Metric nonpositive curvature
Midterm II, 21 Nov 2023, 8:35am-9:55pm

For each of the midterm problems you can get a maximum of 10 points. Your final score from midterms will be equal to the minimum of 40 and the sum of the scores from both midterms.

You are not allowed to use your notes or any other sources.

Problem 1. Let K be the presentation complex of the group

$$\langle a, b, c \mid abca^{-1}b^{-1}c^{-1} = 1\rangle.$$

Find a piecewise Euclidean structure on K of nonpositive curvature, and, if possible, a piecewise hyperbolic structure of curvature ≤ -1.

Solution 1. The complex K consists of 3 loops and a hexagon. The link of the unique vertex of K is a disjoint union of two circles, each divided into 3 edges. We equip the hexagon with the metric of a Euclidean regular hexagon. Then each of the circles in the link has length $3 \cdot \frac{2\pi}{3} = 2\pi$. Thus the link is CAT(1) and so by the link condition K has nonpositive curvature. It is not possible to find a piecewise hyperbolic structure of curvature ≤ -1, since any hyperbolic hexagon has sum of angles $< 4\pi$, and consequently one of the circles in the link would have length $< 2\pi$.

Problem 2. Let X be a complete CAT(0) space, and let γ be an isometry of X. Assume that for some $m > 0$ the isometry γ^m is hyperbolic. Prove that γ is hyperbolic.

Solution 2. The set $\mathrm{Min}(\gamma^m)$ has the form $Y \times \mathbb{R}$. Since γ commutes with γ^m, we have that γ preserves $\mathrm{Min}(\gamma^m)$, as well as its product structure, and has the form $(\gamma_Y, \gamma_{\mathbb{R}})$, where γ_Y is an isometry of Y and $\gamma_{\mathbb{R}}$ is a translation. Since X is complete and Y is closed, we have that Y is complete. Furthermore, because $Y \subset X$ is convex, it is CAT(0). Since γ^m acts trivially on Y, we have that γ_Y has finite order. Thus, by the fixed-point theorem, γ_Y fixes a point $y \in Y$. Then $\{y\} \times \mathbb{R}$ is an axis for γ.

Problem 3. Prove that the group

$$G = \langle a, b, t, s \mid ab = ba, tat^{-1} = ab, sas^{-1} = ab^2\rangle$$

does not act properly and cocompactly on a complete CAT(0) space. You can use the fact that the subgroup of G generated by a and b is isomorphic to $\mathbb{Z}^2$.

Solution 3. This example is due to Gersten. If G acts properly and cocompactly on a complete CAT(0) space X, then all its elements act semi-simply. By the Flat Torus Theorem, we have that $\langle a, b\rangle = \mathbb{Z}^2$ acts properly by translations on an isometrically embedded Euclidean plane $\mathbb{R}^2 \subset X$. For each $g \in \mathbb{Z}^2$, let $v_g \in \mathbb{R}^2$ denote the vector by which g translates. The translation lengths of a, ab, and ab^2 in the action on X coincide with the ones on $\mathbb{R}^2$. They are all equal since a, ab, and ab^2 are conjugate. But this means that the vectors v_a, v_{ab}, v_{ab^2} have equal length. On the other hand, $v_{ab} - v_a = v_b = v_{ab^2} - v_{ab}$. Thus, considered as points, $v_a, v_{ab}, v_{ab^2} \in \mathbb{R}^2$ are colinear. But we cannot have three colinear points in $\mathbb{R}^2$ at the same distance from 0 (i.e. on a common circle centred at 0), contradiction.

Problem 4. Describe the Coxeter complex of the Coxeter group W with generators r, s, t, and $m_{rs} = 2, m_{st} = 3, m_{rt} = 6$. Classify the maximal finite subgroups of W up to conjugation.

Solution 4. Equip the triangle Δ in the definition of the Coxeter complex $X = W \times \Delta/\sim$ with metric of a Euclidean triangle $T \subset \mathbb{R}^2$ of angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$. Let $f\colon W \to \mathrm{Isom}(\mathbb{R}^2)$ be the homomorphism sending r, s, t to the reflections in the sides of T. Consider the map $X \to \mathbb{R}^2$ sending $w \times T$ to the translate $f(w)T$. Then f is a local isometry from the complete space X to a locally convex space $\mathbb{R}^2$. Thus f is a covering map, and since $\mathbb{R}^2$ is simply connected, f is an isometry. We can thus identify X with $\mathbb{R}^2$. By the fixed-point theorem, any finite subgroup $G < W$ fixes a point of $X = \mathbb{R}^2$. After a conjugation, G fixes a point of T. Then G is contained in the stabiliser of one of the three vertices of T. Thus the maximal finite subgroups, up to conjugation, are $\langle r, s\rangle = D_2, \langle s, t\rangle = D_3, \langle r, t\rangle = D_6$.