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RELAXATION TIME LIMITS OF SUBSONIC STEADY STATES FOR HYDRODYNAMIC MODEL OF SEMICONDUCTORS WITH SONIC OR NONSONIC BOUNDARY*

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Abstract. This paper concerns the relaxation time limits for the one-dimensional steady hydrodynamic model of semiconductors in the form of Euler–Poisson equations with sonic or nonsonic boundary. The sonic boundary is the critical and difficult case because of the degeneracy at the boundary and the formation of the boundary layers. In order to avoid the degeneracy of the second order derivatives, we technically introduce an invertible transform to the working equation. This guarantees that the remaining one order degeneracy becomes a good term since the transform used here is strictly increasing. Then we efficiently overcome the degenerate effect. When the relaxation time $\tau \to +\infty$, we first show the strong convergence of the approximate solutions to their asymptotic profiles in L^∞ norm with the order $O(\tau^{-\frac{1}{2}})$. When $\tau \to 0^+$, the boundary layer appears because the boundary data are not equal to each other, and we further derive the uniform error estimates of the approximate solutions to their background profiles in L^∞ norm with the order $O(\tau^{\frac{1}{2}})$ or $O(\tau)$ according to the different cases of boundary data. Unlike the methods adopted in the previous studies, we propose some altogether new techniques of the asymptotic limit analysis to successfully describe the width of the boundary layer, which is almost the order $O(\tau)$ provided $0 < \tau \ll 1$. These original approaches develop and improve the existing studies. Finally, some numerical simulations are carried out, which confirm our theoretical study, in particular, the appearance of boundary layers.

Key words. Euler–Poisson equations, sonic boundary, interior subsonic solutions, infinity-relaxation-time limit, zero-relaxation-time limit, boundary layers

MSC codes. 35B35, 35J70, 35L65, 35Q35

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1. Introduction.

The hydrodynamic model was first derived by Bløtekjær [3] for electrons in a semiconductor. After appropriate simplification the one-dimensional time-dependent system in the isentropic case reads

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$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2 + p(\rho)\big)_x = \rho E - \dfrac{\rho u}{\tau}, \\ \lambda^2 E_x = \rho - d(x), \end{cases} \tag{1.1}$$

where ρ, u, and E denote the electron density, velocity, and electric field, respectively. The function $p = p(\rho) = T\rho^\gamma$ is the pressure, where $T > 0$ is Boltzmann's constant and $\gamma \geq 1$ is the adiabatic exponent. The constant parameter $\tau > 0$ is the momentum relaxation time. The physical parameter $\lambda > 0$ represents the scaled Debye length. The given background density $d(x) > 0$ is called the doping profile standing for a background fixed charge of ions in the semiconductor crystal. The hydrodynamic model (1.1) is also called an Euler–Poisson system with semiconductor effect. For more details we refer to treatises [36, 49] and references therein.

In this paper, we aim at investigating the zero- and infinity-relaxation-time limits with a degenerate boundary, sonic or nonsonic, for this model in the case of one-dimensional isothermal steady-state flows satisfying equations

$$\begin{cases} J \equiv \text{constant}, \\ \left(\dfrac{J^2}{\rho} + p(\rho)\right)_x = \rho E - \dfrac{J}{\tau}, \\ \lambda^2 E_x = \rho - d(x). \end{cases} \tag{1.2}$$

Here, $J = \rho u$ stands for the current density, and $p(\rho) = T\rho$ corresponds to the isothermal ansatz. By the terminology from gas dynamics, we call $c := \sqrt{P'(\rho)} = \sqrt{T} > 0$ the speed of sound. The flow is referred to as subsonic, sonic, or supersonic provided the velocity satisfies

$$|u| < c, \quad |u| = c, \quad \text{or} \quad |u| > c, \quad \text{respectively.} \tag{1.3}$$

Without loss of generality, we set

$$T = 1, \quad \lambda = 1, \text{ and } J = 1, \tag{1.4}$$

and then for smooth solutions with $\rho > 0$, we have $u > 0$, and (1.2) is equivalently reduced to

$$\begin{cases} \left(\dfrac{1}{\rho} - \dfrac{1}{\rho^3}\right)\rho_x = E - \dfrac{1}{\tau\rho}, \\ E_x = \rho - d(x). \end{cases} \tag{1.5}$$

It follows from (1.3) and (1.4) that the flow is subsonic if $\rho > 1$, sonic if $\rho = 1$, or supersonic if $0 < \rho < 1$. Throughout this paper, we are interested in the system (1.5) on $(0, 1)$, which is subjected to the following boundary condition:

$$\rho(0) = a, \quad \rho(1) = b, \quad a, b \geq 1. \tag{1.6}$$

Then the problem we consider turns into

$$\begin{cases} \left(\left(\dfrac{1}{\rho} - \dfrac{1}{\rho^3}\right)\rho_x\right)_x = \rho - d(x) - \left(\dfrac{1}{\tau\rho}\right)_x, \quad x \in (0, 1), \\ \rho(0) = a, \quad \rho(1) = b. \end{cases} \tag{1.7}$$

Namely,

$$\begin{cases} \big(\omega(\rho)\big)_{xx} = \rho - d(x) - \dfrac{1}{\tau}\left(\dfrac{1}{\rho}\right)_x, & x \in (0,1), \\ \rho(0) = a, \quad \rho(1) = b, \end{cases} \tag{1.8}$$

where

$$\omega(\rho) = \ln \rho + \frac{1}{2\rho^2} \tag{1.9}$$

is a strictly increasing function on $[1, +\infty)$.

It also follows from (1.5) that

$$\begin{cases} E = \left(\dfrac{1}{\rho} - \dfrac{1}{\rho^3}\right)\rho_x + \dfrac{1}{\tau\rho}, \\ E_x = \rho - d(x). \end{cases} \tag{1.10}$$

We also assume that the doping profile $d(x)$ is of class $L^\infty(0,1)$, satisfying the subsonic condition $\underline{d} > 1$, where

$$\underline{d} := \inf_{x\in(0,1)} d(x) \text{ and } \bar{d} := \sup_{x\in(0,1)} d(x).$$

Regarding the relaxation time limit $\tau \to +\infty$ or $\tau \to 0^+$, let us denote by (ρ_τ, E_τ) the solution of (1.7) and (1.10) with respect to τ. In what follows, we consider limit problems about relaxation time τ in (1.7). On the one hand, when $\tau \to +\infty$, let us set

$$\bar{\rho} = \lim_{\tau\to+\infty} \rho_\tau \quad \text{and} \quad \bar{E} = \lim_{\tau\to+\infty} E_\tau,$$

which formally satisfy

$$\begin{cases} \left(\left(\dfrac{1}{\bar{\rho}} - \dfrac{1}{\bar{\rho}^3}\right)\bar{\rho}_x\right)_x = \bar{\rho} - d(x), & x \in (0,1), \\ \bar{\rho}(0) = a, \quad \bar{\rho}(1) = b, \end{cases} \tag{1.11}$$

and

$$\begin{cases} \bar{E} = \left(\dfrac{1}{\bar{\rho}} - \dfrac{1}{\bar{\rho}^3}\right)\bar{\rho}_x, \\ \bar{E}_x = \bar{\rho} - d(x). \end{cases} \tag{1.12}$$

It should be pointed out that there is no boundary layer in this case.

On the other hand, when $\tau \to 0^+$, we let $\underline{\rho} = \lim_{\tau\to0^+} \rho_\tau$, which formally satisfies that

$$\underline{\rho}(x) \equiv \text{constant}, \quad x \in (0,1), \tag{1.13}$$

which implies that there might be boundary layers in this case.

Background of study. We now draw a picture of the progress on the studies of well-posedness for a hydrodynamic model of semiconductors. There are major advances in the mathematical theory of steady-state Euler–Poisson equations with/without the semiconductor effect. For the subsonic steady-state flows, Degond and Markowich [8] first proved the existence of the subsonic solution to the one-dimensional steady-state Euler–Poisson equations with the semiconductor effect when its boundary states belong to the subsonic region. Subsequently, Degond and Markowich [9] further showed the existence and local uniqueness of irrotational subsonic flows to the three-dimensional steady-state semiconductor hydrodynamic model. After that, the steady-state subsonic flows were investigated in various physical boundary conditions and different dimensions [2, 14, 22, 39]. As for the supersonic steady-state flows, Peng and Violet [42] established the existence and uniqueness of the supersonic solutions with the semiconductor effect. Lately, Donatelli and Juhász [10] and Donatelli and Marcati [11] investigated the oscillations and defect measures for the quasi-neutral limit and the primitive equations. Regarding transonic steady states, Ascher et al. [1] first examined the existence of the transonic solution to the one-dimensional isentropic Euler–Poisson equations, and then Rosini [46] extended this work to the nonisentropic case. When the doping profile is nonconstant, Gamba [18, 19] studied the one-dimensional and two-dimensional transonic solutions with shocks, respectively. Luo et al. [32] and Luo and Xin [33] further considered the one-dimensional Euler–Poisson equations without the semiconductor effect; a comprehensive analysis on the structure and classification of steady states was carried out in [33]. Meanwhile, both structural and dynamical stability of steady transonic shock solutions was obtained in [32]. And then, He and Huang [24] and Huang et al. [25] studied the nonlinear stability of a large amplitude viscous shock wave and the stability of transonic contact discontinuity for two-dimensional steady compressible Euler flows in a finitely long nozzle. Recently, Li et al. [30, 31] explored the one-dimensional semiconductor Euler–Poisson equations with the sonic boundary condition. Motivated by their pioneering works [30, 31], there is a series of interesting generalizations into the transonic doping profile case in [4], the case of transonic C^∞-smooth steady states in [48], the multidimensional cases in [5, 6], and even the bipolar case [38]. Later on, Feng, Mei, and Zhang [17] demonstrated the structural stability of these smooth transonic steady states by the local singularity analysis.

In addition to these results on the well-posedness, a series of studies were concerned with the asymptotic limits in the hydrodynamic model, such as the Newtonian limits in the speed of light for the relativistic Euler–Poisson equations [34, 35, 37], the quasi-neutral limits [7, 12, 13, 26, 40, 41, 44, 47], the zero-electron-mass limits [20, 21, 29], and the zero-relaxation-time limits [16, 23, 27, 28, 43, 45], for instance. These investigations are important and amazing but do not involve the degeneracy, to the best of our knowledge. Recently, the quasi-neutral limit for a subsonic-sonic solution of system (1.5) with the degenerate sonic boundary was investigated by Chen et al. in [6].

However, noting the difficulty caused by the degeneracy and boundary layers, the relaxation time limit problem for subsonic steady states of system (1.5) with sonic or nonsonic boundary values is still open and challenging, as we know. Hence, the goal of this paper is to answer this question in two directions. The main results are stated as follows.

Main results. We are going to present our main results on the convergence of the original solutions to their asymptotic profiles as the relaxation time $\tau \to \infty$ and $\tau \to 0^+$, respectively.

THEOREM 1.1 (infinity-relaxation-time limit). *Assume that the doping profile $d \in L^\infty(0,1)$ is subsonic such that $\underline{d} > 1$. Let (ρ_τ, E_τ) be the interior subsonic solution* *of system* (1.7) *corresponding to the doping profile* $d(x)$. *Then problem* (1.7) *converges to* (1.11) *with* (1.12) *as* $\tau \to +\infty$ *uniformly in the sense that*

$$\|\rho_\tau - \bar{\rho}\|_{L^\infty(0,1)} \le C\tau^{-\frac{1}{2}} \quad \text{and} \quad \|E_\tau - \bar{E}\|_{H^1(0,1)} \le C\tau^{-\frac{1}{2}}, \tag{1.14}$$

where $C > 0$ *is a constant independent of* $\tau > 0$.

THEOREM 1.2 (zero-relaxation-time limits). *Assume that the doping profile* $d \in L^\infty(0,1)$ *is subsonic such that* $\underline{d} > 1$. *Let* (ρ_τ, E_τ) *be the interior subsonic solution of system* (1.7) *corresponding to the doping profile* $d(x)$. *Then the following two results hold:*

(I) *If* $\rho_\tau(0) = \rho_\tau(1) = a \ge 1$, *then* $\big(\rho_\tau(x), E_\tau(x)\big)$ *converges to its asymptotic state*

$$\big(\underline{\rho}(x), \underline{E}(x)\big) = \left(a, ax - \int_0^x d(s)ds\right)$$

as $\tau \to 0^+$, *without boundary layer for the density* ρ_τ, *but with a huge gap* $\frac{1}{\tau a}$ *between the electric field* E_τ *and its asymptotic state* $\underline{E}$ *over the entire interval* $[0,1]$. *Namely, there exists a constant* $C > 0$ *such that for all* $\tau > 0$ *the following error estimates hold:*

$$\|\rho_\tau - a\|_{L^\infty(0,1)} \le C\tau^{\frac{1}{2}} \quad \text{if } a = 1, \tag{1.15}$$

$$\|\rho_\tau - a\|_{L^\infty(0,1)} \le C\tau \quad \text{if } a > 1, \tag{1.16}$$

$$\left\| E_\tau - \underline{E} - \frac{1}{\tau a} \right\|_{L^\infty(0,1)} \le \begin{cases} C\tau^{\frac{1}{2}} & \text{if } a = 1, \\ C\tau & \text{if } a > 1, \end{cases} \tag{1.17}$$

and

$$\|(E_\tau - \underline{E})_x\|_{L^\infty(0,1)} \le \begin{cases} C\tau^{\frac{1}{2}} & \text{if } a = 1, \\ C\tau & \text{if } a > 1. \end{cases} \tag{1.18}$$

(II) *If* $\rho_\tau(0) = a$ *and* $\rho_\tau(1) = b$ *with* $b > a \ge 1$, *then the density* $\rho_\tau(x)$ *converges to its asymptotic state* $\underline{\rho}(x) \equiv a$ *outside the boundary layer, and the width of the boundary layer becomes thiner as* $\tau \to 0^+$; *the electric field* $E_\tau(x)$ *converges to its asymptotic state*

$$\underline{E}(x) = ax - \int_0^x d(s)ds$$

with a huge correction $\frac{1}{\tau a}$ *over the whole interval* $[0,1]$ *as* $\tau \to 0^+$. *Namely, there exist two constants* $C > 0$ *and* $0 < \tau_0 \ll 1$ *such that for all* $0 < \tau \le \tau_0$ *and* $0 < \varepsilon < 1/2$, *the following error estimates outside the boundary layer hold:*

$$|\rho_\tau(x) - a| \le C\tau \quad \forall x \in [0, 1 - \tau^{1-\varepsilon}] \quad \text{if } a > 1, \tag{1.19}$$

$$|\rho_\tau(x) - a| \le C\tau^{\frac{1}{2}} \quad \forall x \in [0, 1 - \tau^{1-\varepsilon}] \quad \text{if } a = 1, \tag{1.20}$$

and inside the boundary layer $(1 - \tau^{1-\varepsilon}, 1]$ *the density* ρ_τ *starts to rapidly approach the right boundary value* b; *in addition,*

$$\left\|E_\tau - \underline{E} - \frac{1}{\tau a}\right\|_{L^\infty(0,1)} \le \begin{cases} C\tau^{1-\varepsilon} & \text{if } a>1, \\ C\tau^{\frac{1}{2}} & \text{if } a=1, \end{cases} \tag{1.21}$$

and for $1 \le p < 2$,

$$\|(E_\tau - \underline{E})_x\|_{L^p(0,1)} \le \begin{cases} C\tau^{\frac{1-\varepsilon}{p}} & \text{if } a>1, \\ C\tau^{\frac{\eta}{p}} & \text{if } a=1, \end{cases} \tag{1.22}$$

where $\eta = \min\left\{1-\varepsilon, \frac{p}{2}\right\}$.

Remark 1.3. It should be pointed out that we assume $b > a \ge 1$ in case (II) of Theorem 1.2. For $a > b \ge 1$, we can use the same method here to get a similar result. We omit it for simplicity.

Difficulties and strategies. Now, let us mention the main difficult points of the study and outline our strategies in the proof.

In the first part, we investigate the relaxation time limit problem as $\tau \to +\infty$. Due to the boundary degeneracy, the study of the infinity-relaxation-time limit of interior subsonic solutions over $[0,1]$ appears to be challenging. If we use the usual method as that in [15], then we cannot remove the difficulty caused by boundary degeneracy and therefore the uniform estimates about the error function $\rho_\tau - \bar\rho$ in the parameter τ cannot be established. In order to overcome this difficulty, we use the invertible transform defined in (1.9) to reduce the two order degeneracy to the one order degeneracy. Fortunately, this remaining one order degeneracy is a good term since the transform used here is strictly increasing for $\rho \ge 1$. Then we efficiently overcome the degenerate effect.

In the second part, we study the relaxation time limit problem as $\tau \to 0^+$. First, we establish the estimates for n_x in Lemma 4.1 and those for $n - \omega(a)$ and $n - \omega(b)$ in Lemma 4.2, where $n = \omega(\rho_\tau)$. In the case $b > a$, the boundary layer must appear. We figure out the width of the boundary layer and prove a boundary layer estimate near the right endpoint $x = 1$ in Lemma 4.3 as follows:

$$\omega(\rho_\tau(x)) - \omega(a) \le \frac{\bar d}{M}\tau x, \quad x \in [0, 1-\tau^{1-\varepsilon}],$$

where $0 < \varepsilon < 1/2$ is a constant. Then, we study the zero-relaxation-time limit of (1.7) in the two cases.

Case 1. $\rho_\tau(0) = \rho_\tau(1) = a \ge 1$. There is no boundary layer effect in this case. By Lemma 4.2 and in view of the relation between $\omega(\rho_\tau(x)) - \omega(\rho_\tau(0))$ and $\rho_\tau(x) - \rho_\tau(0)$, we obtain the estimates for $\rho_\tau(x) - \rho_\tau(0)$ according to cases in which $a = 1$ or $a > 1$ (see (4.25) and (4.26)). Finally, we study the zero-relaxation-time limit for E_τ. Based on the Poisson equation in (1.5) and the relations between E_τ and ρ_τ (see (1.10) and (1.12)), we show the estimates for E_τ and $(E_\tau)_x$ (see (4.33) and (4.35)).

Case 2. $\rho_\tau(0) = a$ and $\rho_\tau(1) = b$ with $b > a \ge 1$. By Lemma 4.3 we establish the estimates about $\rho_\tau(x) - \rho_\tau(0)$ outside the boundary layer and then prove the estimates for E_τ and $(E_\tau)_x$ in (4.42) and (4.43), respectively.

We conclude this section by stating the arrangement of the rest of this paper. In section 2, we give the important preliminaries such as the existence and regularity of subsonic solutions. In section 3, we analyze the infinity-relaxation-time limit of subsonic steady states and prove Theorem 1.1. In section 4, we investigate zero-relaxation-time limits of subsonic steady states and finish the proof of Theorem 1.2. In section 5, we carry out some numerical simulations in different cases, which perfectly demonstrate our theoretical studies in Theorems 1.1 and 1.2. In particular, the boundary layers in Theorem 1.2 can be clearly observed in numerical forms.

2. Preliminaries. In this section we shall give the important preliminaries for later use. First, we recall the definition of the interior subsonic solution.

DEFINITION 2.1. *We say a pair of functions* (ρ_τ, E_τ) *is an interior subsonic solution of the boundary value problem* (1.5)–(1.6) *or* (1.7) *provided* (i) $(\rho_\tau - 1)^2 \in H_0^1(0,1)$, (ii) $\rho_\tau(x) > 1$ *for all* $x \in (0,1)$, (iii) *the following equality holds for all test functions* $\varphi \in H_0^1(0,1)$,

$$\int_0^1 \left(\frac{1}{\rho_\tau} - \frac{1}{\rho_\tau^3}\right)(\rho_\tau)_x \varphi_x dx + \frac{1}{\tau}\int_0^1 \frac{\varphi_x}{\rho_\tau} dx + \int_0^1 (\rho_\tau - d)\,\varphi dx = 0, \tag{2.1}$$

and (iv) $E_\tau(x)$ *is given by*

$$E_\tau(x) = E_\tau(0) + \int_0^x (\rho_\tau(y) - d(y))\,dy. \tag{2.2}$$

In addition, we continue to recall the existence and uniqueness of interior subsonic solutions, which is excerpted from the first part of Theorem 1.3 in [30].

PROPOSITION 2.2 (existence [30]). *Suppose that the doping profile* $d \in L^\infty(0,1)$ *is subsonic such that* $\underline{d} > 1$. *Then for all* $\tau \in (0,\infty]$ *the boundary value problem* (1.5)–(1.6) *or* (1.7) *admits a unique interior subsonic solution* $(\rho_\tau, E_\tau) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ *satisfying the bounds*

$$1 + m\sin(\pi x) \le \rho_\tau(x) \le \bar{d}, \quad x \in [0,1], \tag{2.3}$$

and

$$\begin{cases} C_1(1-x)^{\frac{1}{2}} \le \rho_\tau(x) - 1 \le C_2(1-x)^{\frac{1}{2}}, \\ -C_3(1-x)^{-\frac{1}{2}} \le (\rho_\tau)_x(x) \le -C_4(1-x)^{-\frac{1}{2}} \end{cases} \quad \textit{for } x \textit{ near } 1, \tag{2.4}$$

where $m = m(\tau, \underline{d}) > 0$, $C_2 > C_1 > 0$, *and* $C_3 > C_4 > 0$ *are certain uniform estimate constants.*

Next, from [30], the regularities of ρ_τ and $n = \omega(\rho_\tau)$ are stated as follows.

PROPOSITION 2.3 (regularity). *For* $1 \le p < 2$, *the subsonic solution* (ρ_τ, E_τ) *obtained in Proposition* 2.2 *satisfies the following properties:*

$$\rho_\tau \in C^{\frac{1}{2}}[0,1] \quad \textit{and} \quad \rho_\tau \in W^{1,p}(0,1). \tag{2.5}$$

And $n = \omega(\rho_\tau)$ *obtained by the transform* (1.9) *satisfies*

$$n \in C^{1,\frac{1}{2}}[0,1] \quad \textit{and} \quad n \in W^{2,p}(0,1). \tag{2.6}$$

3. Relaxation limit as $\tau \to +\infty$. This section is devoted to proving our main result when $\tau \to +\infty$. For the sake of simplicity, we only consider the case in which $a = b = 1$. For all constants $a \geq 1$ and $b \geq 1$, by using the following method, we can get similar results as (1.14) in Theorem 1.1. Then problems (1.8) and (1.11) are, respectively, equivalent to

$$\begin{cases} (\omega(\rho_\tau))_{xx} = \rho_\tau - d(x) - \left(\dfrac{1}{\tau\rho_\tau}\right)_x, \quad x \in (0,1), \\ \omega(\rho_\tau(0)) = \omega(\rho_\tau(1)) = \dfrac{1}{2}, \end{cases} \tag{3.1}$$

and

$$\begin{cases} (\omega(\bar{\rho}))_{xx} = \bar{\rho} - d(x), \quad x \in (0,1), \\ \omega(\bar{\rho}(0)) = \omega(\bar{\rho}(1)) = \dfrac{1}{2}. \end{cases} \tag{3.2}$$

Proof of Theorem 1.1. Let us set

$$V = \omega(\bar{\rho}) - \omega(\rho).$$

By taking the difference between (3.1) and (3.2), we obtain

$$\begin{cases} V_{xx} - (\bar{\rho} - \rho_\tau) = \left(\dfrac{1}{\tau\rho_\tau}\right)_x, \quad x \in (0,1), \\ V|_{x=0} = V|_{x=1} = 0. \end{cases} \tag{3.3}$$

Multiplying the first equation of (3.3) by V and integrating the resulting equation over $[0,1]$, we have

$$\int_0^1 V_{xx} V\,dx - \int_0^1 (\bar{\rho} - \rho_\tau) V\,dx = \int_0^1 \left(\frac{1}{\tau\rho_\tau}\right)_x V\,dx. \tag{3.4}$$

By integration by parts, we get

$$\int_0^1 |V_x|^2 dx + \int_0^1 (\bar{\rho} - \rho_\tau) V\,dx = \int_0^1 \frac{1}{\tau\rho_\tau} V_x dx. \tag{3.5}$$

Noting the strict monotonicity of the function ω, we obtain

$$\int_0^1 (\bar{\rho} - \rho_\tau) V\,dx = \int_0^1 (\bar{\rho} - \rho_\tau)(\omega(\bar{\rho}) - \omega(\rho_\tau)) dx \geq 0. \tag{3.6}$$

Then it follows from (3.5)–(3.6), (2.3), and the Hölder inequality that

$$\int_0^1 |V_x|^2 dx \leq \frac{1}{\tau}\int_0^1 \frac{1}{\rho_\tau} V_x dx \leq \frac{1}{\tau}\int_0^1 |V_x| dx \leq \frac{1}{\tau}\left(\int_0^1 |V_x|^2\right)^{\frac{1}{2}} dx. \tag{3.7}$$

Hence,

$$\|V_x\|_{L^2(0,1)} \leq \tau^{-1}. \tag{3.8}$$

The Poincaré inequality implies that

$$\|V\|_{L^2(0,1)} \leq C\|V_x\|_{L^2(0,1)} \leq C\tau^{-1}. \tag{3.9}$$

Then it follows from the Sobolev imbedding theorems that

$$\|V\|_{L^\infty(0,1)} \le C\|V\|_{H^1(0,1)} \le C\tau^{-1}. \tag{3.10}$$

By (1.9) and the Taylor series of $\omega(\rho_\tau)$ and $\omega(\bar\rho)$ at $\rho_\tau = 1$ and $\bar\rho = 1$, respectively, we have

$$V = \omega(\bar\rho) - \omega(\rho_\tau) = \left(\ln\bar\rho + \frac{1}{2\bar\rho^2}\right) - \left(\ln\rho_\tau + \frac{1}{2\rho_\tau^2}\right) = (\bar\rho - \rho_\tau)(\bar\rho + \rho_\tau - 2) + \cdots. \tag{3.11}$$

Then in view of (2.3), we obtain

$$|V| = |\omega(\bar\rho) - \omega(\rho_\tau)| \ge \begin{cases} C|\rho_\tau - \bar\rho|^2 & \text{as} \quad \rho_\tau \to 1, \bar\rho \to 1, \\ C|\rho_\tau - \bar\rho| & \text{otherwise.} \end{cases} \tag{3.12}$$

Therefore, the first part of (1.14) follows by combining (3.10) and (3.12).

Next, we study the infinity-relaxation-time limit for E_τ. It follows from (1.5), (1.12), and (1.9) that

$$(\omega(\rho_\tau))_x = E_\tau - \frac{1}{\tau\rho_\tau} \quad \text{and} \quad (\omega(\bar\rho))_x = \bar E. \tag{3.13}$$

Then, by noting (2.3) and (3.8), we get

$$\begin{aligned} \|E_\tau - \bar E\|_{L^2(0,1)} &\le \|(\omega(\bar\rho) - \omega(\rho_\tau))_x\|_{L^2(0,1)} + \left\|\frac{1}{\tau\rho_\tau}\right\|_{L^2(0,1)} \\ &\le \|V_x\|_{L^2(0,1)} + \tau^{-1}\left\|\frac{1}{\rho_\tau}\right\|_{L^2(0,1)} \\ &\le C\tau^{-1}. \end{aligned} \tag{3.14}$$

Moreover, by the second equation of (1.5), we have

$$(E_\tau)_x(x) = \rho_\tau(x) - d(x) \quad \text{and} \quad \bar E_x(x) = \bar\rho(x) - d(x). \tag{3.15}$$

Then, by the first part of (1.14) and (3.12), we get

$$\|(E_\tau - \bar E)_x\|_{L^\infty(0,1)} = \|\rho_\tau - \bar\rho\|_{L^\infty(0,1)} \le C\|V\|^{\frac12}_{L^\infty(0,1)} \le C\tau^{-\frac12}. \tag{3.16}$$

This, together with (3.14), implies

$$\|E_\tau - \bar E\|_{H^1(0,1)} \le C\tau^{-\frac12}, \tag{3.17}$$

which is the second part of (1.14). The proof of Theorem 1.1 is completed. ∎

4. Relaxation limits as $\tau \to 0^+$. The main task of this section is to prove our main results when $\tau \to 0^+$. From the first equation of (1.7), we obtain that $\underline{\rho} = \lim_{\tau\to 0^+} \rho_\tau$ is a constant over the entire interval $[0,1]$. Hence, the boundary layers must appear if $a \ne b$. Here, we use a new method to look at the width of a boundary layer.

With the invertible transform ω defined in (1.9), the problem (1.7) is equivalent to

$$\begin{cases} n_{xx} = \rho_\tau - d(x) + \dfrac{1}{\tau}\dfrac{1}{\rho_\tau^2} f'(n) n_x, & x \in (0,1), \\ n(0) = \omega(a), \quad n(1) = \omega(b), \end{cases} \tag{4.1}$$

where f is the inverse transform of ω and satisfies $f'(n) > 0$ for $n = \omega(\rho_\tau)$. Let us define

$$\alpha = \frac{1}{\rho_\tau^2} f'(n), \quad \beta = \rho_\tau - d,$$

which are functions of x. It follows from (1.9) and (2.3) that there exists a constant $M > 0$ independent of $\tau > 0$ such that

$$\alpha \geq M \quad \text{on } [0,1]. \tag{4.2}$$

First, we show the estimates of n_x as follows.

Lemma 4.1. *Assume that the doping profile $d \in L^\infty(0,1)$ is subsonic such that $\underline{d} > 1$. Let n be the solution to the initial value problem (4.1) and $x_0 \in [0,1)$ be an initial point. We denote by $\omega_0 = n_x(x_0)$. Then the following properties hold:*

$$n_x(x) \geq e^{\frac{A(x,x_0)}{\tau}} \Big(\omega_0 - \frac{\bar{d}-1}{M}\tau\Big) + \frac{\bar{d}-1}{M}\tau \quad \forall x \in [x_0, 1] \tag{4.3}$$

and

$$n_x(x) \leq e^{\frac{A(x,x_0)}{\tau}} \Big(\omega_0 + \frac{\bar{d}-1}{M}\tau\Big) - \frac{\bar{d}-1}{M}\tau \quad \forall x \in [x_0, 1], \tag{4.4}$$

where

$$A(x, x_0) = \int_{x_0}^x \alpha(y) dy.$$

Proof. With the definition of α and β, n_x satisfies the following initial value problem for a linear first order differential equation:

$$\begin{cases} (n_x)_x - \dfrac{\alpha}{\tau} n_x = \beta, \quad x \in (x_0, 1], \\ n_x(x_0) = \omega_0. \end{cases}$$

The solution is given by formula

$$n_x(x) = \omega_0 e^{\frac{A(x,x_0)}{\tau}} + r(x), \tag{4.5}$$

where

$$r(x) = e^{\frac{A(x,x_0)}{\tau}} \int_{x_0}^x \beta(y) e^{-\frac{A(y,x_0)}{\tau}} dy.$$

Obviously,

$$1 - \bar{d} \leq \beta \leq \bar{d} - 1 \quad \text{on } [0,1]$$

and

$$A(x, x_0) \geq M(x - x_0) \quad \forall x \in [x_0, 1].$$

Since $1 - \bar{d} < 0$, we have

$$\beta(y) e^{-\frac{A(y,x_0)}{\tau}} \geq \big(1 - \bar{d}\big) e^{-\frac{A(y,x_0)}{\tau}} \geq \big(1 - \bar{d}\big) e^{-\frac{M(y-x_0)}{\tau}}.$$

Therefore,

$$\begin{aligned} r(x) &\geq \big(1-\bar{d}\big)e^{\frac{A(x,x_0)}{\tau}}\int_{x_0}^{x} e^{-\frac{M(y-x_0)}{\tau}}dy \\ &= \big(1-\bar{d}\big)e^{\frac{A(x,x_0)}{\tau}}\frac{\tau}{M}\Big(1-e^{-\frac{M(x-x_0)}{\tau}}\Big) \\ &= -\frac{\big(\bar{d}-1\big)\tau}{M}e^{\frac{A(x,x_0)}{\tau}} + \frac{\bar{d}-1}{M}\tau e^{\frac{A(x,x_0)-M(x-x_0)}{\tau}}. \end{aligned}$$

Since

$$\bar{d}-1>0, \qquad A(x,x_0)-M(x-x_0)\geq 0,$$

we obtain

$$r(x)\geq -\frac{\big(\bar{d}-1\big)\tau}{M}e^{\frac{A(x,x_0)}{\tau}} + \frac{\bar{d}-1}{M}\tau.$$

This together with (4.5) yields (4.3).

Similarly, we have

$$\beta(y)e^{-\frac{A(y,x_0)}{\tau}} \leq \big(\bar{d}-1\big)e^{-\frac{M(y-x_0)}{\tau}}.$$

Therefore,

$$\begin{aligned} r(x) &\leq \big(\bar{d}-1\big)e^{\frac{A(x,x_0)}{\tau}}\int_{x_0}^{x} e^{-\frac{M(y-x_0)}{\tau}}dy \\ &= \frac{\big(\bar{d}-1\big)\tau}{M}e^{\frac{A(x,x_0)}{\tau}} - \frac{\bar{d}-1}{M}\tau e^{\frac{A(x,x_0)-M(x-x_0)}{\tau}} \\ &\leq \frac{\big(\bar{d}-1\big)\tau}{M}e^{\frac{A(x,x_0)}{\tau}} - \frac{\bar{d}-1}{M}\tau, \end{aligned}$$

which proves (4.4). ◻

The next result concerns the estimates for $n-\omega(a)$ and $n-\omega(b)$.

LEMMA 4.2. *Let $b\geq a\geq 1$. Under the assumptions of Lemma 4.1, we have*

$$\omega(\rho_\tau(x))-\omega(a)\geq -\frac{\bar{d}-1}{M}\tau x \quad \forall x\in[0,1] \tag{4.6}$$

and

$$\omega(\rho_\tau(x))-\omega(b)\leq \frac{\bar{d}-1}{M}\tau \quad \forall x\in[0,1]. \tag{4.7}$$

If furthermore, $b=a$, then

$$\omega(\rho_\tau(x))-\omega(a)\leq \frac{\bar{d}-1}{M}\tau x \quad \forall x\in[0,1]. \tag{4.8}$$

Proof. We prove this lemma by contradiction. If (4.6) does not hold, there exists $x_1\in(0,1]$ such that

$$\omega(\rho_\tau(x_1))-\omega(a)< -\frac{\bar{d}-1}{M}\tau x_1. \tag{4.9}$$

By the mean value theorem, there is $\xi_1 \in (0, x_1)$ such that

$$n_x(\xi_1) = \frac{\omega(\rho_\tau(x_1)) - \omega(\rho_\tau(0))}{x_1} < -\frac{\bar{d}-1}{M}\tau. \tag{4.10}$$

Applying (4.4), for all $x \in [\xi_1, 1]$, we have

$$n_x(x) \leq e^{\frac{A(x,\xi_1)}{\tau}} \left(n_x(\xi_1) + \frac{\bar{d}-1}{M}\tau \right) - \frac{\bar{d}-1}{M}\tau < -\frac{\bar{d}-1}{M}\tau < 0, \tag{4.11}$$

which implies that n is strictly decreasing on $[\xi_1, 1]$. It follows from (4.9) that

$$n(1) \leq n(x_1) = \omega(\rho_\tau(x_1)) < \omega(a).$$

On the other hand, ω is strictly increasing on $[1, +\infty)$ and $b \geq a$, and hence

$$n(1) = \omega(b) \geq \omega(a).$$

This is contradictory and (4.6) follows.

Next, we prove (4.7). If it does not hold, then there exists $x_2 \in (0, 1)$ such that

$$\omega(\rho_\tau(x_2)) > \omega(b) + \frac{\bar{d}-1}{M}\tau > \omega(b). \tag{4.12}$$

Since ω is strictly increasing on $[1, +\infty)$, we have $\rho_\tau(x_2) > b$. By the continuity of ρ_τ, there is $y_2 \in [0, x_2)$ such that $\rho_\tau(y_2) = b$. It follows from the mean value theorem that there is $\xi_2 \in (y_2, x_2) \subset (0, 1)$ such that

$$n_x(\xi_2) = \frac{\omega(\rho_\tau(x_2)) - \omega(b)}{x_2 - y_2} \geq \frac{\bar{d}-1}{(x_2 - y_2)M}\tau \geq \frac{\bar{d}-1}{M}\tau. \tag{4.13}$$

Applying (4.3), for all $x \in [\xi_2, 1]$, we have

$$n_x(x) \geq e^{\frac{A(x,\xi_2)}{\tau}} \left(n_x(\xi_2) - \frac{\bar{d}-1}{M}\tau \right) + \frac{\bar{d}-1}{M}\tau \geq \frac{\bar{d}-1}{M}\tau > 0. \tag{4.14}$$

This contradicts (4.12) since $x_2 \in [\xi_2, 1)$ and $n(x_2) > n(1) = \omega(b)$. Hence we get (4.7).

Now we prove (4.8) in a similar way. If it is false, there is $x_3 \in (0, 1]$ such that

$$\omega(\rho_\tau(x_3)) - \omega(a) > \frac{\bar{d}-1}{M}\tau x_3. \tag{4.15}$$

By the mean value theorem, there exists $\xi_3 \in (0, x_3)$ such that

$$n_x(\xi_3) = \frac{\omega(\rho_\tau(x_3)) - \omega(\rho_\tau(0))}{x_3} > \frac{\bar{d}-1}{M}\tau. \tag{4.16}$$

In view of (4.3), for all $x \in [\xi_3, 1]$, we obtain

$$n_x(x) \geq e^{\frac{A(x,\xi_3)}{\tau}} \left(n_x(\xi_3) - \frac{\bar{d}-1}{M}\tau \right) + \frac{\bar{d}-1}{M}\tau \geq \frac{\bar{d}-1}{M}\tau > 0, \tag{4.17}$$

which implies that n is strictly increasing on $[\xi_3, 1]$. Since $x_3 \in (\xi_3, 1]$, it follows from (4.15) that

$$n(1) \geq n(x_3) = \omega(\rho_\tau(x_3)) > \omega(a).$$

This contradicts the fact that

$$n(1) = \omega(b) = \omega(a),$$

and hence (4.8) is proved. ∎

Now we investigate the boundary layer near the right endpoint $x = 1$ in case $b > a \geq 1$. We assume that the width of boundary layer is $\tau^{1-\varepsilon}$, where $0 < \varepsilon < 1/2$ is a constant.

LEMMA 4.3. *Let $b > a \geq 1$. Under the assumptions of Lemma 4.1, we have*

$$\omega(\rho_\tau(x)) - \omega(a) \leq \frac{\bar{d}}{M}\tau x \quad \forall x \in [0, 1-\tau^{1-\varepsilon}]. \tag{4.18}$$

Proof. Indeed, if (4.18) is not correct, there is $x_4 \in (0, 1-\tau^{1-\varepsilon}]$ such that

$$\omega(\rho_\tau(x_4)) - \omega(a) > \frac{\bar{d}}{M}\tau x_4. \tag{4.19}$$

It follows from the mean value theorem that there exists $\xi_4 \in (0, x_4)$ such that

$$n_x(\xi_4) = \frac{\omega(\rho_\tau(x_4)) - \omega(\rho_\tau(0))}{x_4} > \left(\frac{1}{M} + \frac{\bar{d}-1}{M}\right)\tau. \tag{4.20}$$

In view of (4.3), for all $x \in [\xi_4, 1]$, we obtain

$$\begin{aligned} n_x(x) &\geq e^{\frac{A(x,\xi_4)}{\tau}}\left(n_x(\xi_4) - \frac{\bar{d}-1}{M}\tau\right) + \frac{\bar{d}-1}{M}\tau \\ &\geq \frac{\tau}{M}e^{\frac{A(x,\xi_4)}{\tau}} \\ &\geq \frac{\tau}{M}e^{\frac{M}{\tau}(x-\xi_4)}, \end{aligned} \tag{4.21}$$

which implies

$$\begin{aligned} \omega(\rho_\tau(1)) - \omega(\rho_\tau(\xi_4)) &= \int_{\xi_4}^1 n_x(x)dx \\ &\geq \frac{\tau}{M}\int_{\xi_4}^1 e^{\frac{M}{\tau}(x-\xi_4)}dx \\ &= \frac{\tau^2}{M^2}\left(e^{M\tau^{-\varepsilon}} - 1\right) \to +\infty \quad \text{as } \tau \to 0^+. \end{aligned} \tag{4.22}$$

This is impossible and then (4.18) follows. ∎

COROLLARY 4.4. *Let $b \geq a \geq 1$. Under the assumptions of Lemma 4.1, we have*

$$\left|n_x(0)\right| \leq \frac{\bar{d}}{M}\tau. \tag{4.23}$$

Proof. From (4.6) and (4.18), we have

$$\left|\frac{n(x) - n(0)}{x}\right| \leq \frac{\bar{d}}{M}\tau \quad \forall x \in (0, 1-\tau^{1-\varepsilon}],$$

which imlies (4.23). ∎

Proof of Theorem 1.2. We prove this theorem in two cases.

Case 1. $\rho_\tau(0)=\rho_\tau(1)=a\geq 1$, namely, $\omega(\rho_\tau(0))=\omega(\rho_\tau(1))=\omega(a)$.

We first prove (1.15) and (1.16), which show that the limit of $\rho_\tau(x)$ is $\underline{\rho}(x)\equiv a$ over $[0,1]$. Indeed, from (4.6) and (4.8), we have

$$\big|\omega(\rho_\tau(x))-\omega(a)\big|\leq \frac{\bar{d}-1}{M}\tau \quad \forall x\in[0,1]. \tag{4.24}$$

Recalling the Taylor expansion for $\omega(\bar{\rho})-\omega(\rho)$ in (3.11) and using (4.24), for all $x\in[0,1]$, we have

$$|\rho_\tau(x)-\rho_\tau(0)|\leq C\sqrt{|\omega(\rho_\tau(x))-\omega(\rho_\tau(0))|}\leq C\tau^{\frac{1}{2}} \quad \text{if} \quad a=1 \tag{4.25}$$

and

$$|\rho_\tau(x)-\rho_\tau(0)|\leq C\,|\omega(\rho_\tau(x))-\omega(\rho_\tau(0))|\leq C\tau \qquad \text{if} \quad a>1. \tag{4.26}$$

Hence (1.15) and (1.16) follow.

Next, we consider the zero-relaxation-time limit for E_τ. From (1.9) and (1.10), we have

$$\begin{cases} n_x=E_\tau-\dfrac{1}{\tau\rho_\tau},\\ (E_\tau)_x=\rho_\tau-d(x), \end{cases} \tag{4.27}$$

which implies

$$E_\tau(0)=n_x(0)+\frac{1}{\tau\rho_\tau(0)}=n_x(0)+\frac{1}{\tau a}. \tag{4.28}$$

It follows from the second equation of (4.27) that

$$E_\tau(x)=E_\tau(0)+\int_0^x(\rho_\tau(s)-d(s))ds=\frac{1}{\tau a}-D(x)+n_x(0)+\int_0^x\rho_\tau(s)ds, \tag{4.29}$$

where

$$D(x)=\int_0^x d(s)ds, \quad x\in[0,1]. \tag{4.30}$$

For the last term on the right-hand side of (4.29), a straightforward computation gives

$$\int_0^x\rho_\tau(s)ds=\int_0^x\rho_\tau(0)ds+\int_0^x\left(\rho_\tau(s)-\rho_\tau(0)\right)ds=ax+\int_0^x\left(\rho_\tau(s)-\rho_\tau(0)\right)ds. \tag{4.31}$$

In view of (4.25)–(4.26), we obtain

$$\left|\int_0^x\left(\rho_\tau(s)-\rho_\tau(0)\right)ds\right|\leq\int_0^x|\rho_\tau(s)-\rho_\tau(0)|\,ds\leq\begin{cases} C\tau x\leq C\tau & \text{if} \quad \rho_\tau(0)>1,\\ C\tau^{\frac{1}{2}} & \text{if} \quad \rho_\tau(0)=1. \end{cases} \tag{4.32}$$

Combining (4.29) and (4.32) together with (4.23), we have

$$\left|E_\tau(x)-\left(\frac{1}{\tau a}-D(x)+ax\right)\right|\le\begin{cases}C\tau & \text{if}\quad \rho_\tau(0)>1,\\ C\tau^{\frac12} & \text{if}\quad \rho_\tau(0)=1,\end{cases}\tag{4.33}$$

which gives (1.17).

Furthermore, it follows from the second equation of (4.27) that

$$(E_\tau)_x(x)=\rho_\tau(x)-d(x)=\rho_\tau(x)-\rho_\tau(0)+\rho_\tau(0)-d(x).\tag{4.34}$$

This, together with (4.25) and (4.26), gives

$$|(E_\tau)_x(x)-(\rho_\tau(0)-d(x))|=|\rho_\tau(x)-\rho_\tau(0)|\le\begin{cases}C\tau & \text{if}\quad \rho_\tau(0)>1,\\ C\tau^{\frac12} & \text{if}\quad \rho_\tau(0)=1,\end{cases}\tag{4.35}$$

which is (1.18).

Case 2. $\rho_\tau(0)=a$ and $\rho_\tau(1)=b$ with $b>a\ge 1$.

From (4.6) and (4.18), we have

$$|\omega(\rho_\tau(x))-\omega(a)|\le\frac{\bar d}{M}\tau\quad \forall x\in[0,1-\tau^{1-\varepsilon}]\quad \text{if } 0<\tau\ll 1.\tag{4.36}$$

When $\rho_\tau(0)=a>1$, by (4.36) and (3.12), we have

$$|\rho_\tau(x)-\rho_\tau(0)|\le\frac{C\bar d}{M}\tau\quad \forall x\in[0,1-\tau^{1-\varepsilon}],\tag{4.37}$$

which implies (1.19). When $\rho_\tau(0)=a=1$, by (4.36) and (3.12) again, we obtain

$$|\rho_\tau(x)-\rho_\tau(0)|\le C\tau^{\frac12}\quad \forall x\in[0,1-\tau^{1-\varepsilon}],\tag{4.38}$$

which gives (1.20).

Next, we consider the zero-relaxation-time limit for E_τ. From (4.28) and the second equation of (4.27), we have

$$\begin{aligned}E_\tau(x)&=E_\tau(0)+\int_0^x(\rho_\tau(s)-d(s))ds\\ &=n_x(0)+\frac{1}{\tau a}+\int_0^x(\rho_\tau(s)-d(s))ds\\ &=\frac{1}{\tau a}-D(x)+n_x(0)+\int_0^x(\rho_\tau(s)-\rho_\tau(0))\,ds+\int_0^x\rho_\tau(0)ds,\end{aligned}\tag{4.39}$$

where $D(x)$ is defined by (4.30). Then it follows from (4.39) that

$$E_\tau(x)-\left(\frac{1}{\tau a}-D(x)+\int_0^x\rho_\tau(0)ds\right)=n_x(0)+\int_0^x(\rho_\tau(s)-\rho_\tau(0))\,ds.\tag{4.40}$$

From (4.37) and (4.38), we have

$$\begin{aligned}\left|\int_0^x (\rho_\tau(s)-\rho_\tau(0))\,ds\right| &\le \int_0^1 |\rho_\tau(s)-\rho_\tau(0)|\,ds\\ &\le \int_0^{1-\tau^{1-\varepsilon}} |\rho_\tau(s)-\rho_\tau(0)|\,ds + \int_{1-\tau^{1-\varepsilon}}^1 |\rho_\tau(s)-\rho_\tau(0)|\,ds\\ &\le \begin{cases}\dfrac{\bar d}{M}\tau + C(\bar d,a,b)\tau^{1-\varepsilon} \le C\tau^{1-\varepsilon} & \text{if } a>1,\\ C\tau^{\frac12} + C(\bar d,a,b)\tau^{1-\varepsilon} \le C\tau^{\frac12} & \text{if } a=1.\end{cases}\end{aligned} \tag{4.41}$$

This, together with (4.23) and (4.40), implies

$$\left|E_\tau(x) - \left(\frac{1}{\tau a} - D(x) + ax\right)\right| \le \begin{cases}\dfrac{\bar d}{M}\tau + C(\bar d,a,b)\tau^{1-\varepsilon} \le C\tau^{1-\varepsilon} & \text{if } a>1,\\ C\tau^{\frac12} + C(\bar d,a,b)\tau^{1-\varepsilon} \le C\tau^{\frac12} & \text{if } a=1.\end{cases} \tag{4.42}$$

Then (1.21) follows.

Moreover, it follows from (4.34) and (4.41) that for $1 \le p < 2$,

$$\begin{aligned}&\|(E_\tau)_x - (\rho_\tau(0)-d)\|_{L^p(0,1)}\\ &\quad= \left(\int_0^1 |\rho_\tau(x)-\rho_\tau(0)|^p\,dx\right)^{\frac1p}\\ &\quad= \left(\int_0^{1-\tau^{1-\varepsilon}} |\rho_\tau(x)-\rho_\tau(0)|^p\,dx + \int_{1-\tau^{1-\varepsilon}}^1 |\rho_\tau(x)-\rho_\tau(0)|^p\,dx\right)^{\frac1p}\\ &\quad\le \begin{cases}\left(\displaystyle\int_0^1 \frac{\bar d^p}{M^p}\tau^p dx + C(\bar d,a,b,p)\tau^{1-\varepsilon}\right)^{\frac1p} \le C\tau^{\frac{1-\varepsilon}{p}} & \text{if } a>1,\\ \left(\displaystyle\int_0^1 \frac{\bar d^{\frac p2}}{M^{\frac p2}}\tau^{\frac p2} dx + C(\bar d,a,b,p)\tau^{1-\varepsilon}\right)^{\frac1p} \le C\tau^{\frac{\eta}{p}} & \text{if } a=1,\end{cases}\end{aligned} \tag{4.43}$$

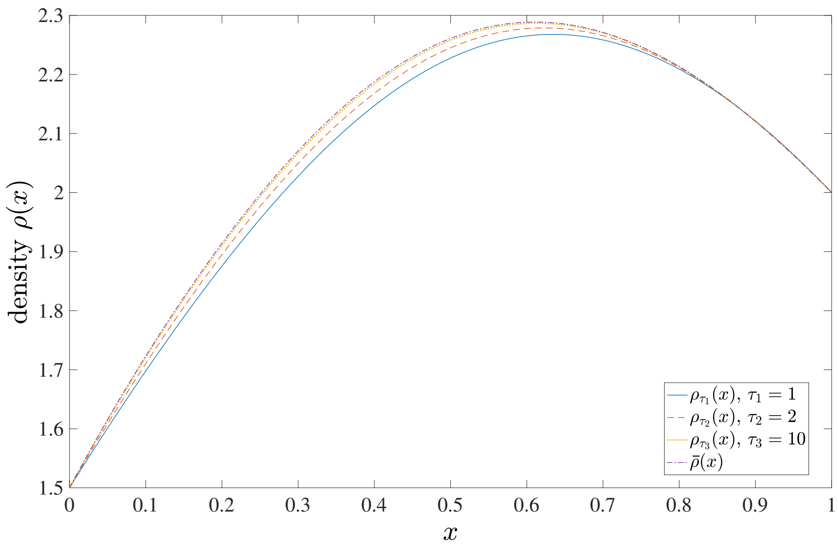
where $\eta = \min\left\{1-\varepsilon, \frac p2\right\}$. This is (1.22). The proof of Theorem 1.2 is finished. $\square$

5. Numerical simulations. In this section, we engage in the numerical verification of our theoretical results. For this purpose, we choose the following doping profile in the problem (1.5) and (1.6),

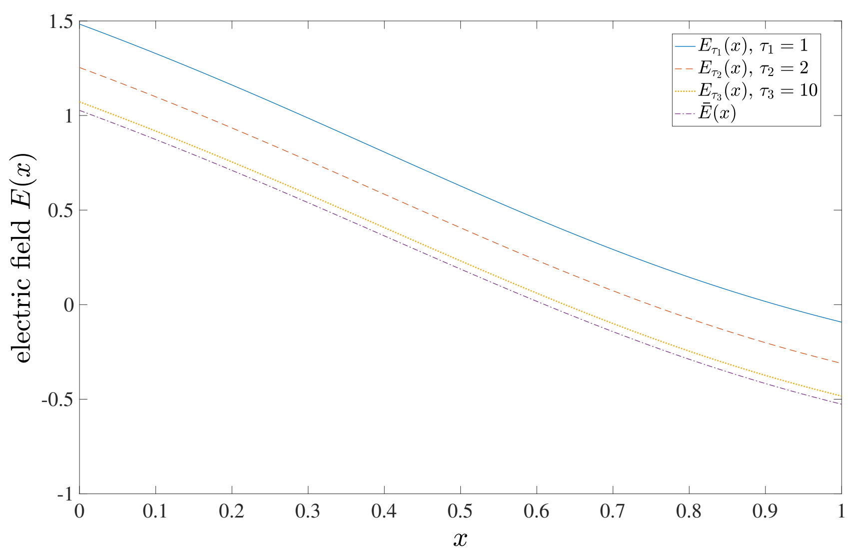
$$d(x) = 3 + \sin(\pi x), \quad x \in [0,1], \tag{5.1}$$

which is unified for the simulations of results in both Theorems 1.1 and 1.2.

First, Theorem 1.1 tells us that the subsonic solution to the problem (1.5) and (1.6) uniformly converges toward the one to the problem (1.11) and (1.12) as the relaxation time τ tends to $+\infty$. This result holds for all boundary data $a, b \ge 1$, and



(a) $\|\rho_\tau - \bar{\rho}\|_{L^\infty(0,1)} \leq C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.

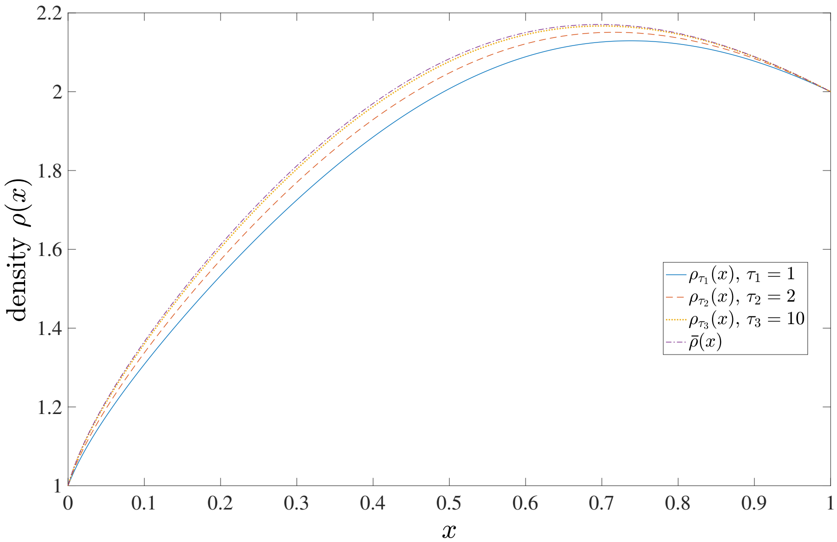


(b) $\|E_\tau - \bar{E}\|_{L^\infty(0,1)} \leq C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.

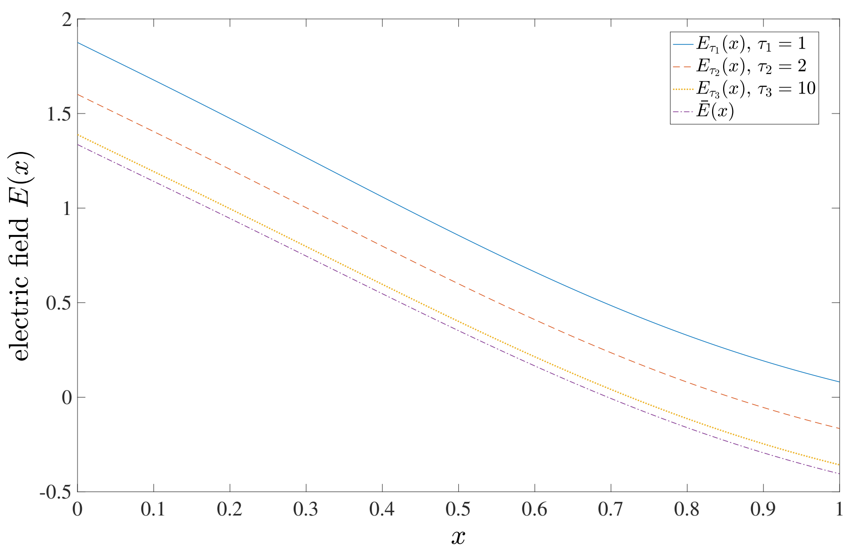
FIG. 1. *Case* 1 *for Theorem* 1.1*:* $a = 1.5$ *and* $b = 2$*.*

as such, we consider the three numerical cases of the boundary data a and b in (1.6) as follows:

1. $a = 1.5$ and $b = 2$; see Figure 1.
2. $a = 1$ and $b = 2$; see Figure 2.
3. $a = b = 1$; see Figure 3.



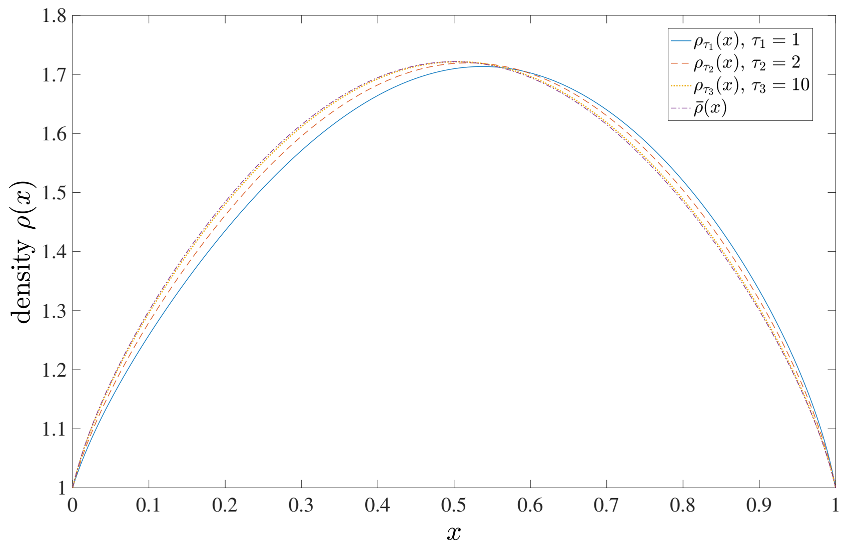
(a) $\|\rho_\tau - \bar{\rho}\|_{L^\infty(0,1)} \le C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.



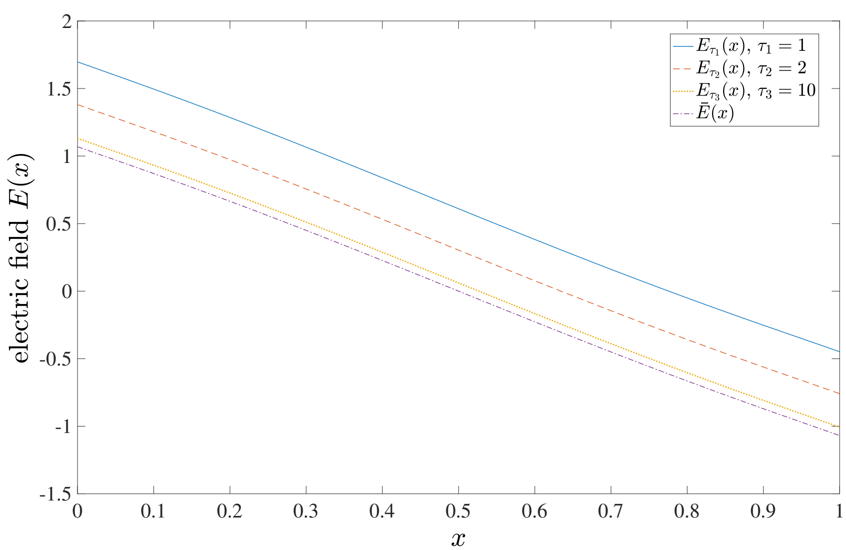
(b) $\|E_\tau - \bar{E}\|_{L^\infty(0,1)} \le C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.

FIG. 2. *Case* 2 *for Theorem* 1.1*:* $a = 1$ *and* $b = 2$.

In all the cases above, we set up the finite approximation sequence of relaxation times as $\tau_1 = 1$, $\tau_2 = 2$, and $\tau_3 = 10$; and the limiting relaxation time τ is $+\infty$. It is worth pointing out that when the relaxation time τ takes the value which is greater than 10, the solution (ρ_τ, E_τ) is already in close proximity to the limiting solution $(\bar{\rho}, \bar{E})$;



(a) $\|\rho_\tau - \bar{\rho}\|_{L^\infty(0,1)} \leq C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.

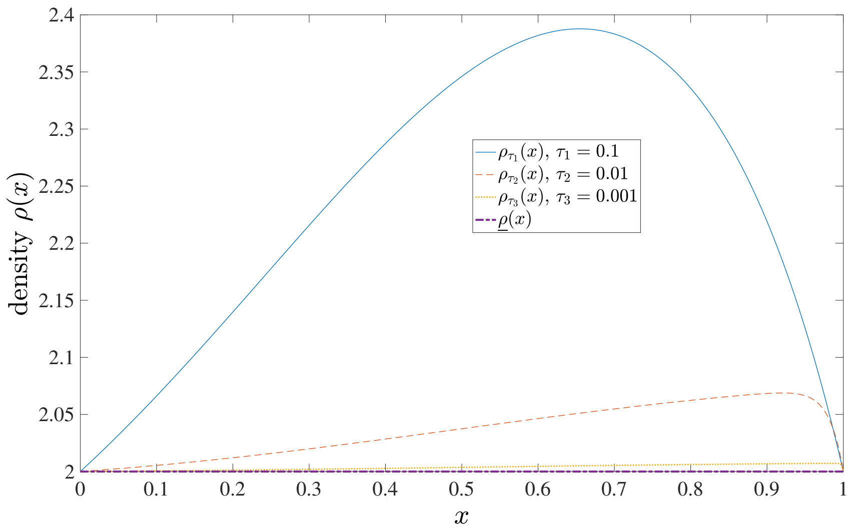


(b) $\|E_\tau - \bar{E}\|_{L^\infty(0,1)} \leq C\tau^{-\frac{1}{2}}$, as $\tau \to +\infty$.

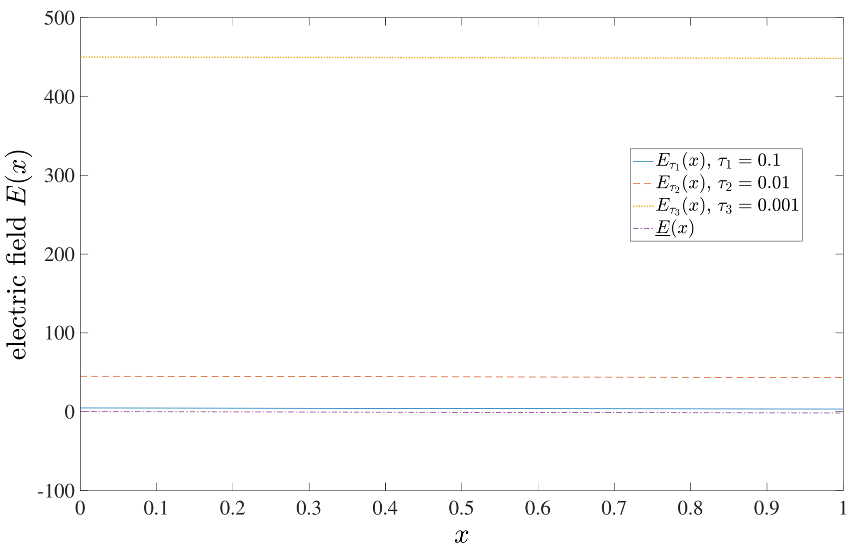
FIG. 3. *Case* 3 *for Theorem* 1.1*:* $a = b = 1$.

see Figures 1, 2, and 3. Besides, we can also see, from the figures, that there are no boundary layers when passing to the limit as $\tau \to +\infty$.

As far as Theorem 1.2 is concerned, we know that whether the boundary data a and b take the same value will make the zero-relaxation-time limit results very different. In fact, when $1 \leq a = b$, the boundary layer does not occur between ρ_τ and



(a) $\|\rho_\tau - \underline{\rho}\|_{L^\infty(0,1)} \leq C\tau$, as $\tau \to 0^+$.



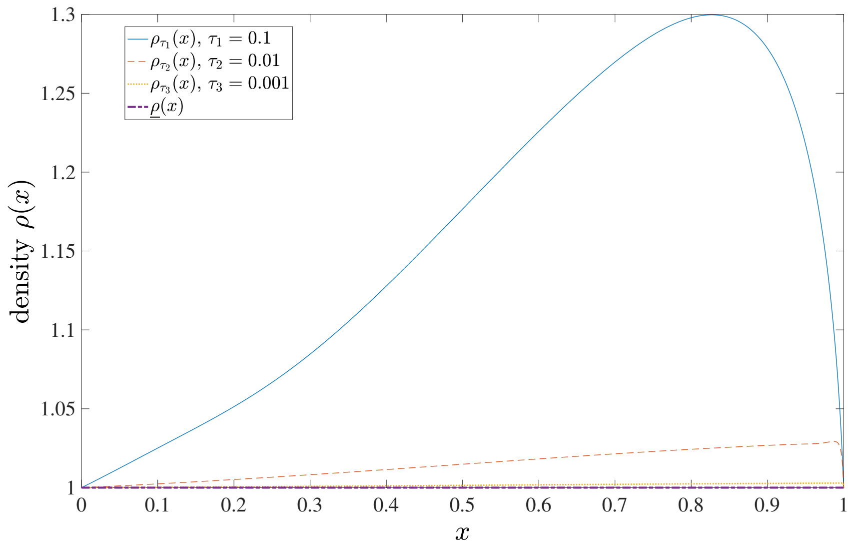
(b) $\|E_\tau - \underline{E} - \frac{1}{\tau a}\|_{L^\infty(0,1)} \leq C\tau$, as $\tau \to 0^+$.

FIG. 4. *Case* 1 *for part* (I) *of Theorem* 1.2*:* $a = b = 2$.

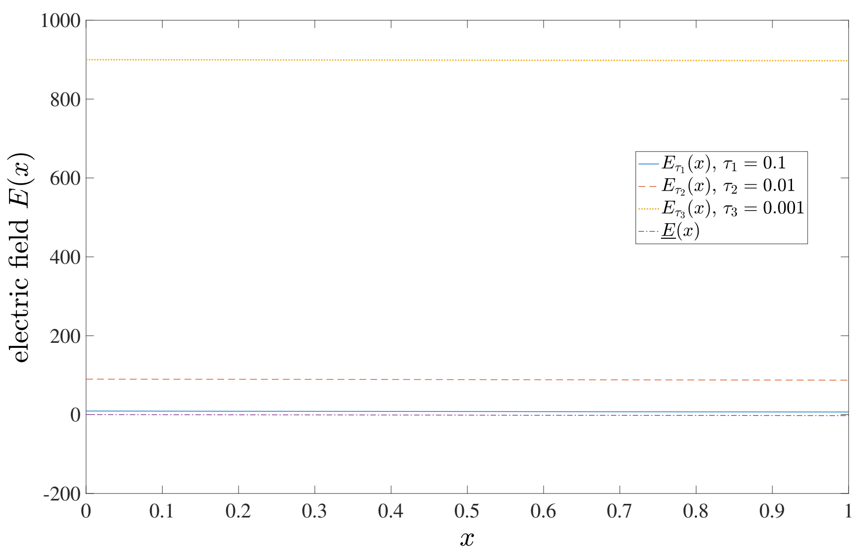
limiting density $\underline{\rho}$ as $\tau \to 0^+$; when $1 \leq a < b$, the boundary layer $(1-\tau^{1-\varepsilon}, 1]$ appears near the right endpoint $x = 1$ with a rough width $\tau^{1-\varepsilon}$. Therefore, we next take into account two numerical cases for part (I) of Theorem 1.2,

1. $a = b = 2$ (see Figure 4),
2. $a = b = 1$ (see Figure 5),

as well as consider two numerical cases for part (II) of Theorem 1.2,



(a) $\|\rho_\tau - \underline{\rho}\|_{L^\infty(0,1)} \leq C\tau^{\frac{1}{2}}, \text{ as } \tau \to 0^+.$

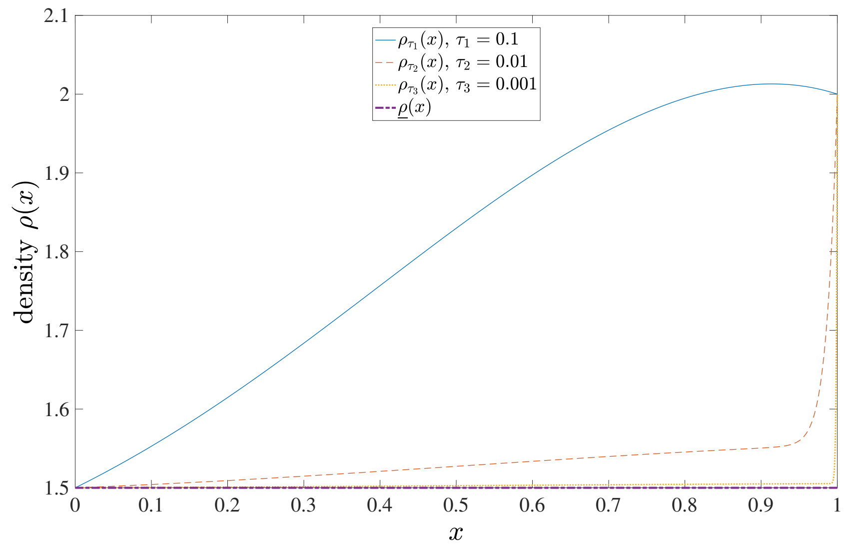


(b) $\|E_\tau - \underline{E} - \frac{1}{\tau a}\|_{L^\infty(0,1)} \leq C\tau^{\frac{1}{2}}, \text{ as } \tau \to 0^+.$

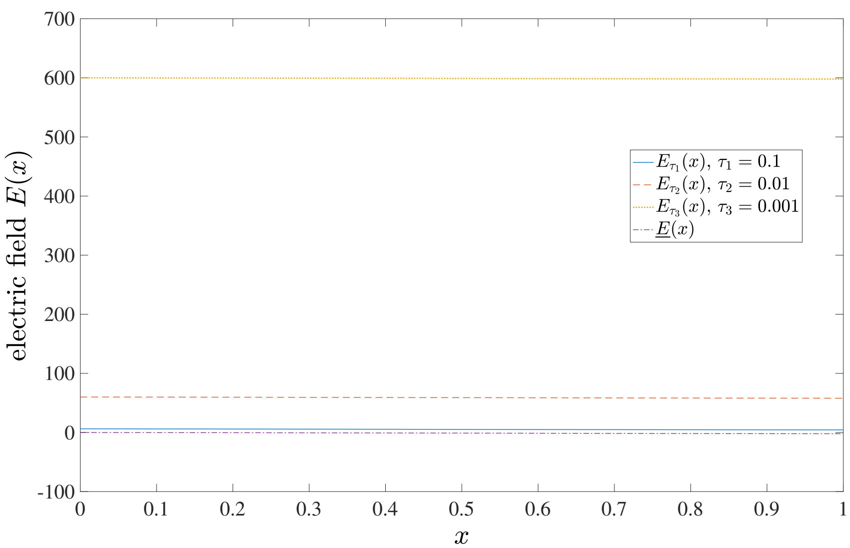
FIG. 5. *Case* 2 *for part* (I) *of Theorem* 1.2*:* $a = b = 1$.

1. $a = 1.5$ and $b = 2$ (see Figure 6),
2. $a = 1$ and $b = 1.25$ (see Figure 7).

In all the four cases for Theorem 1.2, we opt for the finite approximation sequence of relaxation times as $\tau_1 = 0.1$, $\tau_2 = 0.01$, and $\tau_3 = 0.001$, and the limiting relaxation time τ at the moment is 0. We can easily see, in Figures 4 and 5, that there is no



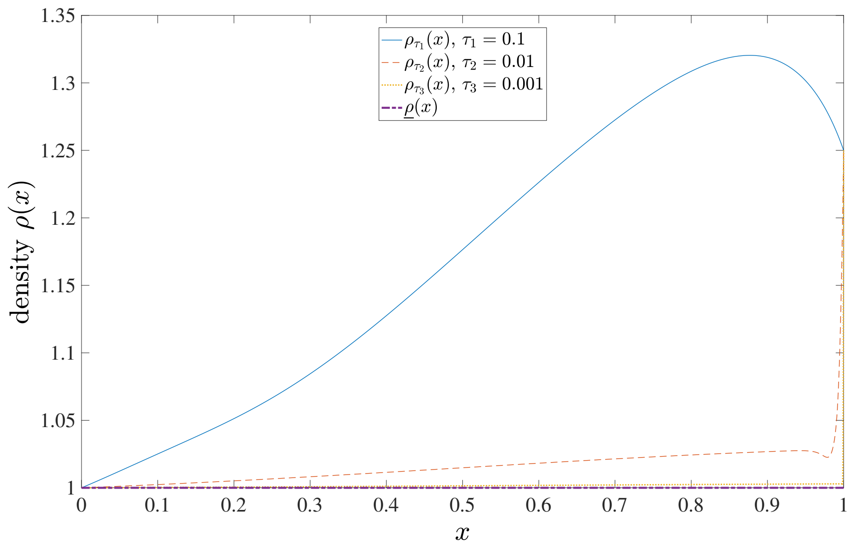
(a) $|\rho_\tau(x) - a| \leq C\tau,\ \ 0 \leq x \leq 1 - \tau^{1-\varepsilon},\ \ \text{as } \tau \to 0^+.$



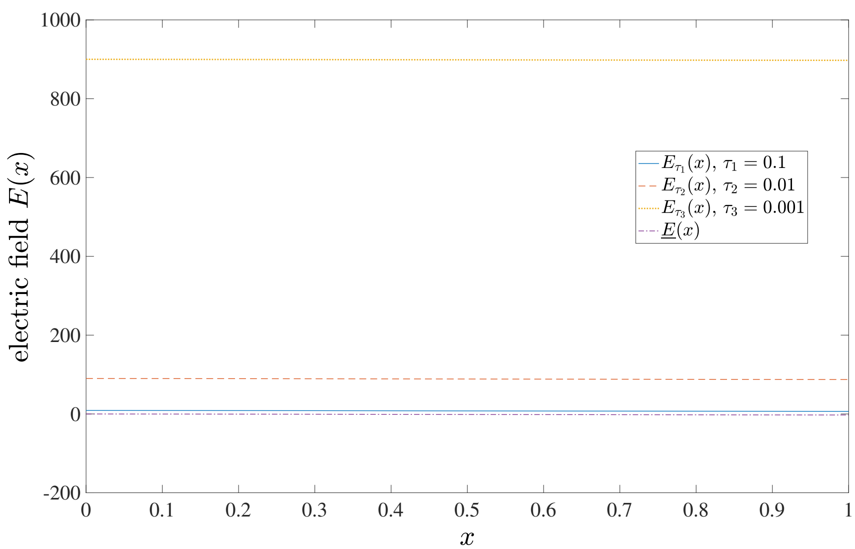
(b) $\|E_\tau - \underline{E} - \frac{1}{\tau a}\|_{L^\infty(0,1)} \leq C\tau^{1-\varepsilon},\ \ \text{as } \tau \to 0^+.$

FIG. 6. *Case* 1 *for part* (II) *of Theorem* 1.2*:* $a = 1.5$ *and* $b = 2$*.*

boundary layer between ρ_τ and $\underline{\rho}$; however, in Figures 6 and 7, we can observe from a numerical perspective that the boundary layer occurs near the right endpoint provided the relaxation time τ is small enough, and the width of boundary layer gets thinner and thinner as the relaxation time τ goes to 0. Also, we can find, in all the four figures for Theorem 1.2, that there is a huge gap $\frac{1}{\tau a}$ between E_τ and the asymptotic profile $\underline{E}$ over the entire interval $[0, 1]$.



(a) $|\rho_\tau(x) - a| \leq C\tau^{\frac{1}{2}},\ \ 0 \leq x \leq 1 - \tau^{1-\varepsilon},\ \ \text{as } \tau \to 0^+.$



(b) $\|E_\tau - \underline{E} - \frac{1}{\tau a}\|_{L^\infty(0,1)} \leq C\tau^{\frac{1}{2}},\ \ \text{as } \tau \to 0^+.$

FIG. 7. *Case* 2 *for part* (II) *of Theorem* 1.2*:* $a = 1$ *and* $b = 1.25$.

All these numerical simulations conducted in this section perfectly support our theoretical results obtained in Theorems 1.1 and 1.2.

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