MATH 599 Nonpositive Curvature
Problem list 5

Definition. The *link* $\mathrm{Lk}(x,S)$ of a point x in a geodesic n–simplex $S \subset M^2_\kappa$ (or in a convex n–polyhedron) is the spherical $(n-1)$–cell consisting of unit tangent vectors at x *pointing towards* S, that is, initial vectors of geodesic segments contained in S.

The *link* $\mathrm{Lk}(x,K)$ of a point x in a simplicial (or polyhedral) complex K is the piecewise spherical complex whose cells are $\mathrm{Lk}(x,S)$, with $y \sim y'$ for $y \in \mathrm{Lk}(x,S), y' \in \mathrm{Lk}(x,S')$ whenever S and S' share a face in K and y corresponds to y' under this identification.

Problem 1. (technical) Let x be a point in an M_κ-simplicial (or polyhedral) complex K with $\varepsilon(x) > 0$. Prove that $B(x, \frac{\varepsilon}{2})$ is isometric with the open ball of radius $\frac{\varepsilon}{2}$ around the cone point in $C_\kappa(\mathrm{Lk}(x,K))$.

Hints: for a pair of points $y = (t,u), y' = (t',u') \in C_\kappa(\mathrm{Lk}(x,K))$ split the proof into two steps:

(i) If $d(y,y') < t + t'$, then $d(u,u') < \pi$ and $d_{cone}(y,y') \leq d(y,y')$.

(ii) If $d(u,u') < \pi$, then $d(y,y') \leq d_{cone}(y,y') < t + t'$.

In (i), given a string (y_i) between y and y', construct a fan of comparison triangles for xy_iy_{i+1} to obtain a bound on $d(u,u')$ and consequently on $d_{cone}(y,y')$.

Problem 2. Let K be an M_κ–simplicial complex (or in general a polyhedral complex) with finitely many isometry types of cells. Prove that K has curvature $\leq \kappa$ if and only if for each vertex $v \in K$ the complex $\mathrm{Lk}(v,K)$ is CAT(1).

Definition. The *presentation complex* of a group presentation $\langle S|R\rangle$ consists of one vertex, $|S|$ directed edges starting and ending at the unique vertex, and a 2–dimensional cell corresponding to each relator $r \in R$, whose boundary circle is identified with the closed edge-path spelled by r.

It is an exercise in topology to prove that the fundamental group of the presentation complex is the group $\langle S|R\rangle$, which thus acts freely by deck transformations on the universal cover of the presentation complex.

Problem 3. Let K be the presentation complex of the group:

(i) $\langle s, t \mid s^2t^2 = ts \rangle$,

(ii) $\langle s, t, r \mid sts^{-1} = t^{-1}, rsr^{-1} = t \rangle$,

(iii) $\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$.

Find a piecewise Euclidean structure on K of nonpositive curvature, and, if possible, a piecewise hyperbolic structure of curvature ≤ -1.

In the further problems you can use the following variant of Cartan–Hadamard theorem for spaces of curvature ≤ 1.

Theorem. *Let X be a piecewise spherical complex of curvature ≤ 1 with finitely many isometry types of cells and no isometrically embedded circles of length $< 2\pi$. Then X is* CAT(1).

Definition. A simplicial complex is *flag* if every set of edges pairwise connected by edges spans a simplex.

Problem 4. (i) Let K be a piecewise spherical simplicial complex, each of whose edges has length $\frac{\pi}{2}$. Prove that K is CAT(1) if and only if it is flag. Hint: Proceed by induction on dimension.

(ii) Let K be a simply connected polyhedral complex build of Euclidean cubes. Prove that K is CAT(0) if and only if each $\mathrm{Lk}(v, K)$ is flag.

Problem 5. Let Γ be a finite graph with vertices $v_1, \ldots, v_n$. The *right-angled Artin group defined by* Γ is the group presented as $A_\Gamma = \langle S|R \rangle$, where $S = \{s_1, \ldots, s_n\}$ and R consists of relations $s_is_j = s_js_i$ for all v_i, v_j joined by an edge. Find a CAT(0) cube complex with a free and cocompact action of A_Γ. Hint: if Γ has no triangles, then it suffices to consider the universal cover of the presentation complex.