MATH 599 Nonpositive Curvature
Problem list 6

Problem 1. Let X be CAT(0) and let $C \subset X$ be convex and complete. Show that for every $x \in X$ there exists a unique point (called the *projection of x to C*) $\pi(x) \in C$ satisfying $d(x, \pi(x)) = \inf_{y \in C} d(x, y)$.

Problem 2. Let X be CAT(0) and let $C \subset X$ be convex and complete, as in Problem 1. Suppose $x \notin C$ and $y \in C$, where $y \neq \pi(x)$. Show that the Alexandrov angle at $\pi(x)$ between the geodesic segments $\pi(x)x$ and $\pi(x)y$ is $\geq \frac{\pi}{2}$.

Problem 3. Prove that the projection map $x \to \pi(x)$ from Problem 1 does not increase distances, i.e. $d(\pi(x), \pi(x')) \leq d(x, x')$.

Problem 4. Let f, g be isometries of X. Show that $|g| = |fgf^{-1}|$ and $\mathrm{Min}(fgf^{-1}) = f\big(\mathrm{Min}(g)\big)$.

Problem 5. Show that every isometry of $\mathbb{R}^n$ is semi-simple. Hint: every isometry is of form $x \to Ax + b$, where $A \in O(n)$. If it is not elliptic, then $-b$ is not in the image of $A - \mathrm{Id}$.

Problem 6. An $\mathbb{R}$–*tree* is a metric space that is CAT(κ) for every κ. Show that every isometry of an $\mathbb{R}$–tree is semi-simple.

Problem 7. Prove that if a group acts on a metric space properly and cocompactly, then all its elements are semi-simple.