A HIGH DIMENSIONAL BRAMBLE LEMMA

SERGEY NORIN

1. Introduction

The goal of this note is to prove a Helly-type result involving a higher-dimensional analogue of brambles initially defined by Seymour and Thomas [ST93].

We define brambles on simplicial complexes in terms of cohomology with coefficients in $\mathbb{Z}_2$. Hence from now on we work by default over $\mathbb{Z}_2$. We follow the algebraic topological terminology from Munkres [Mun84], but let us recall some of the basics in the setting we are interested in. (In particular, as some definitions simplify over $\mathbb{Z}_2$.)

We work exclusively with finite simplical complexes. A *d-dimensional simplicial complex K in $\mathbb{R}^N$* is a finite collection of simplices of dimension at most d closed under taking faces such that intersection of any two of simplices in K is a face of both of them. We denote by $K^{(i)}$ the set of i-dimensional simplices of K, and by $|K|$ the union of simplices in K.

Let $\mathcal{C}^i(K)$ denotes the $\mathbb{Z}_2$-vector space of formal linear combinations of elements of $K^{(i)}$, i.e. the *i-dimensional cochains of X*. The *coboundary map* $\delta : \mathcal{C}^i(K) \to \mathcal{C}^{i+1}(K)$ is defined by

$$\delta\sigma = \sum_{\sigma' \in K^{(i+1)}, \sigma \subset \sigma'} \sigma'$$

for every $\sigma \in K^{(i)}$ and extended linearly to $\mathcal{C}^i(K)$. Let $Z^i(K) = \{C \in \mathcal{C}^i(K) \mid \delta C = 0\}$ denote the group of *cocycles* and let $B^i(K) = \{\delta C \mid C \in \mathcal{C}^{i-1}(K)\}$ denote the group of *coboundaries*. Then $B^i(K)$ is a subgroup of $Z^i(K)$, as $\delta^2 = 0$. The group

$$H^i(K) = Z^i(K)/B^i(K)$$

is the *i-th cohomology group* $H^i(K)$ of K. The cohomology groups of a complex K are topological invariants, i.e. while they are defined in terms of the abstract simplicial complex corresponding to K, they depend only on $|K|$ (see [Mun84, Theorem 44.2]).

By convention, we set $K^{-1} = \emptyset$, and so $B^0(K)$ is isomorphic to $\mathbb{Z}_2$ and consists of a zero cochain and a cochain $\mathbf{1}_K := \sum_{\sigma \in K^{(0)}} \sigma$.

We are now ready to define a bramble and state our main result. A collection $\mathcal{B}$ of subcomplexes of a d-dimensional complex K is a *(d-dimensional) bramble on K* if $\mathcal{B}$ is closed under unions and the cohomology of every element of $\mathcal{B}$ is trivial in dimensions from 0 up to $d-1$. Explicitly, for every $Z \in \mathcal{B}$, $0 \le i \le d-1$ and $C \in \mathcal{C}^i(Z)$ such that $\delta C = 0$ we have $C = \delta(C')$ for some $C' \in \mathcal{C}^{i-1}(Z)$. For $i = 0$, in particular, we have under these conditions $C = 0$ or $C = \mathbf{1}_Z$. Note that a collection $\mathcal{B}$ of subcomplexes of a simplicial complex closed under unions is a 1-dimensional bramble if and only if every element of $\mathcal{B}$ is connected, and $\mathcal{B}$ is a 2-dimensional bramble if and only if every element of it is simply connected.

Theorem 1. *Let X and Y be d-dimensional simplicial complexes, such that $H^d(Y) = 0$. Let $\mathcal{B}$ be a d-dimensional bramble on X and let $f : |X| \to |Y|$ be continuous then*

$$\cap_{Z \in \mathcal{B}} f(|Z|) \neq \emptyset.$$

Note that the condition $H^d(Y) = 0$ holds, in particular when Y is contractible.

Let X and Y be simplicial complexes. A map $f : X \to Y$ is *simplicial* if it maps the vertices $X^{(0)}$ of X to $Y^{(0)}$ and extends affinely from vertices of every simplex to its interior. The simplicial approximation theorem can be used to reduce Theorem 1 to the case when f is simplicial as we will show lates in this section and the main effort is establishing the following version Theorem 1 for simplicial maps, which is done in the next section.

Theorem 2. *Let X and Y be d-dimensional simplicial complexes, such that $H^d(Y) = 0$. Let $\mathcal{B}$ be a d-dimensional bramble on X and let $f : X \to Y$ be simplicial. Then there exist $y \in Y^{(0)}$ such that $y \in \cap_{Z \in \mathcal{B}} f(Z)$.*

A simplicial complex X' is a *subdivision* of a complex X if each simplex of X' is contained in a simplex of X and each simplex of X is a union of some collection of simplices of X'. We use the following form of the simplicial approximation theorem.

Theorem 3 ([Mun84, Theorems 15.4 and 16.1])**.** *Let X and Y be simplicial complexes and and let $f : |X| \to |Y|$ be continuous. Then for every $\varepsilon > 0$ there exist subdivisions X' of X and Y' of Y and a simplicial map $h : X' \to Y'$ such that $d(h(x), f(x)) \leq \varepsilon$ for all $x \in |X|$, where $d(\cdot, \cdot)$ is the Euclidean metric on $|Y|$.*

Theorem 1 is implied by Theorem 2 and Theorem 3, as follows.

Proof of Theorem 1 assuming Theorem 2. Suppose that $\cap_{Z \in \mathcal{B}} f(|Z|) = \emptyset$. Then there exists $\varepsilon > 0$ such that $\cap_{Z \in \mathcal{B}} B_\varepsilon(f(|Z|)) = \emptyset$, where $B_\varepsilon(S) = \{y \in |Y| \mid d(y, S) \leq \varepsilon\}$ is the ε-neighborhood of S for any $S \subseteq Y$. Let subdivisions X' of X and Y' of Y and a simplicial map $h : X' \to Y'$ be chosen to satisfy the conclusion of Theorem 3, i.e. $d(h(x), f(x)) < \varepsilon$ for all $x \in |X|$. Note that $\mathcal{B}$ naturally corresponds to a bramble $\mathcal{B}'$ on X' where every element Z' of $\mathcal{B}'$ is a subdivision of the corresponding element Z of $\mathcal{B}$, an so $|Z| = |Z'|$ and consequently $H^{(i)}(Z) = H^{(i)}(Z')$ for every i as noted above. In particular, we have $\cap_{Z' \in \mathcal{B}'} B_\varepsilon(f(|Z'|)) = \emptyset$. As $h(|Z'|) \subseteq B_\varepsilon(f(|Z'|))$ for every $Z' \in \mathcal{B}'$, it follows that $\cap_{Z' \in \mathcal{B}'} h(|Z'|) = \emptyset$, in contradiction with Theorem 2 applied to h and $\mathcal{B}'$. □

2. Proof of Theorem 2

Our proof of Theorem 2 is inspired by the proof by Dotterrer, Kaufman and Wagner [DKW18] of Gromov's Topological Overlap theorem [Gro10]. However, the proof in [DKW18] involves working with both chains and co-chains, and the homotopy map constructed in [DKW18], analogous to the map H constructed in Lemma 4 below, is subject to quantitative restriction on the norm of images of simplices, while our restrictions are qualitative and related to the position of images with respect to the bramble.

Given a pair of simplicial complexes X and Y simplicial map $f : X \to Y$ we define a pullback map $f^\sharp : \mathcal{C}^i(Y) \to \mathcal{C}^i(X)$, by defining $f^\sharp(\sigma) = \sum_{\sigma' \in X^{(i)}, f(\sigma') = \sigma} \sigma'$ for every $\sigma \in Y^{(i)}$ and extending linearly to $\mathcal{C}^i(X)$.

Crucially, the coboundary map commutes with pullbacks, i.e.

$$f^\sharp \delta = \delta f^\sharp.$$

Given a subcomplex Z of X let $\iota_{Z,X} : Z \to X$ denote the inclusion map from Z to X. We denote the corresponding pullback map $\iota^{\sharp}_{Z,X} : \mathcal{C}^i(X) \to \mathcal{C}^i(Z)$ by $\downarrow_Z$. This map has a natural explicit definition $\downarrow_Z (\sigma) = \sigma$ for every $\sigma \in Z^{(i)}$ and $\downarrow_Z (\sigma) = 0$ for every $\sigma \in X^{(i)} - Z^{(i)}$. **??** in particular implies that

$$\downarrow_Z \delta = \delta \downarrow_Z .$$

A linear map $H : \mathcal{C}^*(Y) \to \mathcal{C}^{*-1}(X)$ is an *$f^\sharp$-null homotopy* if $f^\sharp = \delta H + H\delta$.

Lemma 4. *Let $f : X \to Y$ be a simplicial map between d-dimensional simplicial complexes, let $\mathcal{B}$ be a d-dimensional bramble in X. Suppose that $H^d(Y)$ is trivial, i. e. for every $\sigma \in Y^{(d)}$ there exists $C \in \mathcal{C}^{(d-1)}(Y)$ such that $\sigma = \delta(C)$.*

Then there exist an $f^\sharp$–null homotopy H such that for every $C \in \mathcal{C}^*(Y)$ and $Z \in \mathcal{B}$ if $\downarrow_Z C = 0$ then $\downarrow_Z H(C) = 0$.

Proof. Assume without loss of generality that $X = \cup_{Z \in \mathcal{B}} Z$. In particular, $X \in \mathcal{B}$, as $\mathcal{B}$ is closed under unions. We define $H(\sigma)$ for $\sigma \in Y^{(i)}$ for $0 \le i \le d$ by induction on $d - i$, so that

(i) $f^\sharp(\sigma) = \delta H(\sigma) + H(\delta\sigma)$, and
(ii) $\downarrow_Z H(\sigma) = 0$ for every $Z \in \mathcal{B}$ such that $\downarrow_Z C = 0$ then $\downarrow_Z H(\sigma) = 0$.

We then extend H to $C^i(Y)$ linearly.

For the base case consider $\sigma \in Y^{(d)}$. Then there exists $C \in \mathcal{C}^{d-1}(Y)$ such that $\delta(C) = \sigma$. Let $Z \in \mathcal{B}$ be maximum such that $\sigma \notin f(Z)$. Then

$$\delta \downarrow_Z f^\sharp(C) = \downarrow_Z f^\sharp(\delta(C)) = \downarrow_Z f^\sharp(\sigma) = 0.$$

As $H^{d-1}(Z)$ is trivial, there exists $C' \in \mathcal{C}^{d-2}(Z)$ such that $\downarrow_Z \delta(C') = \downarrow_Z f^\sharp(C)$. Define $H(\sigma) = f^\sharp(C) - \delta(C')$. Then $\downarrow_Z H(\sigma) = 0$, by the above and

$$\delta H(\sigma) + H(\delta\sigma) = \delta f^\sharp(C) - \delta^2(C') = f^\sharp(\delta(C)) = f^\sharp(\sigma).$$

For the induction step consider now $\sigma \in Y^{(i)}$. Note that we have already defined H on $C^{i+1}(Y)$. Let $C = f^\sharp(\sigma) - H(\delta(\sigma)) \in \mathcal{C}^i(X)$. We have

$$\delta H(\delta\sigma) = (\delta H + H\delta)(\delta\sigma) = f^\sharp(\delta\sigma) = \delta(f^\sharp(\sigma)).$$

Thus $\delta(C) = 0$. As $H^i(X)$ is trivial there exists $C' \in C^{i-1}(X)$ such that $C = \delta C'$. Again let Z be be maximum such that $\sigma \notin f(Z)$. Then $\delta(\sigma)$ is a sum of simplices $\sigma' \in Y^{(i+1)}$ such that $\sigma \subseteq \sigma'$ and so $\sigma' \notin f(Z)$. By the induction hypothesis this implies, $\downarrow_Z H(\sigma') = 0$ for each such σ', and so $\downarrow_Z H(\delta\sigma) = 0$. Thus

$$\delta(\downarrow_Z C') = \downarrow_Z (\delta C') = \downarrow_Z C = \downarrow_Z f^\sharp(\sigma) - \downarrow_Z H(\delta\sigma) = 0.$$

As $H^{i-1}(Z)$ is trivial, there exists $C'' \in \mathcal{C}^{i-2}(Z)$ such that $\downarrow_Z \delta C'' = \downarrow_Z C'$. Finally define $H(\sigma) = C' - \delta C''$. Then $\downarrow_Z H(\sigma) = 0$, and

$$\delta H(\sigma) + H(\delta\sigma) = \delta C' - \delta\delta C'' + H(\delta\sigma) = C + H(\delta\sigma) = f^\sharp(\sigma),$$

as desired. □

Finally, Theorem 2 readily follows from Lemma 4.

Proof of Theorem 2. As in Lemma 4 we assume that $X \in \mathcal{B}$.

Let H be as in Lemma 4. Consider arbitrary $y \in Y^{(0)}$. Then

$$\delta(H(\delta(y)) = f^\sharp(\delta(v)) + H(\delta(\delta(y))) = \delta(f^\sharp(y)).$$

Thus $\delta(H(\delta(y)) + f^\sharp(y)) = 0$. Thus either $H(\delta(y)) + f^\sharp(y) = \mathbf{1}_X$ or $H(\delta(y)) + f^\sharp(y) = 0$. In the first case, if follows that $\downarrow_Z (H(\delta(y)) + f^\sharp(y)) = \mathbf{1}_Z$ for every $Z \in \mathcal{B}$. We claim that this in turn implies that $y \subseteq f(Z)$. Indeed, suppose not, then $\downarrow_Z (f^\sharp(\sigma)) = 0$ for every $\sigma \in Y$ such that $y \subseteq \sigma$. Thus as in the proof of Lemma 4 we have $\downarrow_Z (H(\delta(y))) = 0$ and so $\downarrow_Z (H(\delta(y)) + f^\sharp(y)) = 0$, a contradiction.

It follows that if $H(\delta(y)) + f^\sharp(y) = \mathbf{1}_X$ for some $y \in Y^{(0)}$ then the theorem holds. Assume then for a contradiction that no such y exists and $H(\delta(y)) + f^\sharp(y) = 0$ for every $y \in Y^{(0)}$. Summing over all such y we get

$$0 = H(\delta(\mathbf{1}_Y)) + f^\sharp(\mathbf{1}_Y) = 0 + \mathbf{1}_X = \mathbf{1}_X,$$

yielding the desired contradiction. □

References

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