Math 599: Nonpositive curvature
Requirements for the exam on December 13th 2-5pm

1. The definition of a metric space, an isometry, a geodesic line, ray, and segment. The definition of a metric on $\mathbb{S}^n$, with the proof of the triangle inequality, the classification of geodesic segments, and the law of cosines. The definition of $\mathbb{H}^n$ in the hyperboloid model with $d(x, y) = \operatorname{arccosh}\big(-(x, y)\big)$ and the proof that it is well defined, i.e. that $(x, y) \leq -1$. The definition of the orthogonal complement and the proof that $(\cdot, \cdot)$ is positive definite on $x^{\perp}$ for $x \in \mathbb{H}^n$. Definitions of hyperbolic segments, angles, the law of cosines, and the proof of the triangle inequality.

2. The definition of a hyperplane and a reflection in $\mathbb{S}^n$ and $\mathbb{H}^n$. The proof of the proposition that any k-tuple of points can be mapped by an isometry to any other k-tuple with the same distances. The Klein model of $\mathbb{H}^n$. The complete proof of the formula for the distance with the cross-ratio. The proof of the proposition describing orthogonality in the Klein model.

3. The definition of the complex hyperbolic space $\mathbb{CH}^n$ with its metric. The proof of the reverse Schwartz inequality. Examples of triangles isometric to triangles in $\mathbb{H}^2$ and in $(\mathbb{H}^2, \frac{d}{2})$. The definition of model spaces. The proof of the existence of comparison triangles. The definition of a CAT(κ) space. The definition of the Alexandrov angle. The proof of the Alexandrov Lemma (see recitation).

4. The definition of a Riemannian metric and its associated metric. Riemanian metrics on $\mathbb{S}^n$ and $\mathbb{H}^n$. The proof that for $\mathbb{H}^2$ this agrees with d from the first class, using the statement of the lemma about the form of the Riemannian metric under the exp map (without its proof). The proof that CAT(κ) implies CAT(κ') for $\kappa < \kappa' < 0$. The proof that $\mathbb{CH}^2$ is CAT(-1). The definition of the symmetric space $P(n, \mathbb{R})_1$ with its metric, and the isometric action of $\mathbf{SL}(n, \mathbb{R})$.

5. The definition of a κ-cone, and the statement of Berestovskii's theorem. The definition of a geodesic simplex, an M_κ–simplicial complex, and its intrinsic pseudo-metric. The definition of the function ϵ, and the proof of the criterion that $\epsilon > 0$ implies that the pseudo-metric is a metric. The statement of Bridson's theorem that a simplicial complex

with finitely many isometry types of simplices is a complete geodesic metric space, with the proof of completeness.

6. The definition of a space with curvature $\leq \kappa$, a convex space, and a locally convex space. The complete proof of the Cardan–Hadamard Theorem.

7. The proof of the existence of the centre of a bounded set. The proof of the corollary about conjugacy classes of finite subgroups. Definitions of proper and cocompact actions. Definitions of displacement, translation length, and $\mathrm{Min}(g)$. Examples in $\mathbb{H}^2$. The proof of the Axis Theorem.

8. The proof of the theorem that $\mathrm{Min}(g) = Y \times \mathbb{R}$ (without analysing the action of an f commuting with g), including the proof of the flat triangle lemma, but not the quadrangle lemma. The statement of the Flat Torus Theorem.

9. The definition of a Coxeter matrix, its cosine matrix, and the associated Coxeter group. The definition of the Tits representation and the proof that the image of each $W_{s,t}$ is the dihedral group $D_{m_{st}}$. The definition of the Coxeter complex and the Cayley graph of a Coxeter group.

10. The definition of a simplex of groups. The definition of developability (as coming from an action of G on K, where K is a complex glued out of simplices), the equivalence with the algebraic definition. The definition of the fundamental group of a simplex of groups. The definition of the local development at a vertex and the statement of the theorem of Bridson and Haefliger. The sketch of Poincaré's proof for Coxeter groups with positive definite $(\cdot, \cdot)$.

11. The definition of the Davis complex, and the piecewise Euclidean structure that makes it CAT(0) for $|W| = \infty$. The definition of a chamber complex and a building. Examples. The definition of an apartment and the statement of the proposition about each 2 chambers lying in a single apartment. The definition of a CAT(0) metric on the appropriate subcomplex of a building for $|W| = \infty$.

12. The definition of the boundary of a CAT(0) space. The cone topology. The proof of the proposition that it does not depend on the basepoint. The definition of the angle metric, and the proof that it is a metric.