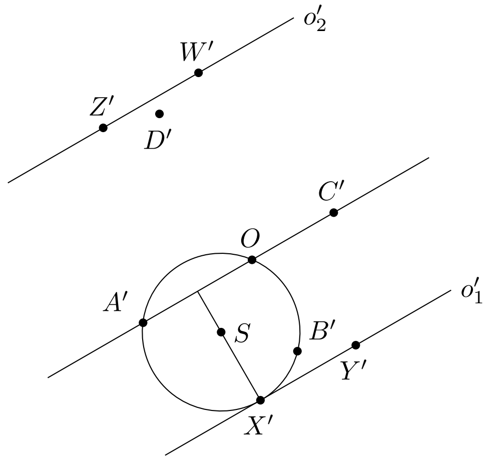
Therefore, our task now is to show that $X'Y'W'Z'$ is a rectangle.



The first observation we make is that, since o_1, o_2 and AC are tangent at O, o_1', o_2' and $(AC)' = AC$ become parallel lines (informally, thinking of lines as "circles" containing the point ∞, o_1', o_2' and $(AC)'$ are "circles" tangent at $O' = \infty$, hence they are parallel lines). Further, X', Y' lie on o_1' and Z', W' lie on o_2', and A', C' lie on AC.



The second observation is that the perpendicular to o_1' raised from X' bisects the segment $A'O$. To establish this claim, we observe that the line AB becomes a circle $(AB)'$ when we apply the inversion, and that $(AB)'$ is tangent to the line o_1' at X' and contains A' and O. If S denotes the center of $(AB)'$, then the perpendicular bisector of the chord $A'O$ intersects S (why?). Similarly the perpendicular to o_1' raised from X' also intersects S (since $(AB)'$ is tangent to o_1' at X'). Therefore the perpendicular to o_1' raised from X' must be the perpendicular bisector of $A'O$ (since they are both perpendicular to o'_1 and intersect at S).

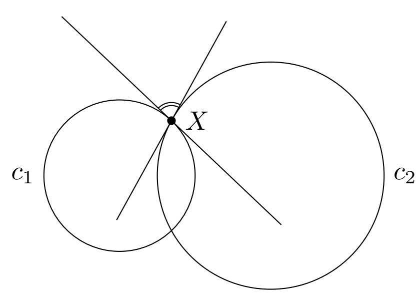


A similar argument shows that the perpendicular to o'_2 raised from Z' bisects $A'O$, and hence $Z'X'$ is perpendicular to o'_1 and o'_2. Similarly, $Y'W'$ is perpendicular to o'_1 and o'_2. Therefore $Z'X' \parallel Y'W'$, and so $X'Y'W'Z'$ forms a rectangle, since it is a parallelogram with perpendicular edges.

Following the argument given in the second paragraph of this solution, we conclude that X, Y, W, Z lie on a circle. □

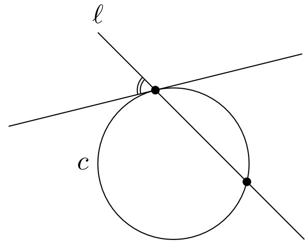
12.1. **More properties of the inversion.** So far, all of our application of the inversion in a circle relied on the simple fact that it mapped circles/lines to circles/lines, and preserved "incidence," i.e. if we could show that the images of some points lie on a circle/line, then we could deduce that the original points also lie on a circle/line. However, the inversion in a circle also preserves the "angles" between circles/lines! In order to make this precise, we need to define what the angle between two circles/lines is.

Definition 12.8. Let c_1 and c_2 be two circles intersecting at a point X. The **angle between c_1 and c_2 at X** is the angle between their respective tangent lines at X.



Note that the ambiguity in the choice of angles between tangent lines forces us to consider α and $180° - \alpha$ as equivalent. If we wish to differentiate the angles α and $180° - \alpha$, we could endow our circles with an "orientation," but this is not necessary for what follows.

Exercise 12.9. Give a similar definition for the angle between a circle c and a line ℓ intersecting at a point Y.



Exercise 12.10. Suppose a circle/line a intersects another circle c in two points X, Y. Prove that the angle between a and c at X equals the angle between a and c at Y.

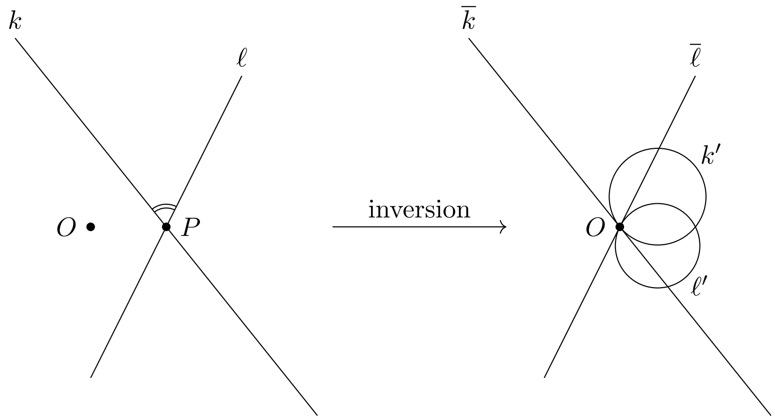
Exercise 12.11. Two lines ℓ_1 and ℓ_2 always intersect at ∞. Define the angle between ℓ_1 and ℓ_2 at ∞ in such a way so that (i) the angle between parallel lines is 0 and (ii) Exercise 12.10 generalizes to the case when a line ℓ_1 intersects a line ℓ_2 in two points.

Theorem 12.12. The inversion in any circle preserves the angles between circles/lines, i.e. if a, b are intersecting circles/lines, and a', b' denote their images under some inversion, then a' and b' intersect and the angle between a' and b' equals the angle between a and b.

Proof. In the following proof we fix an inversion, and we denote the image of any figure F by the primed symbol F'.

Our first observation is that it suffices to check that the angle between ℓ and k equals the angle between their images ℓ' and k', whenever ℓ, k are lines. To see this, we claim that if c is a circle and t is its tangent line at X, then c' and t' are tangent at X'; this is true because c' and t' being tangent at X' is equivalent to them intersecting only at X', which follows from c, t being tangent at X. Therefore, if c_1, c_2 intersect at X with respective tangent lines t_1 and t_2 at X, then the angle between t_1' and t_2' at X' equals the angle between c_1' and c_2' at X'. Thus, without loss of generality, we will only prove that the angle between ℓ' and k' equals the angle between ℓ and k whenever ℓ and k are lines.

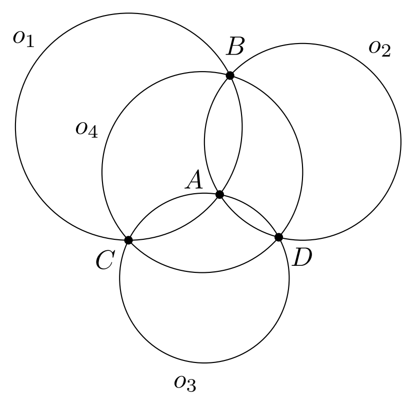
Let O denote the centre of the inversion circle, and suppose ℓ, k are two lines intersecting at a point P, with an angle of α between them. First suppose that neither ℓ, k contain O. Then ℓ' and k' are circles intersecting at P' and O; we will calculate the angle between them at O. To do this, let $\overline{\ell}, \overline{k}$ denote the tangent lines to ℓ', k' at O. The discussion following the definition of the inversion established that $\overline{\ell} \parallel \ell$ and $\overline{k} \parallel k$ (Definition 12.1). Hence the angle between $\overline{k}$ and $\overline{\ell}$ equals the angle between k and ℓ.



The other cases to consider are when k or ℓ contains O. Both of these cases are easier than the one we proved, and so we leave them as exercises for the reader to complete. This completes the proof of the theorem. □

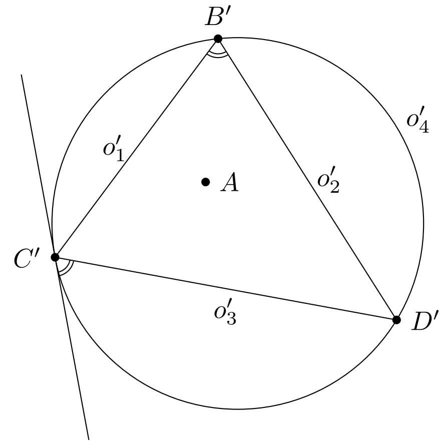
Remark. One can define the angle between arbitrary differentiable curves intersecting at a point X, and one can show that the inversion is a differentiable map which preserves the angle between any two differentiable curves.

Problem 12.13. Consider four circles o_1, o_2, o_3, o_4, each three of which intersect in a common point $A \neq B \neq C \neq D$, as in the figure below. Prove that the angle between o_1 and o_2 equals the angle between o_3 and o_4.



Solution. The idea is to consider an inversion through any circle centered at A. Since the circles o_1, o_2 and o_3 contain A, their images o_1', o_2' and o_3' are straight lines. Then, the points B', C', D' form a triangle whose edges lie on the lines o_1', o_2' and o_3', and, in particular, the angle at vertex B' equals the angle between o_1' and o_2'. Further, since o_4 is a circle which does not contain A, o_4' is still a circle, and since it contains B', C' and D', o_4' is the circumscribed circle of $B'C'D'$.

We claim that $\angle D'B'C'$ equals the angle between the line $C'D'$ and the tangent line to o'_4 at C'.



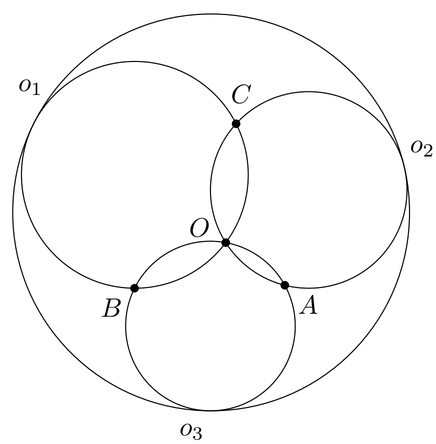
This is actually proved in a previous problem (Problem 3.12), and so we won't repeat its proof. We conclude that the angle between o'_3 and o'_4 equals $\angle D'B'C'$ (which is the angle between o'_1 and o'_2).

The fact that inversions preserve angles implies that $\angle D'B'C'$ = angle between o_1 and o_2, and

$$\text{angle between } o'_3 \text{ and } o'_4 = \text{angle between } o_3 \text{ and } o_4,$$

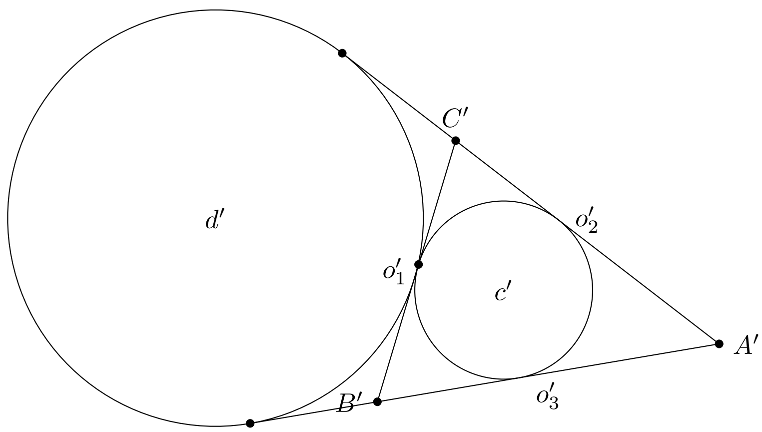
and so we deduce that the angle between o_3 and o_4 equals the angle between o_1 and o_2, as desired. □

Problem 12.14. Let o_1, o_2 and o_3 be circles intersecting in a common point, and suppose they are pairwise non-tangent. Find a circle or line tangent to o_1, o_2 and o_3.



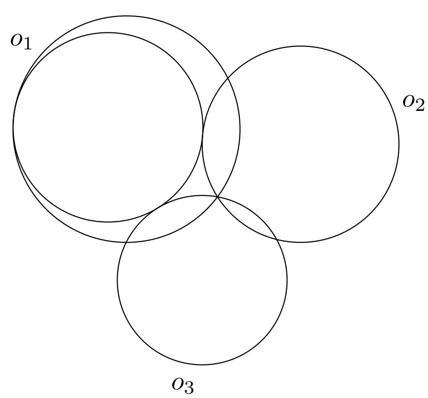
Solution. Let O be the single point of intersection. Since o_1, o_2, o_3 are pairwise non-tangent, each pair intersects twice; label the other intersection points $A \neq B \neq C \neq O$ as in the figure above.

As is customary with the problems in this section, the key step in the solution is to take the inversion in some circle - in this case, we will consider the inversion in any circle centered at O. If $A', B', C', o_1', o_2', o_3'$ denote the images of A, B, C, o_1, o_2, o_3 under this inversion, then o_1', o_2', o_3' are lines and $A'B'C'$ is a triangle whose edges lie on o_1', o_2', o_3'. Since the inscribed circle c' of $A'B'C'$ is tangent to o_1', o_2', o_3', its image c'' is a circle or line tangent to o_1, o_2, o_3. If c'' is a circle, then we are done. If c'' is a line, then O lies on c', and hence O is contained in the interior of $A'B'C'$. Therefore O cannot lie on any escribed circle of $A'B'C'$, and so, if we take d' an escribed circle of $A'B'C'$, then its image d'' is a circle which is tangent to o_1, o_2, o_3; either way we have completed the solution.

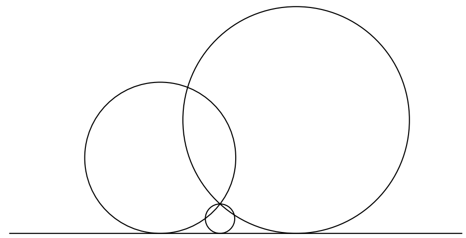


□

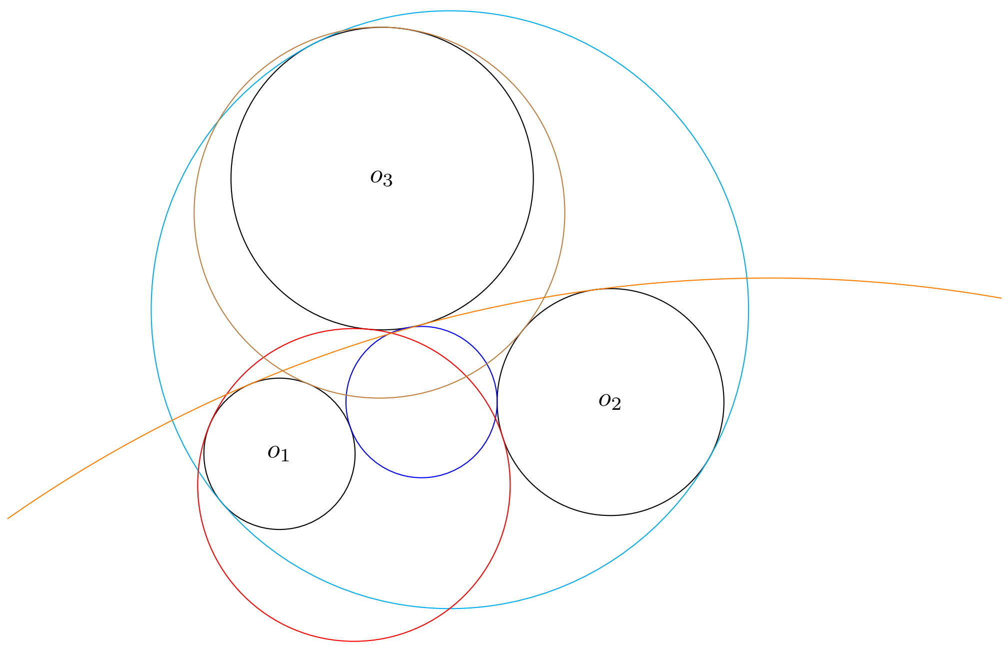
Remark. (1) In the figure given in the preceeding problem statement, we chose the inscribed circles of $A'B'C'$. If we chose an escribed circle, the figure would look like this:



(2) Following the notation of the preceeding solution, if c' is the inscribed circle of $A'B'C'$, it is possible for c'' to be line, as the following figure shows:

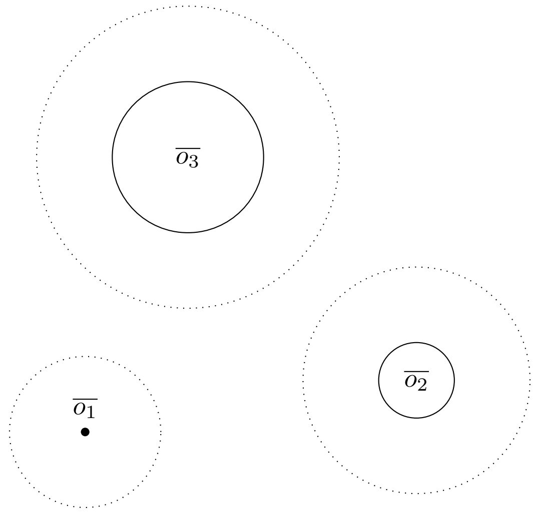


Problem 12.15 (Apollonius)**.** Given any three circles o_1, o_2, o_3 lying outside each other, find a common tangent circle or line. (Note: in the accompanying figure we have drawn multiple solutions to the problem.)

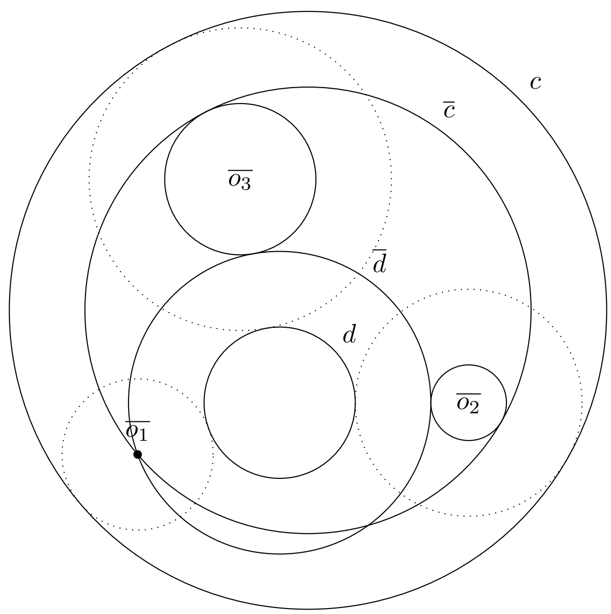


Solution. Let o_1 be the smallest circle, and let r be its radius. Shrink o_2 and o_3 by radius r and turn o_1 to a point - let $\overline{o_1}, \overline{o_2}, \overline{o_3}$ denote these shrunken circles. Note that $\overline{o_2}$ and $\overline{o_3}$

may also be single points.

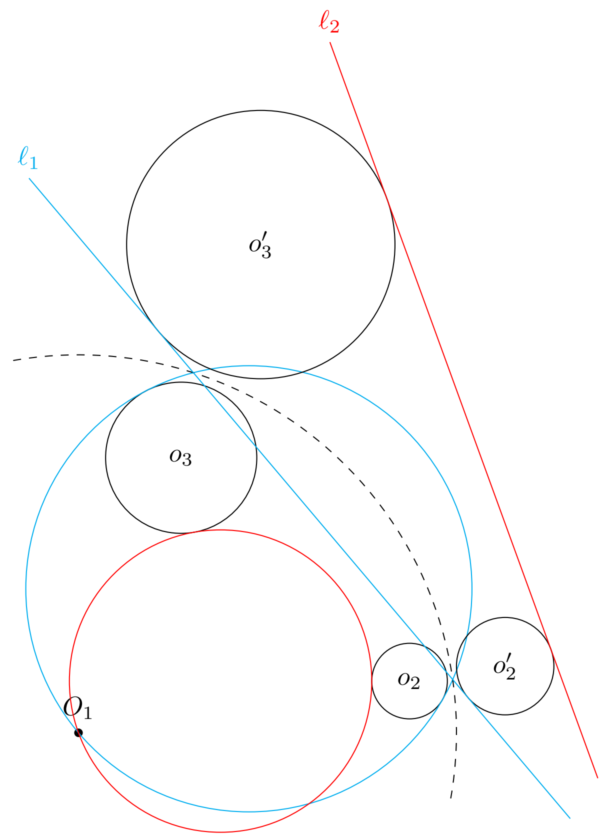


We consider three cases: (i) If we can find a circle $\bar{c}$ internally tangent to $\overline{o_2}$ and $\overline{o_3}$ and containing $\overline{o_1}$,then expanding $\bar{c}$ by radius r produces a circle c tangent to o_1, o_2 and o_3. (ii) If, instead we find a circle $\bar{d}$ so that $\bar{d}$ contains $\overline{o_1}$ but externally tangent to $\overline{o_2}$ and $\overline{o_3}$, then shrinking $\bar{d}$ by radius r produces a circle d tangent to o_1, o_2, o_3 (why is the radius of $\bar{d}$ at least r?).



Finally, (iii) if instead we find a line ℓ containing $\overline{o_1}$ and tangent to both $\overline{o_2}, \overline{o_3}$ from the same side, then shifting ℓ in a perpendicular direction by distance r produces a line tangent to o_1, o_2, o_3.

Therefore, we have reduced the problem to the case when one of the circles, o_1, is actually just a single point O_1. Now the key is to consider an inversion in some circle centered at O_1 - in the figure below, an arc of the inversion circle is shown as a dashed curve. Let the images of o_2 and o_3 be circles o_2' and o_3'. Then, finding a tangent circle/line to o_2, o_3 containing O_1 is equivalent to finding a tangent line to o_2' and o_3'. Let ℓ be a tangent line to o_2', o_3', tangent to them from the same side (as in the figure below). We consider three cases: (1) If ℓ separates O_1 from o_2' and o_3', then the circle ℓ' is a circle through O_1 which contains both o_2, o_3 in its interior. (2) If ℓ contains O_1, o_2' and o_3' all on one side, then ℓ' is a circle through O_1 which doesn't contain o_2 or o_3. (3) If ℓ contains O_1, then ℓ' is a line tangent to o_2 and o_3 passing through O_1. Since cases (1), (2) and (3) exhaust all possibilities, and in each case we have constructed a solution to the problem (corresponding to the cases (i), (ii), (iii) given above), we have completed the solution of the problem. Note that in the figure below, we have drawn two choices of ℓ, one which satisfies case (1) and one which satisfies case (2).



□

Exercise 12.16. In the statement of Problem 12.15, we have drawn a figure presenting five different solutions of the problem. However, in our solution, we present only two solutions (corresponding to the two common tangent lines to o_2' and o_3' which are tangent from the same side). Which circles in the figure are not described by our solution? Explain how you would modify the solution so that it could describe all of the circles presented in the Problem statement's figure.