MATH 599 Nonpositive Curvature
Problem list 3

Problem 1. Let X be a CAT(κ) space. Prove that there is a unique geodesic segment joining each pair of points $x, y \in X$ (provided $d(x,y) < \mathrm{diam} M_\kappa$ if $\kappa > 0$), and this geodesic segment varies continuously with its endpoints.

Problem 2. Let X be a finite graph (with possibly varying length of edges). When is X

(i) CAT(0),

(ii) CAT(-1),

(iii) CAT(1)?

Definition. A map $c\colon [0,a] \to X$ into a metric space X is *local geodesic* if every $p \in [0,a]$ has a neighbourhood U such that the restriction of c to U is a geodesic.

Problem 3. Let X be a CAT(κ) space. Prove that every local geodesic in X of length at most $\mathrm{diam} M_\kappa$ is a geodesic.

Definition. Let x, y, z be points in a metric space X with $y, z \neq x$. The *comparison angle* between y and z at x is the angle at $\bar{x}$ of the comparison triangle $\bar{x}\bar{y}\bar{z}$ in $\mathbb{R}^2$, and is denoted by $\overline{\angle}_x(y,z)$.

Let $c\colon [0,a] \to X, c'\colon [0,a'] \to X$ be two geodesics starting at $c(0) = c'(0) = x$. The *Alexandrov angle* between c and c' at x is

$$\angle(c,c') = \lim_{\varepsilon \to 0} \sup_{0<t,t'<\varepsilon} \overline{\angle}_x(c(t), c'(t')).$$

Problem 4. Suppose that the concatenation of c^{-1} and c' in the definition above is a geodesic. Prove $\angle(c,c') = \pi$.

Problem 5. Let c, c', c'' be three geodesics with $c(0) = c'(0) = c''(0)$. Prove $\angle(c,c') + \angle(c',c'') \geq \angle(c,c'')$.

Problem 6. Prove that in $\mathbb{H}^2$ and $\mathbb{CH}^2$ the hyperbolic angle is equal to the Alexandrov angle.

Problem 7 (Alexandrov's Lemma)**.** Consider four distinct points, $x, y, y', z \in M_\kappa$ for $\kappa \leq 0$. Suppose that y, y' lie on the opposite sides of the geodesic line through xz. Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the angles in the geodesic triangles $xyz, xy'z$. Assume $\gamma + \gamma' \geq \pi$. Consider a triangle $\bar{x}\bar{y}\bar{y'}$ in M_κ with distances between vertices $d(x, y), d(x, y'), d(z, y) + d(z, y')$ (show that they satisfy the triangle inequality). Denote its angles by $\bar{\alpha}, \bar{\beta}, \bar{\beta}'$. Let $\bar{z}$ be a point on $\bar{y}\bar{y'}$ at distance $d(z, y)$ from $\bar{y}$. Prove $\bar{\alpha} \geq \alpha + \alpha'$, $\bar{\beta} \geq \beta$, $\bar{\beta}' \geq \beta$ and $d(x, z) \leq d(\bar{x}, \bar{z})$.

Problem 8. Prove that X is CAT(0) if and only if for every geodesic triangle xyz and a point p on xy, its comparison point $\bar{p}$ in the comparison triangle $\bar{x}\bar{y}\bar{z}$ satisfies $d(p, z) \leq d(\bar{p}, \bar{z})$. Hint: Use monotonicity in the law of cosines.

Problem 9. Prove that X is CAT(0) if and only the Alexandrov angle between two sides of each geodesic triangle is no greater than the angle between the corresponding sides of the comparison triangle in $\mathbb{R}^2$. Hint: use Problems 7 and 8.