MATH 599 Nonpositive Curvature
Preparation problems to midterm I

Problem 1. Consider $\mathbb{R}^2$ with the metric.

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt[3]{|x_1 - y_1|^3 + \ldots + |x_n - y_n|^3}.$$

Prove that this metric space is geodesic. For which x, y are geodesics joining x, y unique?

Problem 2. Prove that hyperbolic segments in $\mathbb{H}^n$ are geodesic segments (we omitted doing this in class).

Problem 3. Consider an equilateral geodesic triangle in M^2_κ of side length $c < \frac{2}{3}\mathrm{diam} M^2_\kappa$. Compute its angle (as a function of c and κ).

Problem 4. Let $x \in \mathbb{H}^n$ in the hyperboloid model and let $u \in \mathbb{R}^{n+1}$ be a unit vector. Prove that the distance between x and the hyperplane $u^\perp \cap \mathbb{H}^n$ equals $\mathrm{arcsinh}|(x, u)|$.

Problem 5. Let $u, v \in \mathbb{R}^{n+1}$ be unit vectors w.r.t. the form $(\cdot, \cdot)$. Give an interpretation of (u, v) depending on the relative position of the hyperplanes $u^\perp \cap \mathbb{H}^n$ and $v^\perp \cap \mathbb{H}^n$.

Problem 6. In $\mathbb{H}^2$ consider a hyperbolic triangle formed of geodesic segments with angles A, B, C and distances between vertices a, b, c. Prove

(i) the *dual hyperbolic law of cosines*:

$$\cos C = -\cos A \cos B + \sin A \sin B \cosh c,$$

(ii) the *hyperbolic law of sines*:

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}.$$

Problem 7. In $\mathbb{H}^2$ consider a right-angled hexagon formed of geodesic segments of lengths A, b, C, a, B, c. Prove

$$\frac{\sinh a}{\sinh A} = \frac{\sinh b}{\sinh B} = \frac{\sinh c}{\sinh C}.$$

Problem 8. Let $D \subset \mathbb{R}^2$ be the unit disc considered as the Klein model of $\mathbb{H}^2$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product. Show that for $x, y \in D$ we have

$$d(x,y) = \operatorname{arccosh} \frac{1 - \langle x, y \rangle}{\sqrt{1 - \langle x, x \rangle^2}\sqrt{1 - \langle y, y \rangle^2}}$$

Problem 9. Prove that an isometry of $\mathbb{H}^2$ preserves the cross-ratio of the points in the boundary circle of the Klein model (or the Poincaré model)

Problem 10. Prove that any ordered triple of distinct points in the boundary circle of the Klein (or Poincaré) model of $\mathbb{H}^2$ can be mapped by an isometry to any other such triple, and that such an isometry is unique.

Problem 11. Let ϕ be an isometry of $\mathbb{H}^2$ that is not *elliptic*, i.e. fixing no point of $\mathbb{H}^2$. Prove that then ϕ either fixes exactly 2 points of the boundary circle of the model and acts as a translation on the geodesic line joining them (we call such ϕ *hyperbolic*), or fixes exactly 1 point of the boundary circle of the model (we call such ϕ *parabolic*).

Problem 12. In the Poincaré model B of $\mathbb{H}^2$ a *horocycle* at $\xi \in \partial B$ is a circle in B tangent to ∂B at ξ. Prove that a parabolic isometry of $\mathbb{H}^2$ fixing ξ preserves each horocycle at ξ.

Problem 13. Prove that in a CAT(κ) space balls of radius $\leq \frac{1}{2}\mathrm{Diam} M^2_\kappa$ are convex.

Problem 14. Prove that for $\kappa \leq 0$ a CAT(κ) space is contractible.

Problem 15. Show that a geodesic metric space X is CAT(κ) if and only if for every geodesic triangle in X with vertices x, y, z (of perimeter $< 2M^2_\kappa$ if $\kappa > 0$) and the midpoint m of the side xy we have $d(m, z) \leq d(\bar{m}, \bar{z})$ (where $\bar{m}, \bar{z}$ are in the comparison triangle in M^2_κ).

Problem 16. Show that a geodesic metric space X is CAT(0) if and only if for every triple of points $x, y, z \in X$ and for every point $m \in X$ satisfying $d(m, x) = d(m, y) = \frac{1}{2}d(x, y)$ we have

$$d(x,z)^2 + d(y,z)^2 \geq 2d(m,z)^2 + \frac{1}{2}d(x,y)^2.$$

Problem 17. In Alexandrov's Lemma, show that if one of the inequalities in its conclusion is an equality, then they are all equalities.

Problem 18. Give an example of a metric space and two geodesic segments that are disjoint except at the common starting point, with Alexandrov angle 0.

Problem 19. Prove that in $\mathbb{S}^2$ the spherical angle is equal to the Alexandrov angle.

Problem 20. Prove that in a CAT(κ) space we have

$$\angle(c, c') = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d(c(t), c'(t)).$$

Problem 21. Prove the law of cosines in $\mathbb{CH}^n$.

Problem 22. A subspace V of $\mathbb{C}^{n+1}$ regarded as a vector space over $\mathbb{R}$ is *totally real* if $(u, v) \in \mathbb{R}$ for all $u, v \in V$. Let $p\colon \mathbb{C}^{n+1} \to \mathbb{P}^n$ be the natural projection. If $p(V) \cap \mathbb{CH}^n \neq \emptyset$, then it called a *totally real subspace of dimension n in* $\mathbb{CH}^n$. Prove that subspaces of $\mathbb{CH}^n$ isometric to $\mathbb{H}^n$ are exactly the totally real subspaces of dimension n.

Problem 23. Characterize all subspaces of $\mathbb{CH}^n$ isometric to M^2_{-4}, which is $\mathbb{H}^2$ with the metric rescaled by $\frac{1}{2}$.

Problem 24. Let $p \in P(n, \mathbb{R})_1$ and consider the map $S\colon P(n, \mathbb{R})_1 \to P(n, \mathbb{R})_1$ defined as $S(q) = pq^{-1}p$. Show that S is an isometry and that it acts as $-\mathrm{Id}$ on the tangent space to $P(n, \mathbb{R})_1$ at p.

Problem 25. A *flat* in $P(n, \mathbb{R})_1$ is a subspace isometric to $\mathbb{R}^k$. Show that every flat is contained in a flat of dimension $k = n$. Prove that $\mathbf{SL}(n, \mathbb{R})$ acts transitively on the set of pairs (F, p), where F is a maximal flat and $p \in F$.

Problem 26. Let $p \in P(n, \mathbb{R})_1$. Show that the following formula involving matrix exponentiation defines a geodesic line in $P(n, \mathbb{R})_1$.

$$c(t) = e^{tp}$$

Prove that $c(t)$ is contained in a unique maximal flat if and only if all the eigenvalues of p are distinct.

Problem 27. Verify that the function d we defined on the cone $X = C_\kappa Y$ is indeed a metric (and that it is complete if and only if Y is complete).

Hint: To verify $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$, let $x_i = (y_i, t_i)$. If all t_i are nonzero, consider first the case $d(y_1, y_2) + d(y_2, y_3) < \pi$. In the second case, use Aleksandrov Lemma.

Problem 28. Characterize the geodesics in $C_\kappa Y$, given the geodesics in Y.

Problem 29. Prove the theorem of Berestovskii that $C_\kappa Y$ is CAT(κ) if and only if Y is CAT(1).

Hint: use the previous problem. For the *if* direction consider three cases:

- $d(y_1, y_2) + d(y_2, y_3) + d(y_3, y_1) < 2\pi$ (here use Problem 8 from list 3 and the fact that $C_\kappa \mathbb{S}^2 \subset M^3_\kappa$),
- $d(y_1, y_2) + d(y_2, y_3) + d(y_3, y_1) \geq 2\pi$, but all $d(y_i, y_j) < \pi$ (here use Problems 5,7, and 9 from list 3),
- there are some $d(y_i, y_j) \geq \pi$ (use Problems 5 and 7).