a relationship to hold between arguments (with premises) and the corresponding conditionals' being tautologies. We use the notation '$\Gamma \vdash p$' to mean that p follows (syntactically) from the set of formulas in Γ as premises. Again as a special case where Γ is empty, we use $\vdash p$ to indicate that p can be proved from zero premises.

As mentioned before, there is a correspondence between the syntax and the semantics of propositional logic. Just as there is a semantic deduction theorem, so too is there a syntactic deduction theorem saying: $\Gamma \vdash p$ if and only if the corresponding conditional can be proved (without premises). Again, if q, r and s are all the formulas in Γ, then $\Gamma \vdash p$ if and only if $\vdash [(q \wedge r \wedge s) \longrightarrow p]$.

So we see that bot'. syntactically and semantically there is a way of converting "argument forms" -- i.e., logical implications -- into individual sentences, and vice versa. Sometimes it is handy to do this, as we shall see in the syntactic case; but other times we can adapt our existing methods to handle the more general case of having premises and conclusions. For example, we *could* use the deduction theorem with our tree method of evaluating sentences. All we would have to do is convert the argument with premises into the corresponding conditional, and then apply the tree method of negating this conditional and branching. But it would be just as easy, or easier, to follow the method outlined in the section on arguments: make the root of the tree contain all the premises and the negation of the conclusion.

Section 3.11. An Explicit Formal System of Propositional Logic

A syntactic presentation of a system of logic is generally made by stating

(1) a set of axioms

(2) a set of rules of inference

(3) a definition of a proof, that is, a definition of "from premises $p_1, p_2, \cdots p_m$ we can prove q".

Generally speaking, an axiom is a formula (or formula-type) which is simply presented without justification, and a rule of inference is a way of converting (one or more) already presented formulas into another (and thus allowing the rules of inference to operate on it). Using this method, one usually defines "a proof of q from $p_1, p_2, \cdots p_m$" (remember we abbreviated this as $\Gamma \vdash q$, where $p_1, p_2, \ldots p_m$ are the members of Γ) like this: the proof is a finite sequence of lines of formulas, each one of which either (a) is a member of Γ, or (b) is an axiom, or (c) is the result of applying a rule of inference to previous members of the sequence; and (d) the formula q occurs as the last line of the sequence. Such systems of.logic are called *axiomatic systems*.

Now, as it turns out, it is possible for there to be systems of logic with no axioms, at the cost of having a larger set of rules of inference and a somewhat more complicated definition of "proof". Such systems are called ***natural deduction systems***, and are what we shall present here.

Certain rules of inference are quite reasonable and natural, for example, if you have $(p \wedge q)$ at your disposal then you should be allowed to infer p (and also infer q). The reason that this rule is natural is because of the meaning, or point, of '$\wedge$', which is to be understood as 'and'. Similarly, if we are presented with $\sim\sim p$ we should be allowed to infer p ("from a double negative, infer the positive"). Another obvious rule of inference is that if you have been given two things, p and $(p \longrightarrow q)$ then you should be able to infer q. A different kind of rule of inference, but one which is still intuitively valid, goes like this: "I want to prove that p is true. Just imagine what would happen if it weren't. Why, then q would be true, and hence r would be false and so s would have to be true. But this is impossible because s is contradictory. So our assumption that p is not true has to be wrong; therefore p is true." This method of argumentation is extremely popular and useful; it is often called an *indirect proof* (of p) or a *reductio ad absurdum* (of not-p) or a *proof by contradiction* (of p). One version of this rule was incorporated in our truth trees, when we negated the conclusion and discovered that all paths closed. (That was our contradiction). We concluded that the assumption of the negation had to be wrong.

Let us investigate the structure of this last argument a little more. We already knew some facts (e.g., that *s* was false), and we wished to prove from these known facts that *p* had to be true. What we did here was to *temporarily* make the assumption that *p* was false (i.e., that $\sim p$ was true), and then we looked at what this assumption led to. What we discovered was that it led to a contradiction. Since it leads to a contradiction, the assumption $\sim p$ cannot be right. So we should (and did) infer from this chain of reasoning (from this *subargument*), that *p* must be right. But also we have to now give up our assumption -- after all it was just an assumption for a subargument, to see where it would lead. The justification for *p* is the entire subargument, but once we have inferred *p* we need to "erase" or "eliminate" or at least never be able to use our assumption $\sim p$ (and also of course not use any of the reasoning which started with $\sim p$).

Clearly the use of assumptions is a familiar, legitimate and important part of ordinary argumentation, but just as clearly, to *assume* something (for the purpose of argument) is quite different from *proving* it. So an important feature of any natural deduction system ought to be a clear and simple device for distinguishing sentences which are assumed from sentences which are derived and for making it clear just what assumptions (if any) a sentence is "dependent" on. There are many ways in which this can be done; the way that we will choose is the simplest of those of which we are aware. Before we turn to that device, we may begin with some simpler considerations.

Suppose we have as premisses of an argument p, q, and r. We begin by listing them in the following manner:

| p
q
r
| ———— ⟸ *This line indicates the end of the list of premisses.*
|
| ⟸ *This line indicates the "scope" of the premisses--i.e. anything written to the RIGHT of this line is "dependent" on the premisses--i.e. is claimed to follow from the premisses, is claimed to be true IF the premisses are true.*
| *Indeed, any sentence written UNDER (in the literal sense) the premisses is a sentence which is claimed to be true UNDER (in a metaphorical sense) the assumption that the premisses are true.*

Before proceeding to discuss further the format for handling additional assumptions

made in the course of an argument (or *derivation*, as we shall call our formal arguments), we consider a few simple rules and a few simple derivations.

We introduce eleven rules; here are a few of the simplest ones:

CONJUNCTION ELIMINATION ($\wedge E$)

| .
 .
 .
 $p \wedge q$
 .
 .
 .
 p | | .
 .
 .
 $p \wedge q$
 .
 .
 .
 q

⟸ *The format here is to indicate that if you already have a conjunction either as a premise or assumption or previously derived line, then directly below it you may write down either conjunct.*

CONJUNCTION INTRODUCTION ($\wedge I$)

| .
 .
 .
 p
 .
 .
 .
 q
 .
 .
 .
 $p \wedge q$ | | .
 .
 .
 p
 .
 .
 .
 q
 .
 .
 .
 $q \wedge p$

⟸ *The format here is to indicate that if you already have p and q as lines, then their conjunction may be written, in either order, directly below them.*

Each of these two rules has two forms. In the rule ($\wedge E$) we can "simplify" or "eliminate" from either side of a conjunction. In the ($\wedge I$) rule, we can form our conjunction in either direction, so long as we have both conjuncts available. When writing a proof, you not only apply these rules of inference, but also you include another column (which is called the *annotation*) in which you state what previous lines were used and which rule of inference was applied. It should also be noted that these rules can be used on a line of a proof which was an assumption (the formula above a horizontal line). In presenting a proof we give line numbers off to the left so that we can refer to preceding lines.

Example: We derive $P \wedge Q$ from $Q \wedge P$:

1.	$Q \wedge P$	
2.	Q	1, $\wedge E$
3.	P	1, $\wedge E$
4.	$P \wedge Q$	2,3 $\wedge I$

Example: We derive $P\wedge(Q\wedge R)$ from $(R\wedge Q)\wedge R$:

1.	$(P\wedge Q)\wedge R$	
2.	$P\wedge Q$	1, $\wedge E$
3.	P	2, $\wedge E$
4.	Q	2, $\wedge E$
5.	R	1, $\wedge E$
6.	$Q\wedge R$	4,5 $\wedge I$
7.	$P\wedge(Q\wedge R)$	3,6 $\wedge I$

IMPLICATION ELIMINATION ($\longrightarrow E$) ***(Traditionally known as Modus Ponens)***

p
$p \longrightarrow q$
q

$p \longrightarrow q$
p
q

Example: We derive R *from* P, $P\longrightarrow Q$, $Q\longrightarrow R$:

1.	P	
2.	$P\longrightarrow Q$	
3.	$Q\longrightarrow R$	
4.	Q	1,2 $\longrightarrow E$
5.	R	3,4 $\longrightarrow E$

Example: We derive R from $P\vee S$, $(P\vee S)\longrightarrow Q$, $Q\longrightarrow R$:

1.	$P\vee S$	
2.	$(P\vee S)\longrightarrow Q$	
3.	$Q\longrightarrow R$	
4.	Q	1,2 $\longrightarrow E$
5.	R	3,4 $\longrightarrow E$

Example: We derive R from P, Q, $(P\wedge Q)\longrightarrow(R\wedge S)$:

1.	P	
2.	Q	
3.	$(P\wedge Q)\longrightarrow(R\wedge S)$	
4.	$P\wedge Q$	1,2 $\wedge I$
5.	$R\wedge S$	3,4 $\longrightarrow E$
6.	R	5, $\wedge E$

DISJUNCTION INTRODUCTION ($\vee I$)

p
$p \vee q$

p
$q \vee p$

Example: We derive R from P and $(P\vee Q)\longrightarrow R$:

1.	P	
2.	$(P\vee Q)\longrightarrow R$	
3.	$P\vee Q$	1, $\vee I$
4.	R	2,3 $\longrightarrow E$

NEGATION ELIMINATION ($\sim E$) ***(often called Double Negation)***

$\sim\sim p$
p

Example: We derive $P \wedge Q$ *from* $P \wedge \sim\sim Q$:

1.	$P \wedge \sim\sim Q$	
2.	P	1, $\wedge E$
3.	$\sim\sim Q$	1, $\wedge E$
4.	Q	3, $\sim E$
5.	$P \wedge Q$	2,4 $\wedge I$

Example: We derive S from Q, $P \wedge T$, $((P \wedge Q) \vee R) \longrightarrow \sim\sim S$:

1.	Q	
2.	$P \wedge T$	
3.	$((P \wedge Q) \vee R) \longrightarrow \sim\sim S$	
4.	P	2, $\wedge E$
5.	$P \wedge Q$	1,4 $\wedge I$
6.	$(P \wedge Q) \vee R$	5, $\vee I$
7.	$\sim\sim S$	3,6 $\longrightarrow E$
8.	S	7, $\sim E$

It is important to identify the main connective of a sentence when using inference rules-- **the inference rules never apply to internal parts of a sentence.** For example, you can never derive a conjunction by disjunction introduction, or derive something from a disjunction by implication elimination, even if there are conjunctions or disjunctions embedded within the formulas.

Here are some examples of the ***misuse*** of the inference rules. In the first one, we incorrectly did a $\vee I$ on the antecedent, rather than to the whole formula. In the second we incorrectly did $\longrightarrow E$ on a subpart of the formula on line 2 (to correctly use $\longrightarrow E$ on line 2 we would need the formula $P \longrightarrow Q$, and then we would infer R rather than $(Q \longrightarrow R)$. In the third example, we incorrectly applied $\wedge E$ to an embedded occurrence of $\wedge$. And in the last example we applied $\wedge E$ to the antecedent of line 1, whereas the main connective is $\longrightarrow$.

1.	$P \longrightarrow Q$	
2.	$(P \vee R) \longrightarrow Q$	*INCORRECT* $\vee I$ *on 1*

1.	$((P \longrightarrow Q) \longrightarrow R) \wedge P$	
2.	$(P \longrightarrow Q) \longrightarrow R$	1, $\wedge E$
3.	P	1, $\wedge E$
4.	$Q \longrightarrow R$	*INCORRECT* $\longrightarrow E$ *on 2,3*

1.	$(P \wedge Q) \wedge R$	
2.	P	*INCORRECT* $\wedge E$ *on 1*

1.	$(P \wedge Q) \longrightarrow R$	
2.	$Q \longrightarrow R$	*INCORRECT* $\wedge E$ *on 1*

EQUIVALENCE ELIMINATION $(\longleftrightarrow E)$

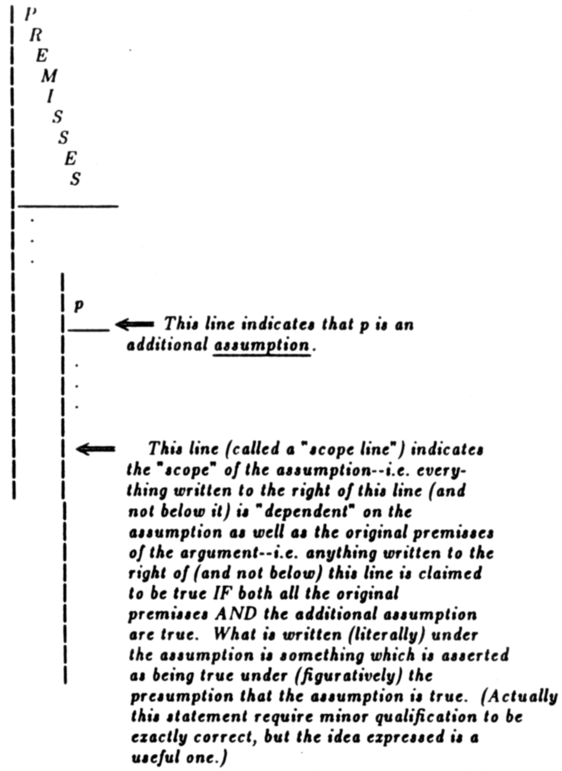
.	.	.	.
$p \longleftrightarrow q$	$p \longleftrightarrow q$	p	q
.	.	.	.
p	q	$p \longleftrightarrow q$	$p \longleftrightarrow q$
.	.	.	.
q	p	q	p

Example: we derive Q from $P \longleftrightarrow (Q \longleftrightarrow R)$ *and* $P \wedge R$:

1.	$P \longleftrightarrow (Q \longleftrightarrow R)$	
2.	$P \wedge R$	
3.	P	2, $\wedge E$
4.	$Q \longleftrightarrow R$	1,3 $\longleftrightarrow E$
5.	R	2, $\wedge E$
6.	Q	4,5 $\longleftrightarrow E$

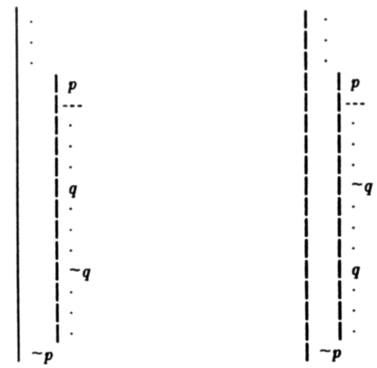
We now turn to the problem of formulating a rule of ***Negation Introduction*** -- there are a number of ways this could be done, but our choice will be a form of Reductio Ad Absurdum argument. In such a case we prove $\sim p$ by ***assuming*** p and deriving a contradiction. On the basis of the existence of the subargument or ***subderivation*** we conclude that $\sim p$, simultaneously giving up our assumption of p.

We noted earlier that it is important to distinguish assumptions from derived sentences etc. To achieve this we further employ an earlier device--we indicate that we are ***assuming*** p by the following:



Using this new device, we may represent negation introduction as follows:

NEGATION INTRODUCTION (~I)



Note: ~p is *not* written under the assumption p as it is not dependent on it; instead it is "popped out" one level.

Note: As a special case either *q* or ~*q* may be *p* itself.

In arguing under an assumption we do, of course, refer back to things we have already proved. For example, we derive ~*P* from *P*⟶*Q* and *P*⟶~*Q*:

```
1. | P⟶Q
2. | P⟶~Q
   | ------
3. |  | P
   |  | ---
4. |  | P⟶Q      1, ?
5. |  | P⟶~Q     2, ?
6. |  | Q        3,4 ⟶E
7. |  | ~Q       3,5 ⟶E
8. | ~P          3-7 ~I
```

Note: In the last line we refer to the WHOLE SUBDERIVATION 3-7, RATHER THAN TO SPECIFIC LINES, because it is the existence of the subderivation that justifies the conclusion ~P.

But how do we justify lines 4 and 5? Intuitively, they *are* justified, because we are still arguing under our original premisses, and certainly anything we were entitled to assert on the basis of those premisses, we are still entitled to assert even though we now have an additional assumption.

We introduce the rule of Reiteration (R) to allow us to repeat lines already derived from the premisses (or the premisses themselves) or lines already derived from the premisses and additional assumptions ***provided that*** we are still arguing under the scope of those assumptions. (We could in fact eliminate this rule, but only at the cost of making certain rules that we will introduce in the next chapter very messy to state.)

Intuitively, the rule is straightforward--any premise or assumption, or anything derived from the premisses plus previous assumptions may be asserted at any point, ***provided that*** we are still arguing under the scope of those previous assumptions.

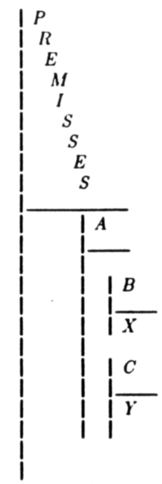
Although intuitively clear, the rule of reiteration is a little difficult to state rigorously.

REITERATION (R)

> A line X of a derivation may be written again as line Y provided that every scope line to the left of X continues unbroken to the left of Y and provided that X occurs above Y in the derivation.

Don't be put off by this definition; the idea behind it is simple: You can reiterate any line previously occurring in a derivation provided that all the assumptions "under" which it occurred are still "operative". For those of you who have programmed in block structured languages, this should be familiar. Reiteration (until a new wrinkle that comes up in the next chapter) is merely the usual statement of when an identifier is valid: it is valid in its own block and in any block that it contains. The scope lines in this system of propositional logic can therefore be viewed like the blocks in a statically-scoped programming language.

Reiteration is ***never*** done from below or from the right. It may occur only from above or from above and to the left, and ***not always*** then as the following example illustrates:



Here line X may not be reiterated at line Y, because there is a scope line to the left of X which does not continue unbroken to the left of Y. And intuitively speaking, it would wrong to do so since X may have depended on the assumption B, whereas line Y is in a position that is not dependent on B.

Now that we have our reiteration rule we can complete the annotation of the derivation on the previous page:

```
1. | P-->Q
2. | P-->~Q
   ------
3. |   | P
   |   |-------
4. |   | P-->Q      1, R
5. |   | P-->~Q     2, R
6. |   | Q          3,4 -->E
7. |   | ~Q         3,5 -->E
8. | ~P             3,7 ~I
```

Example: We derive $\sim P$ from $P \longrightarrow Q$ and $\sim Q$:

```
1. | P-->Q
2. | ~Q
   -------
3. |   | P
   |   |------
4. |   | P-->Q    1, R
5. |   | Q        3,4 -->E
6. |   | ~Q       2, R
7. | ~P           3-6, ~I
```

Example: We give two different derivations of $\sim Q$ from $P \rightarrow (Q \rightarrow R)$ and $P \wedge \sim R$:

```
1.  | P→(Q→R)                      1.  | P→(Q→R)
2.  | P∧~R                         2.  | P∧~R
    | ------                           | ------
3.  | P           2, ∧E            3.  |   | Q
4.  | ~R          2, ~E            4.  |   |---             2, R
5.  | Q→R         1,3 →E           5.  |   | P              4, ∧E
6.  |   | Q                        6.  |   | P→(Q→R)        1, R
    |   |----                          |   |
7.  |   | Q→R     5, R             7.  |   | Q→R            5,6 →E
8.  |   | R       6,7 →E           8.  |   | R              3,7 →E
9.  |   | ~R      4, R             9.  |   | ~R             4, ∧E
10. | ~Q          6-9 ~I           10. | ~Q                 3-9 ~I
```

Negation Elimination and Negation Introduction can be used together to create another sort of Reductio Ad Absurdum argument: as the following two examples illustrate, we can sometimes prove a formula p by assuming $\sim p$, deriving a contradiction, concluding $\sim\sim p$ by NI, and then p by NE.

Example: We derive P from $\sim P \rightarrow Q$ and $\sim Q$:

```
1. | ~P→Q
2. | ~Q
   | ------
3. |   | ~P
   |   |---
4. |   | ~P→Q      1, R
5. |   | Q         3,4 →E
6. |   | ~Q        2, R
7. | ~~P           3-6, ~I
8. | P             7, ~E
```

Example: We derive $P \vee Q$ from $\sim P \rightarrow Q$:

```
1.  | ~P→Q
    | --------
2.  |   | ~(P∨Q)
    |   |_______
3.  |   |   | P
    |   |   |---
4.  |   |   | P∨Q        3, ∨I
5.  |   |   | ~(P∨Q)     2, R
6.  |   | ~P             3-5, ~I
7.  |   | ~P→Q           1, R
8.  |   | Q              6,7 →E
9.  |   | P∨Q            8, ∨I
10. | ~~(P∨Q)            2-9, ~I
11. | P∨Q                10, ~E
```

We are now in a position to provide derivations for more interesting arguments. Consider the following argument in English.

God, if he exists, is omnipotent, and omniscient as well. Moreover, he is

benevolent, provided that he exists. If God can prevent evil, then if he knows that evil exists, he is not benevolent if he does not prevent it. If he is omnipotent, he can prevent evil. And if he is omniscient, he knows that evil exists if it does exist. Evil does not exist if God prevents it. However, evil does exist. Therefore, God does not exist.

Using the following scheme of abbreviation

G: God exists
P: God is omnipotent
S: God is omniscient
B: God is benevolent
C: God can prevent evil
K: God knows that evil exists
V: God does prevent evil
E: Evil exists

We can translate the premises of the argument as

$G \rightarrow P \wedge S$
$G \rightarrow B$
$C \rightarrow (K \rightarrow (\sim V \rightarrow \sim B))$
$P \rightarrow C$
$S \rightarrow (E \rightarrow K)$
$V \rightarrow \sim E$
E

And the conclusion as $\sim G$. (You should try to do this to see if you get the same translations). A derivation for this argument is:

```
1.  | G→P∧S
2.  | G→B
3.  | C→(K→(~V→~B))
4.  | P→C
5.  | S→(E→K)
6.  | V→~E
7.  | E
    | -----
8.  |  | G
    |  |-----
9.  |  | G→P∧S                  1, R
10. |  | P∧S                    8,9 →E
11. |  | P                      10, ∧E
12. |  | S                      10, ∧E
13. |  | G→B                    2, R
14. |  | B                      8,13 →E
15. |  | P→C                    4, R
16. |  | C                      11,15 →E
17. |  | C→(K→(~V→~B))          3, R
18. |  | K→(~V→~B)              16, 17 →E
19. |  | S→(E→K)                5, R
20. |  | E→K                    12, 19 →E
21. |  | E                      7, R
22. |  | K                      20, 21 →E
23. |  | ~V→~B                  18, 22 →E
24. |  |  | V
    |  |  |-----
25. |  |  | V→~E               6, R
26. |  |  | ~E                 24, 25 →E
27. |  |  | E                  7, R
28. |  | ~V                     24-27 ~I
29. |  | ~B                     23, 28 →E
30. | ~G                        8-29 ~I
```

The next rule we introduce is implication introduction. Often we want to derive conclusion which is a conditional statement. For example, we might want to derive $P \rightarrow R$ from $P \rightarrow Q$ and $Q \rightarrow R$; or $(P \wedge Q) \rightarrow R$ from $P \rightarrow (Q \rightarrow R)$. The normal technique employed in proving a conditional (frequently called 'conditional proof') is to assume the antecedent of the condtional that we wish to prove and to derive the consequent from it, concluding, on the basis of this subderivation, that the conditional holds. This suggests the following rule:

IMPLICATION INTRODUCTION ($\rightarrow I$)

```
| .
| .
| .
|  | p
|  |----
|  | .
|  | .
|  | .
|  | q
| p→q
```

Example: We derive $P \rightarrow R$ from $P \rightarrow Q$ and $Q \rightarrow R$:

```
1. | P→Q
2. | Q→R
   | -----
3. |  | P
   |  |----
4. |  | P→Q    1, R
5. |  | Q      3,4 →E
6. |  | Q→R    2, R
7. |  | R      5,6 →E
8. | P→R      3-7 →I
```

(Again we cite an entire subderivation, as it is its existence which justifies our conclusion.)

Example: We derive $Q \rightarrow P$ from P:

```
1. | P
   | -----
2. |  | Q
   |  |---
3. |  | P    1, R
4. | Q→P     2-3 →I
```

Example: We derive $P \rightarrow Q$ from $\sim P$:

```
1. | ~P
   | -----
2. |  | P
   |  |---
3. |  |  | ~Q
   |  |  |-----
4. |  |  | P     2, R
5. |  |  | ~P    1, R
6. |  | ~~Q      3-5 ~I
7. |  | Q        6, ~E
8. | P→Q         2-7 →I
```

Example: We derive $(P \wedge Q) \rightarrow R$ from $P \rightarrow (Q \rightarrow R)$:

```
1. | P→(Q→R)
   | -----
2. |    | P∧Q
   |    |------
3. |    | P               2, ∧E
4. |    | P→(Q→R)         1, R
5. |    | Q→R             3,4 →E
6. |    | Q               2, ∧E
7. |    | R               5,6 →E
8. | (P∧Q)→R              2-7 →I
```

Example: We derive $P \rightarrow (Q \rightarrow R)$ from $(P \wedge Q) \rightarrow R$:

```
1. | (P∧Q)→R
   | -------
2. |    | P
   |    |---
3. |    |    | Q
   |    |    |---
4. |    |    | P             2, R
5. |    |    | P∧Q           3,4 ∧I
6. |    |    | (P∧Q)→R       1, R
7. |    |    | R             5,6 →E
8. |    | Q→R                3-7 →I
9. | P→(Q→R)                 2-8 →I
```

Sometimes we can derive a sentence from no premises at all. This is possible if and only if the derived sentence is a *theorem* (of propositional logic). As we indicated in the section above on "Syntax, Semantics and the Deduction Theorem", it can be proved that $\vdash p$ if and only if $\models p$. That is, a formula can be proved from no premises ($\vdash p$) just in case it is a tautology ($\models p$). In such a case, since we have no premisses, we merely use a vertical line without a horizontal line.

Example: We derive $P \rightarrow P$:

```
1. |    | P
   |    |_
2. |    | P      1, R
3. | P→P         1-2 →I
```


Example: We derive $P \rightarrow (Q \rightarrow P)$:

```
1. |    | P
   |    |----
2. |    |    | Q
   |    |    |---
3. |    |    | P      1, R
4. |    | Q→P         2-3 →I
5. | P→(Q→P)         1-4 →I
```

Example: We derive $\sim(P \wedge \sim P)$:

```
1. |    | P∧~P
   |    |------
2. |    | P          1, ∧E
3. |    | ~P         1, ∧E
4. | ~(P∧~P)         1-3 ~I
```

Example: We derive $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$:

```
1.  |    | P→(Q→R)
    |    |------
2.  |    |    | P→Q
    |    |    |-----
3.  |    |    |    | P
    |    |    |    |---
4.  |    |    |    | P→Q                  2, R
5.  |    |    |    | Q                    3,4 →E
6.  |    |    |    | P→(Q→R)              1, R
7.  |    |    |    | Q→R                  3,6 →E
8.  |    |    |    | R                    5,7 →E
9.  |    |    | P→R                       3-8 →I
10. |    | (P→Q)→(P→R)                    2-9 →I
11. | (P→(Q→R))→((P→Q)→(P→R))             1-10 →I
```

The following rule should be intuitively plausible.

EQUIVALENCE INTRODUCTION ($\leftrightarrow I$)

```
| .                  | .
| .                  | .
| p --> q            | p --> q
| .                  | .
| .                  | .
| q --> p            | q --> p
| .                  | .
| .                  | .
| p <--> q           | q <--> p
```

Normally, when trying to prove an equivalence, you do not have at your disposal the conditionals each way. Therefore, it is normal to use $\leftrightarrow I$ in conjunction with $\rightarrow I$. One recognizes that it is desired to prove the equivalence, and so sets out first to prove each conditional.

Example: We derive $P \leftrightarrow \sim\sim P$

```
1. |     | P
   |     |---
2. |     |   | ~P
   |     |   |----
3. |     |   | P        1, R
4. |     | ~~P          2-3 ~I
5. | (P --> ~~P)        1-4 -->I
6. |     | ~~P
   |     |-----
7. |     | P            7 ~E
8. | ~~P --> P          6-7 -->I
9. | P <--> ~~P         5,8 <-->I
```

Our final rule of inference is disjunction elimination. There are various ways to state such a rule, all of which are equivalent in the presence of our other rules. Later we shall present some of these others as "derived rules". We choose what we think is the most simple version of disjunction elimination.


```
| .                  | .
| .                  | .
| p v q              | p v q
| .                  | .
| .                  | .
| ~p                 | ~q
| .                  | .
| .                  | .
| q                  | p
```

In this course, this rule is not allowed, unless you prove it using V-Elim.

The intuitive justification for such a rule is this: the $p \vee q$ line says "at least one of p and q are true", and the negation line says that it isn't p that's true (or, it isn't q that's true). So it must be the other one which is true.

We now have all the rules required in order to be able to derive the conclusion of any truth functionally valid argument from its premisses. (We haven't proved this but you can take our word for it.)

In order to "abbreviate" proofs, we could show that from these rules other "derived" rules are also acceptable. If we were to do a lot of proofs, it would be worth our while to prove ten or fifteen derived rules. We shall content ourselves with mentioning some of the most common other rules. One familiar argument form employs 'sub-arguments'. This is frequently called "Argument by Cases".

Example: Either John's mother or his girlfriend will be at the recital. If his girlfriend is at the recital, and John makes noise, he will be unable to hear the music, and so will be displeased. John doesn't want to displease his girlfriend. If his mother is at the recital and John makes noise, whe will be mortified and punish him. Therefore, if John is smart he'll keep quiet at the recital.

This argument depends on many implicit premisses, and so is not directly amenable to a truth functional analysis, but one would naturally argue to the conclusion by showing that it follows in either case--i.e. whether his girlfriend or his mother attends the recital. Note that the order in which the two cases are presented does not matter, and neither does it matter whether the "either...or..." is stated before or after or between the two subproofs.

Argument by cases is particularly common in mathematical contexts.

Our comments are suggestive of the following rule:

DISJUNCTION ELIMINATION ($\vee E$)

```
   |  .
   |  .
   |  .
k  |  p v q
   |  .
   |  .
   |  .
l  |     | p      Hyp(v-E)
   |     |---
   |     | .
   |     | .
   |     | .
m  |     | r
   |  .
   |  .           Order here doesn't matter--i.e. there are 6
   |  .           simple versions of this argument form.
n  |     | q      Hyp(v-E)
   |     |---
   |     | .
   |     | .
   |     | .
t  |     | r
   |  .
   |  .
t' |  r           v-E, k, l-m, n-t
```

Example: We derive Q from $P \longrightarrow Q$, $R \longrightarrow Q$ and $P \vee R$:

```
1.  | P-->Q
2.  | R-->Q
3.  | PvR
    | -----
4.  |    | P
    |    |---
5.  |    | P-->Q     1, R
6.  |    | Q         4,5 -->E
    |
7.  |    | R
    |    |---
8.  |    | R-->Q     2, R
9.  |    | Q         7,8 -->E
10. |  Q             3, 4-6, 7-9 SC    <== here we cite the disjunction and the two subderivations.
```


Example: We derive $Q \vee P$ from $P \vee Q$:

```
1. | PvQ
   | -----
2. |    | P
   |    |---
3. |    | QvP    2, vI
   |
4. |    | Q
   |    |---
5. |    | QvP    4, vI
6. | QvP         1, 2-3, 4-5 SC
```

Sometimes we argue not from just two cases, but from three or more. The following examples illustrate how to approach such situations.

```
1.  | P-->Q
2.  | R-->Q
3.  | S-->Q
4.  | (PvR)vS
    | ------
5.  |    | S
    |    |---
6.  |    | S-->Q          3, R
7.  |    | Q              5,6 -->E
    |
8.  |    | PvR
    |    |---
9.  |    |    | P
    |    |    |---
10. |    |    | P-->Q     1, R
11. |    |    | Q         9,10 -->E
    |    |
12. |    |    | R
    |    |    |---
13. |    |    | R-->Q     2, R
14. |    |    | Q         12,13 -->E
15. |    | Q              8, 9-11, 12-14 SC
16. | Q                   4, 5-7, 8-15 SC
```

Example: We derive $P\vee(Q\vee R)$ from $(P\vee Q)\vee R$:

```
1.   | (P∨Q)∨R
     | ------
2.   |      | P∨Q
     |      |------
3.   |      |     | P
     |      |     |---
4.   |      |     | P∨(Q∨R)    4, ∨I
     |      |
5.   |      |     | Q
     |      |     |---
6.   |      |     | Q∨R        5, ∨I
7.   |      |     | P∨(Q∨R)    6, ∨I
8.   |      | P∨(Q∨R)          2, 3-4, 5-7 SC
     |
9.   |      | R
     |      |---
10.  |      | Q∨R              9, ∨I
11.  |      | P∨(Q∨R)          10, ∨I
12.  | P∨(Q∨R)                 1, 2-8, 9-11 SC
```

Example: We derive $S\longrightarrow R$ from $P\vee Q$, $P\longrightarrow R$, $Q\longrightarrow(S\longrightarrow R)$:

Choosing Implication Introduction as our main attack, we get this:

```
1.   | P∨Q
2.   | P⟶R
3.   | Q⟶(S⟶R)
     | -------
4.   |      | S
     |      |---
5.   |      | P∨Q              1, R
6.   |      |     | P
     |      |     |---
7.   |      |     | P⟶R        2, R
8.   |      |     | R          6,7 ⟶E
     |      |
9.   |      |     | Q
     |      |     |---
10.  |      |     | S          4, R
11.  |      |     | Q⟶(S⟶R)    3, R
12.  |      |     | S⟶R        9,11 ⟶E
13.  |      |     | R          10,12 ⟶E
14.  |      | R                5, 6-8, 9-13 SC
15.  | S⟶R                     4-14 ⟶I
```


Choosing Separation of Cases as our main attack, we get this:

```
1.   | P∨Q
2.   | P⟶R
3.   | Q⟶(S⟶R)
     | ------
4.   |      | P
     |      |---
5.   |      | P⟶R              2, R
6.   |      | R                4,5 ⟶E
7.   |      |     | S
     |      |     |---
8.   |      |     | R          6, R
9.   |      | S⟶R              7-8 ⟶I
     |
10.  |      | Q
     |      |---
11.  |      | Q⟶(S⟶R)          3, R
12.  |      | S⟶R              10,11 ⟶E
13.  | S⟶R                     1, 4-9, 10-12 ∨E
```

(From this last example, with the two different proofs, you should infer that we can have many distinct proofs for the same logical inference.)

Another derived rule is the theorem ($p\vee{\sim}p$), sometimes called the Law of Excluded Middle.

LAW OF EXCLUDED MIDDLE (EM)

```
| .
| .        i.e. p∨~p may be written down as a line of a
| .        derivation at any point (for any formula p)
| p∨~p
```

Justification: EM is to be understood as an abbreviation of the following schema for a derivation:

1.	│ │ $\neg(p \vee \sim p)$	
	│ │------	
2.	│ │ │ p	
	│ │ │---	
3.	│ │ │ $p \vee \sim p$	2, $\vee I$
4.	│ │ │ $\neg(p \vee \sim p)$	1, R
5.	│ │ $\sim p$	2-4 $\sim I$
6.	│ │ $p \vee \sim p$	5, $\vee I$
7.	│ $\sim\neg(p \vee \sim p)$	1-6 $\sim I$
8.	│ $p \vee \sim p$	7, $\sim E$

EM is often handy in arguing by cases:

Example: We derive Q from $P \longrightarrow Q$ and $\sim P \longrightarrow Q$:

1.	│ $P \longrightarrow Q$	
2.	│ $\sim P \longrightarrow Q$	
	│ --------	
3.	│ $P \vee \sim P$	EM
4.	│ │ P	
	│ │---	
5.	│ │ $P \longrightarrow Q$	1, R
6.	│ │ Q	4,5 $\longrightarrow E$
7.	│ │ $\sim P$	
	│ │---	
8.	│ │ $\sim P \longrightarrow Q$	2, R
9.	│ │ Q	7,8 $\longrightarrow E$
10.	│ Q	3, 4-6, 7-9 SC

Another derived rule of inference is this, often called Modus Tollens.

Modus Tollens (MT)

│ .
│ .
│ .
│ $p \longrightarrow q$
│ .
│ .
│ $\sim q$
│ .
│ .
│ .
│ $\sim p$

Here are some more derived rules of inference which you might find handy in proving some of the homework problems.

DeMorgan's Laws: (DM)

│ $\neg(p \vee q)$	│ $\neg(p \wedge q)$	│ $(\sim p \wedge \sim q)$	│ $\sim p \vee \sim q$
│ .	│ .	│ .	│ .
│ .	│ .	│ .	│ .
│ .	│ .	│ .	│ .
│ $(\sim p \wedge \sim q)$	│ $\sim p \vee \sim q$	│ $\neg(p \vee q)$	│ $\neg(p \wedge q)$

Divergence (Div)

│ $p \leftrightarrow q$	│ $p \leftrightarrow q$
│ .	│ .
│ .	│ .
│ .	│ .
│ $\sim p$	│ $\sim q$
│ .	│ .
│ .	│ .
│ .	│ .
│ $\sim q$	│ $\sim p$