MATH 599 Nonpositive Curvature
Problem list 7

Problem 1. Consider a geodesic quadrilateral in a CAT(0) space. Prove that if the sum of its Alexandrov angles is $\geq 2\pi$, then its convex hull is isometric to the convex hull of a convex quadrilateral in $\mathbb{R}^2$.

Problem 2. Let X, Y be metric spaces. Let f be an isometry of $X \times Y$ mapping each $\{x\} \times Y$ to a $\{x'\} \times Y$. Prove that then f is of the form $f_1 \times f_2$, where f_1 is an isometry of X and f_2 is an isometry of Y.

Problem 3. Let g be a hyperbolic isometry of a CAT(0) space X and suppose that f is an isometry commuting with g. Let $\mathrm{Min}(g) = Y \times \mathbb{R}$ be the natural decomposition from class. Prove that f preserves $\mathrm{Min}(g)$ and is of the form $f_1 \times f_2$, where f_1 is an isometry of Y and f_2 a (possibly trivial) translation of $\mathbb{R}$. Prove that if X is complete and f is hyperbolic, then its restriction to $\mathrm{Min}(g)$ is hyperbolic as well, with the same translation length.

Problem 4. Let G be a finitely generated group acting on a CAT(0) space, with centre $A \cong \mathbb{Z}$ acting by hyperbolic isometries (apart from the identity element). Prove that there is a finite index subgroup $G' < G$ containing A as a direct factor.

Remark. The problem remains true for a complete CAT(0) space if $A \cong \mathbb{Z}^n$.

Problem 5. Let A be a free abelian group of rank n acting properly by semi-simple isometries on a complete CAT(0) space with $\mathrm{Min}(A) = Y \times \mathbb{R}^n$. Prove that

(i) an isometry g normalizing A preserves $\mathrm{Min}(A)$ and the product decomposition,

(ii) a group G of isometries normalising A has a subgroup G' of finite index centralising A. (In particular, by Remark, if G is finitely generated, then there is such G' containing A as a direct factor. You are allowed to use this in Problem 7.)

Problem 6. Prove that if G acts properly and cocompactly on a complete CAT(0) space, then G has no ascending sequence of virtually abelian subgroups. (In particular if G is abelian, then it is finitely generated.)

Hint: use the flat torus theorem, Problem 5(i), and the fixed point theorem.

Problem 7. Prove that if G acts properly and cocompactly on a complete CAT(0) space, then every virtually solvable subgroup $S < G$ is finitely generated and virtually abelian.

Hints: use Problem 6 to reduce to the finitely generated case. Induct on the derived length of S and use Problem 5(ii) to show that there is $S' < S$ of finite index with finite commutator.