MATH 599 Nonpositive Curvature
Problem list 8

Problem 1. Let G be a group generated by order-two elements s_1, s_2 such that the order of $s_1 s_2$ equals m. Show that G is isomorphic to D_m (where $D_2 = \mathbb{Z} \oplus \mathbb{Z}$).

Problem 2. Let Γ be the Cayley graph of the Coxeter group of W. The *Cayley complex* of W is obtained from Γ by gluing a $2m$-gon along each loop in Γ corresponding to a finite coset of a two-generator special subgroup $W_{\{s,t\}}$ of order $2m$ with $m < \infty$.

(i) Prove that the Cayley complex is simply-connected. Hint: use the natural map from the universal cover of the presentation complex to the Cayley complex.

(ii) Prove that for $|S| \geq 3$ the Coxeter complex is simply connected.

Problem 3. Describe the Coxeter complexes of Coxeter groups:

(i) with three generators and all three exponents equal ∞,

(ii) with three generators and all three exponents equal 4,

(iii) with generators s_1, s_2, t_1, t_2 and $m_{s_i t_j} = 2$ for all i, j, and $m_{s_1 s_2} = m_{t_1 t_2} = \infty$,

(iv) with four generators and all six exponents equal 3,

(v) with generators p, r, s, t, and $m_{pr} = m_{rs} = m_{st} = m_{tp} = 3,\ m_{ps} = m_{rt} = 2$.

You are allowed to use Problem 2(ii).

Definition. Let W be a Coxeter group and let $X = W \times \triangle / \sim$ be its Coxeter complex. The *Davis complex* Σ is obtained as $W \times \triangle_\Sigma / \sim$, where $\triangle_\Sigma \subset \triangle$ is the subcomplex of the barycentric subdivision of $\triangle$ spanned on all the vertices corresponding to faces labelled by T with finite W_T.

Problem 4. Let W be a Coxeter group with all m_{ij} even or ∞. Prove:

(i) For each $T \subset S$ there is a natural homomorphism $W \to W_T$ and hence we have $W_T \subset W$.

(ii) W is *virtually torsion-free*, meaning it has a finite index subgroup all of whole nontrivial elements are of infinite order. You are allowed to use the theorem that for infinite W the Davis complex carries a W-invariant piecewise-Euclidean metric that is CAT(0).

Problem 5. Find two isomorphic Coxeter groups with distinct Coxeter matrices (interchanging the generators does not count).