MATH 599 Nonpositive Curvature
Preparation problems to midterm II

Problem 1. Prove that a CAT(0) space is convex.

Problem 2. Prove that a local isometry from a complete metric space to a locally convex metric space is a covering map.

Below $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

Problem 3. Let K be the presentation complex of the group

(i) $\langle a, b, c, d, e, f \mid [a, b][c, d][e, f] = 1\rangle$,

(ii) $\langle a, b, c \mid ab = ba, cac^{-1} = b\rangle$,

(iii) $\langle a, b, c \mid a^2bc = ba, c^2ac = b^2\rangle$,

(iv) $\langle a, b, c \mid ab = c^2, ba = c^2\rangle$.

Find a piecewise Euclidean structure on K of nonpositive curvature, and, if possible, a piecewise hyperbolic structure of curvature ≤ -1.

Problem 4. Realise each of the following groups as the fundamental group of a nonpositively curved metric space.

(i) $\langle a, b \mid aba = bab\rangle$,

(ii) $\langle a, b \mid abab = baba\rangle$,

(iii) $\langle a, b, t, s \mid ab = ba, tat^{-1} = ab^2\rangle$,

(iv) $\langle a, b, t, s \mid ab = ba, tat^{-1} = ab, sas^{-1} = b\rangle$.

Problem 5. Let X be a simplicial complex with finitely many isometry types of cells and nonpositive curvature. Let Y be its locally convex subcomplex. Prove that the natural map $\pi_1(Y) \to \pi_1(X)$ is an embedding.

Problem 6. Let g be an isometry of a metric space X. Show that

$$\lim_{n\to\infty} \frac{1}{n} d(x, g^n(x))$$

exists for all x, and is independent of x.

Hint: subadditive sequence.

Problem 7. Suppose that a group G acts properly and cocompactly on a metric space X. Prove that the set of translation distances of the elements of G is a discrete subset of $\mathbb{R}$.

Problem 8. Prove that the following groups do not act properly and cocompactly on a CAT(0) space:

(i) the Heisenberg group $\langle s, t, r \mid [s,t] = [s,r] = 1,\ [t,r] = s\rangle$,

(ii) the solvable group $\langle \mathbb{Z}^2, t \mid t^{-1}vt = Av \text{ for all } v \in \mathbb{Z}^2\rangle$, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

(iii) the fundamental group of the unit tangent bundle of the orientable genus 2 surface.

In the following problems you are allowed to use the theorem below.

Theorem. *Let G, H be groups and ϕ_1, ϕ_2 two embeddings of H into G. The* HNN–extension *defined by this data is the group presented as $\langle G, t \mid t\phi_1(h)t^{-1} = \phi_2(h) \text{ for all } h \in H\rangle$. Then the natural map from G to the* HNN*–extension is an embedding.*

The theorem follows from viewing the HNN–extension as a loop of groups, which has nonpositive curvature.

Problem 9. Prove that the group $\langle a, b \mid bab^{-1} = a^2\rangle$ does not act properly and cocompactly on a CAT(0) space.

Problem 10. Prove that the group

$$\langle a, b, t, s \mid ab = ba, tat^{-1} = ab^2, sas^{-1} = b\rangle$$

does not act properly and cocompactly on a complete CAT(0) space.

Problem 11. Describe the Coxeter complex and the Davis complex Σ for Coxeter groups:

(i) with three generators and exponents $2, \infty, \infty$,

(ii) with three generators and exponents $3, 3, \infty$,

(iii) with three generators and exponents $2, 3, 6$,

(iv) with generators p, r, s, t, and $m_{pr} = 6$, $m_{rs} = m_{st} = 3$, $m_{ps} = m_{pt} = m_{rt} = 2$,

(v) with generators p, r, s, t, and $m_{pr} = m_{st} = 4$, $m_{rs} = 3$, $m_{ps} = m_{pt} = m_{rt} = 2$.

Problem 12. Prove that every finite subgroup of a Coxeter group lies in a conjugate of a finite special subgroup.

Problem 13. Let Σ be the Davis complex of a Coxeter group with all $m_{ij} \in \{2, \infty\}$. Show that after assembling the simplices of Σ into cubes, the links of the resulting cube complex are flag.

Problem 14. Describe the action of the Coxeter group

(i) D_∞,

(ii) with three generators and all exponents equal to 3,

on $\mathbb{R}^{|S|}$ via the dual representation to the Tits representation. Hint: look at the translates of the simplex spanned by the basis dual to the standard basis.

Problem 15. Consider the full symmetry group G of the square tiling of $\mathbb{R}^2$. What is the simplex of groups structure of $\mathbb{R}^2/G$?

Problem 16. Let abc be a triangle with the following simplex of groups structure. The edge and vertex groups are $G_{ab} = \mathbb{Z}_3$, $G_{ac} = G_{bc} = \mathbb{Z}_2$, $G_c = D_3$, $G_a = G_b = \langle s, t \mid s^2 = t^3 = (st)^3 = 1 \rangle$ and G_{abc} is trivial. Moreover, suppose that the maps φ send generators of edge groups to s and t in G_a, G_b, and to arbitrary generators in G_c. What are the local developments? Is this simplex of groups developable?