MATH 599 Nonpositive Curvature
Problem list 1

Problem 1. Consider $\mathbb{R}^n$ with one of the following metrics.

(i) $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = |x_1 - y_1| + \ldots + |x_n - y_n|$

(ii) $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$

Prove that these metric spaces are geodesic. Given x, y, are geodesics joining x, y unique?

Definition. A sequence (x_n) of points in a metric space is called *Cauchy* if for every ε there is n_0 such that for all $m, n > n_0$ we have $d(x_n, x_m) < \varepsilon$. A metric space is *complete* if every Cauchy sequence has a *limit*, that is, a point x such that for every ε there is n_0 such that for all $n > n_0$, we have $d(x_n, x) < \varepsilon$.

Problem 2. Let X be a complete metric space with the property that for each pair of points $x, y \in X$ there is a point m with $d(x, m) = d(y, m) = \frac{1}{2}d(x, y)$. Prove that X is a geodesic metric space.

Definition. A subset C of a metric space X is *convex* if every pair of points $x, y \in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C. The *open ball* $B(x, r)$ *of radius* r *centered at* x is the subset of X of all points at distance $< r$ from x.

Problem 3. Prove that open balls in $\mathbb{S}^n$ of radius $\leq \frac{\pi}{2}$ are convex. Hint: points on the geodesic segment joining non-antipodal $x, y \in \mathbb{S}^n$ can be written as $sx + ty$ where $s + t \geq 1$ and $s, t \geq 0$.

Problem 4. In $\mathbb{S}^2$ consider a spherical triangle formed of geodesic segments with angles A, B, C and distances between vertices a, b, c. Prove the *dual spherical law of cosines*:

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Hint: consider the *dual triangle* whose vertices are unit vectors

- orthogonal to the planes containing the sides of the original triangle, and
- lying in the halfspaces containing that triangle.

Problem 5. Prove that a hyperplane in $\mathbb{H}^n$ (with distance inherited from $\mathbb{H}^n$) is isometric to $\mathbb{H}^{n-1}$.

Problem 6. Let φ be an isometry of $\mathbb{H}^n$. Prove that

(i) if φ is not the identity, then the set of points which it fixes is contained in a hyperplane,

(ii) if φ acts as the identity on a hyperplane H, then φ is either the identity or the reflection through H,

(iii) φ can be written as the composition of $n + 1$ or fewer reflections through hyperplanes.