MATH 599 Nonpositive Curvature
Problem list 2

Problem 1. Construct a common orthogonal geodesic line to two non-intersecting godesic lines in the Klein model of $\mathbb{H}^2$.

Problem 2. Construct a right-angled pentagon with geodesic sides in the Klein model of $\mathbb{H}^2$.

Problem 3. Let $p, x, y, z, w \in \mathbb{R}^2$ so that

(i) either they all lie on a circle, or

(ii) the points x, y, z, w are colinear.

Show that the cross ratio

$$(x, y, z, w) = \frac{|x-w|}{|x-z|}\frac{|y-z|}{|y-w|}$$

can be expressed as

$$\frac{\sin|\measuredangle xpw|}{\sin|\measuredangle xpz|}\frac{\sin|\measuredangle ypz|}{\sin|\measuredangle ypw|}.$$

Definition. In $\mathbb{R}^n$, an *inversion* in the sphere with radius r centered at a point p is the map $\mathbb{R}^n \cup \infty \to \mathbb{R}^n \cup \infty$ mapping each point q to the point q' lying on the geodesic ray from p containing q and satisfying $|pq||pq'| = r^2$. In particular, points of the sphere are fixed and its inside is exchanged with its outside (and p is exchanged with ∞).

Problem 4. Prove that an inversion of $\mathbb{R}^2$

(i) maps any circle or a line to a circle or a line,

(ii) preserves angles between lines and circles,

(iii) preserves the cross-ratio of points on lines and circles.

Problem 5. In $\mathbb{R}^2$ consider the semicircle

$$S = \{(x, y) \text{ with } x^2 + y^2 = 1 \text{ and } y \geq 0\}$$

and the interval

$$I = \{(x, y) \text{ with } -1 \leq x \leq 1 \text{ and } y = 0\},$$

and denote their common endpoints by $z_3 = (-1, 0)$ and $z_4 = (1, 0)$. Consider the orthogonal projection $\pi\colon S \to I$ defined by $\pi((x, y)) = (x, 0)$. Prove that for any $z_1, z_2 \in S$ we have

$$(\pi(z_1), \pi(z_2), z_3, z_4) = (z_1, z_2, z_3, z_4)^2.$$

Definition. Let

$$D = \{x \in \mathbb{R}^{n+1} \text{ with } x_{n+1} = 1 \text{ and } x_1^2 + \ldots + x_n^2 < 1\}$$

be the Klein model for hyperbolic space. Consider the upper-hemisphere

$$S = \{x \in \mathbb{R}^{n+1} \text{ with } x_{n+1} > 0 \text{ and } |x|^2 = 1\}$$

and the orthogonal projection $\pi\colon S \to D$ defined by $\pi(x) = (x_1, \ldots, x_n, 1)$. This correspondence gives the *hemisphere model* S of $\mathbb{H}^n$. Let

$$B = \{x \in \mathbb{R}^{n+1} \text{ with } x_{n+1} = 0 \text{ and } x_1^2 + \ldots + x_n^2 < 1\}.$$

Consider the inversion i of $\mathbb{R}^{n+1}$ in the sphere of radius $\sqrt{2}$ centred at $(0, \ldots, 0, -1)$, which maps S to B. This correspondence gives the *Poincaré disc model* B of $\mathbb{H}^n$.

Problem 6. (i) Prove that both in S and in B geodesic lines are contained in circles orthogonal to the boundary of the model.

(ii) Given x, y in one of these models and the geodesic line passing through x, y with endpoints x_∞, y_∞, prove $d(x, y) = \log(x, y, x_\infty, y_\infty)$.

(iii) Prove that reflections through hyperplanes in the hyperboloid model correspond to inversions in the circles orthogonal to the boundary of B.

Problem 7. Prove that the composition of the maps from the hyperboloid model to the Klein model, then to the hemisphere model, and then to the Poincaré disc model is simply obtained as the central projection with center $(0, \ldots, 0, -1)$ from the hyperboloid to B.